Differential Equations

SECOND ORDER (homogeneous)

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A Tutorial Module for learning to solve 2nd order (homogeneous) differential equations

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1. Theory

In this Tutorial, we will practise solving equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

i.e. second order (the highest derivative is of second order),
linear (y and/or its derivatives are to degree one) with
constant coefficients (a, b and c are constants that may be zero).

There are no terms that are constants and no terms that are only a function of x. If such terms were present, it would be conventional to collect them together on the right-hand-side of the equation. Here, we simply have a zero on the right-hand-side of the equals sign and this type of ordinary differential equation (o.d.e.) is called "homogeneous".

Since the o.d.e. is second order, we expect the general solution to have two arbitrary constants (these will be denoted A and B).

A trial solution of the form $y = Ae^{mx}$ yields an "auxiliary equation":

$$am^2 + bm + c = 0,$$

that will have two roots (m_1 and m_2).

The general solution y of the o.d.e. is then constructed from the possible forms (y_1 and y_2) of the trial solution. The auxiliary equation may have:

i) real different roots,
m_1 and $m_2 \rightarrow y = y_1 + y_2 = Ae^{m_1 x} + Be^{m_2 x}$

or ii) real equal roots,
$m_1 = m_2 \rightarrow y = y_1 + xy_2 = (A + Bx)e^{m_1 x}$

or iii) complex roots,
$p \pm iq \rightarrow y = y_1 + y_2 \equiv e^{px}(A\cos qx + B\sin qx)$

2. Exercises

Find the general solution of the following equations. Where boundary conditions are