Vectors

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

SCALAR PRODUCT

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A Tutorial Module for learning about the scalar product of two vectors

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Table of contents

1. Theory

The purpose of this tutorial is to practice using the scalar product of two vectors. It is called the 'scalar product' because the result is a 'scalar', i.e. a quantity with **magnitude** but no associated direction.

The **SCALAR PRODUCT** (or 'dot product') of $\underline{a}$ and $\underline{b}$ is

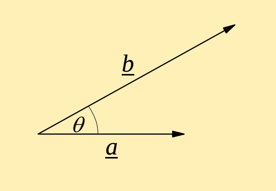
$$\begin{aligned}\underline{a} \cdot \underline{b} &= |\underline{a}|\,|\underline{b}|\cos\theta \\ &= a_x b_x + a_y b_y + a_z b_z\end{aligned}$$

where θ is the angle between $\underline{a}$ and $\underline{b}$

and

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}.$$



Note that when

$$\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$$

and

$$\underline{b} = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}$$

then the magnitudes of $\underline{a}$ and $\underline{b}$ are

$$|\underline{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

and

$$|\underline{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}\ ,$$

respectively.

2. Exercises

Click on Exercise links for full worked solutions (there are 16 exercises in total)

Exercise 1. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 2\underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{b} = \underline{i} + 2\underline{j} + 8\underline{k}$

Exercise 2. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 4\underline{i} - 7\underline{j} + 2\underline{k}$, $\underline{b} = 5\underline{i} - \underline{j} - 4\underline{k}$

Exercise 3. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 2\underline{i} + 3\underline{j} + 3\underline{k}$, $\underline{b} = 3\underline{i} - 2\underline{j} + 5\underline{k}$

Exercise 4. Calculate $\underline{a} \cdot \underline{b}$ when $\underline{a} = 3\underline{i} + 6\underline{j} - \underline{k}$, $\underline{b} = 8\underline{i} - 3\underline{j} - \underline{k}$

EXERCISE 5. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = \underline{i} + \underline{j} + 3\underline{k}, \quad \underline{b} = \underline{i} - 7\underline{j} + 2\underline{k}$$

EXERCISE 6. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = \underline{i} + 23\underline{j} + 7\underline{k}, \quad \underline{b} = 26\underline{i} + \underline{j} - 7\underline{k}$$

EXERCISE 7. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = \underline{i} + \underline{j} + 3\underline{k}, \quad \underline{b} = 2\underline{i} + 7\underline{j} - 3\underline{k}$$

EXERCISE 8. Show that $\underline{a}$ is perpendicular to $\underline{b}$ when

$$\underline{a} = 39\underline{i} + 2\underline{j} + \underline{k}, \quad \underline{b} = \underline{i} - 23\underline{j} + 7\underline{k}$$

Exercise 9. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 7\,\mathrm{N}, \quad |\underline{s}| = 3\,\mathrm{m}, \quad \theta = 0^\circ$

Exercise 10. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 4\,\mathrm{N}, \quad |\underline{s}| = 2\,\mathrm{m}, \quad \theta = 27^\circ$

Exercise 11. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 5\,\mathrm{N}, \quad |\underline{s}| = 4\,\mathrm{m}, \quad \theta = 48^\circ$

Exercise 12. Calculate the work done $\underline{F} \cdot \underline{s}$ given $|\underline{F}|$, $|\underline{s}|$ and θ (the angle between the force $\underline{F}$ and the displacement $\underline{s}$) when
$|\underline{F}| = 2\,\mathrm{N}, \quad |\underline{s}| = 3\,\mathrm{m}, \quad \theta = 56^\circ$

Exercise 13. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when
$$\underline{a} = 2\underline{i} - \underline{j} + 2\underline{k}\ , \qquad \underline{b} = \underline{i} + \underline{j} + \underline{k}$$

Exercise 14. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when
$$\underline{a} = \underline{i} + \underline{j} + \underline{k}\ , \qquad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$$

Exercise 15. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when
$$\underline{a} = \underline{i} - 2\underline{j} + 2\underline{k}\ , \qquad \underline{b} = 2\underline{i} + 3\underline{j} + \underline{k}$$

Exercise 16. Calculate the angle θ between vectors $\underline{a}$ and $\underline{b}$ when
$$\underline{a} = 5\underline{i} + 4\underline{j} + 3\underline{k}\ , \qquad \underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$$

3. Answers

1. 36,
2. 19,
3. 15,
4. 7,
5. Hint: If $\theta = 90°$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
6. Hint: If $\theta = 90°$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
7. Hint: If $\theta = 90°$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
8. Hint: If $\theta = 90°$ then what will $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$ be?
9. 21 J,
10. 7.128 J,
11. 13.38 J,
12. 3.355 J,
13. 54.7°,

14. 90°,
15. 100.3°,
16. 79.6°.

4. Tips on using solutions

● When looking at the THEORY, ANSWERS, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

5. Alternative notation

● Here, we use symbols like $\underline{a}$ to denote a vector.
In some texts, symbols for vectors are **in bold** (eg **a** instead of $\underline{a}$)

● In this Tutorial, vectors are given in terms of the unit Cartesian vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$.

For example, $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ implies that $\underline{a}$ can be **decomposed** into the sum of the following three vectors:

	$\underline{i}$	(*one* step along the x-axis)
PLUS	$2\,\underline{j}$	(*two* steps along the y-axis)
PLUS	$3\,\underline{k}$	(*three* steps along the z-axis)

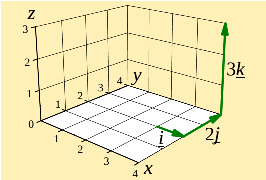
See the figures on the next page ...

$$\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$$

one step along the x-axis

two steps along the y-axis

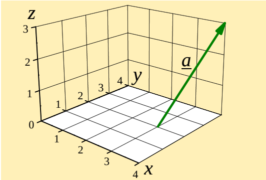
three steps along the z-axis



$\underline{a}$ is the (vector) sum

of $\underline{i}$ and $2\underline{j}$

and $3\underline{k}$



- A common alternative notation for expressing $\underline{a}$ in terms of these **Cartesian components** is given by $\underline{a} = (1, 2, 3)$

Full worked solutions

Exercise 1.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 2\underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{b} = \underline{i} + 2\underline{j} + 8\underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (2)(1) + (-3)(2) + (5)(8) \\ &= 2 - 6 + 40 \\ &= 36. \end{aligned}$$

Return to Exercise 1

Exercise 2.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 4\underline{i} - 7\underline{j} + 2\underline{k}, \;\; \underline{b} = 5\underline{i} - \underline{j} - 4\underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (4)(5) + (-7)(-1) + (2)(-4) \\ &= 20 + 7 - 8 \\ &= 19. \end{aligned}$$

Return to Exercise 2

Exercise 3.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 2\underline{i} + 3\underline{j} + 3\underline{k}$, $\underline{b} = 3\underline{i} - 2\underline{j} + 5\underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (2)(3) + (3)(-2) + (3)(5) \\ &= 6 - 6 + 15 \\ &= 15. \end{aligned}$$

Return to Exercise 3

Exercise 4.

$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$, where

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\underline{b} = b_x \underline{i} + b_y \underline{j} + b_z \underline{k}$$

$\underline{a} = 3\underline{i} + 6\underline{j} - \underline{k}, \quad \underline{b} = 8\underline{i} - 3\underline{j} - \underline{k}$ gives

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (3)(8) + (6)(-3) + (-1)(-1) \\ &= 24 - 18 + 1 \\ &= 7. \end{aligned}$$

Return to Exercise 4

Exercise 5.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$
\begin{array}{ll}
\underline{a} \text{ perpendicular to } \underline{b} \text{ gives} & \theta = 90^\circ \\
\text{i.e.} & \cos\theta = 0 \\
\text{i.e.} & \underline{a} \cdot \underline{b} = 0
\end{array}
$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$
\begin{aligned}
\text{i.e.} \qquad \underline{a} \cdot \underline{b} &= (1)(1) + (1)(-7) + (3)(2) \\
&= 1 - 7 + 6 \\
&= 0.
\end{aligned}
$$

Return to Exercise 5

Exercise 6.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$
\begin{aligned}
\underline{a} \text{ perpendicular to } \underline{b} \text{ gives} \quad & \theta = 90^\circ \\
\text{i.e.} \quad & \cos\theta = 0 \\
\text{i.e.} \quad & \underline{a} \cdot \underline{b} = 0
\end{aligned}
$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$
\begin{aligned}
\text{i.e.} \qquad \underline{a} \cdot \underline{b} &= (1)(26) + (23)(1) + (7)(-7) \\
&= 26 + 23 - 49 \\
&= 0.
\end{aligned}
$$

Return to Exercise 6

Exercise 7.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$\begin{array}{lrl} \underline{a} \text{ perpendicular to } \underline{b} \text{ gives} & \theta & = 90^\circ \\ \text{i.e.} & \cos\theta & = 0 \\ \text{i.e.} & \underline{a}\cdot\underline{b} & = 0 \end{array}$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$\begin{aligned} \text{i.e.} \qquad \underline{a}\cdot\underline{b} &= (1)(2) + (1)(7) + (3)(-3) \\ &= 2 + 7 - 9 \\ &= 0. \end{aligned}$$

Return to Exercise 7

Exercise 8.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between $\underline{a}$ and $\underline{b}$

$$\begin{aligned} \underline{a} \text{ perpendicular to } \underline{b} \text{ gives} \quad \theta &= 90^\circ \\ \text{i.e.} \quad \cos\theta &= 0 \\ \text{i.e.} \quad \underline{a} \cdot \underline{b} &= 0 \end{aligned}$$

To show that this is true, use $\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z$

$$\begin{aligned} \text{i.e.} \qquad \underline{a} \cdot \underline{b} &= (39)(1) + (2)(-23) + (1)(7) \\ &= 39 - 46 + 7 \\ &= 0. \end{aligned}$$

Return to Exercise 8

Exercise 9.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$,

$$|\underline{F}| = 7\,\mathrm{N}$$

$$|\underline{s}| = 3\,\mathrm{m}$$

$$\theta = 0^\circ \text{ gives } \cos\theta = 1$$

$\therefore\ \underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta = (7\,\mathrm{N})(3\,\mathrm{m})(1) = 21\,\mathrm{J}.$

<u>Note:</u> When the angle θ is zero then

$$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}|$$

and one simply multiplies the magnitudes of $\underline{F}$ and $\underline{s}$.

Return to Exercise 9

Exercise 10.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta,$

$$|\underline{F}| = 4\ \mathrm{N}$$

$$|\underline{s}| = 2\ \mathrm{m}$$

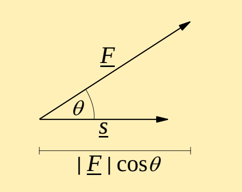
$$\theta = 27^\circ \text{ gives } \cos\theta \simeq 0.8910$$

$\therefore\ \underline{F} \cdot \underline{s} \simeq (4\ \mathrm{N})(2\ \mathrm{m})(0.8910) \simeq 7.128\ \mathrm{J}.$

$$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$$

and

$$\underline{F} \cdot \underline{s} = (|\underline{F}| \cos\theta)\ |\underline{s}|$$



Note: $\underline{F} \cdot \underline{s}$ is the product of $|\underline{s}|$ and the **projected component** of force $|\underline{F}| \cos\theta$ along the direction of $\underline{s}$.

Return to Exercise 10

Exercise 11.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$,

$$|\underline{F}| = 5\,\mathrm{N}$$

$$|\underline{s}| = 4\,\mathrm{m}$$

$$\theta = 48^\circ \text{ gives } \cos\theta \simeq 0.6691$$

$\therefore\ \underline{F} \cdot \underline{s} \simeq (5\,\mathrm{N})(4\,\mathrm{m})(0.6691) \simeq 13.38\,\mathrm{J}.$

Return to Exercise 11

Exercise 12.

$\underline{F} \cdot \underline{s} = |\underline{F}||\underline{s}| \cos\theta$,

$$|\underline{F}| = 2\text{ N}$$

$$|\underline{s}| = 3\text{ m}$$

$$\theta = 56^\circ \text{ gives } \cos\theta \simeq 0.5592$$

$\therefore\ \underline{F} \cdot \underline{s} \simeq (2\text{ N})(3\text{ m})(0.5592) \simeq 3.355\text{ J}.$

Return to Exercise 12

Exercise 13.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a} \cdot \underline{b} &= a_x b_x + a_y b_y + a_z b_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = 2\underline{i} - \underline{j} + 2\underline{k}\ , \quad \underline{b} = \underline{i} + \underline{j} + \underline{k}$$

i.e. $a_x = 2,\ a_y = -1,\ a_z = 2$ and $b_x = 1,\ b_y = 1,\ b_z = 1$

then

$$\underline{a} \cdot \underline{b} = (2)(1) + (-1)(1) + (2)(1) = 2 - 1 + 2 = 3$$

$$|\underline{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

$$|\underline{b}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\therefore \quad \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{3}{3\sqrt{3}} \simeq 0.5774$$

so $\theta \simeq \cos^{-1}(0.5774) \simeq 54.7^\circ$.

Return to Exercise 13

Exercise 14.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x \underline{i} + a_y \underline{j} + a_z \underline{k} \\ \underline{b} &= b_x \underline{i} + b_y \underline{j} + b_z \underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a} \cdot \underline{b} &= a_x b_x + a_y b_y + a_z b_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = \underline{i} + \underline{j} + \underline{k}\ , \quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$$

i.e. $a_x = 1,\ a_y = 1,\ a_z = 1$ and $b_x = 2,\ b_y = -3,\ b_z = 1$

then

$$\underline{a} \cdot \underline{b} = (1)(2) + (1)(-3) + (1)(1) = 2 - 3 + 1 = 0$$

$$|\underline{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\underline{b}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \frac{0}{\sqrt{3}\sqrt{14}} = \frac{0}{\sqrt{42}} = 0$$

so $\theta = \cos^{-1}(0) = 90^\circ.$

Return to Exercise 14

Exercise 15.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \tfrac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a}\cdot\underline{b} &= a_xb_x + a_yb_y + a_zb_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = \underline{i} - 2\underline{j} + 2\underline{k}\ , \quad \underline{b} = 2\underline{i} + 3\underline{j} + \underline{k}$$

i.e. $a_x = 1,\ a_y = -2,\ a_z = 2$ and $b_x = 2,\ b_y = 3,\ b_z = 1$

then

$$\underline{a} \cdot \underline{b} = (1)(2) + (-2)(3) + (2)(1) = 2 - 6 + 2 = -2$$

$$|\underline{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|\underline{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \tfrac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \tfrac{-2}{3\sqrt{14}} \simeq -0.1782$$

so $\quad \theta \simeq \cos^{-1}(-0.1782) \simeq 100.3^\circ,$

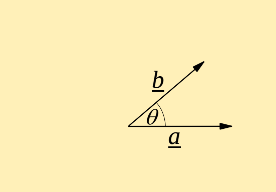
see also the Note on the next page ...

Note:

When $\underline{a} \cdot \underline{b}$ is positive

$\cos\theta$ is positive

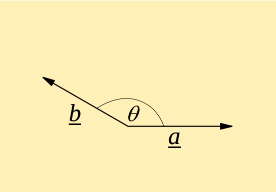
and θ is an acute angle



When $\underline{a} \cdot \underline{b}$ is negative

$\cos\theta$ is negative

and θ is an obtuse angle



End of Note.

Return to Exercise 15

Exercise 16.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$, where θ is the angle between vectors $\underline{a}$ and $\underline{b}$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$$

$$\begin{aligned} \text{If} \quad \underline{a} &= a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \\ \underline{b} &= b_x\underline{i} + b_y\underline{j} + b_z\underline{k} \end{aligned}$$

then $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$ where

$$\begin{aligned} \underline{a}\cdot\underline{b} &= a_xb_x + a_yb_y + a_zb_z \\ |\underline{a}| &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\ |\underline{b}| &= \sqrt{b_x^2 + b_y^2 + b_z^2} \end{aligned}$$

$$\underline{a} = 5\underline{i} + 4\underline{j} + 3\underline{k}\ , \quad \underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$$

i.e. $a_x = 5,\ a_y = 4,\ a_z = 3$ and $b_x = 4,\ b_y = -5,\ b_z = 3$

then

$$\underline{a} \cdot \underline{b} = (5)(4) + (4)(-5) + (3)(3) = 20 - 20 + 9 = 9$$

$$|\underline{a}| = \sqrt{5^2 + 4^2 + 3^2} = \sqrt{25 + 16 + 9} = \sqrt{50}$$

$$|\underline{b}| = \sqrt{4^2 + (-5)^2 + 3^2} = \sqrt{16 + 25 + 9} = \sqrt{50}$$

$$\therefore \quad \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \frac{9}{\sqrt{50}\sqrt{50}} = \frac{9}{50} = 0.18$$

so $\theta = \cos^{-1}(0.18) \simeq 79.6^\circ$.

Return to Exercise 16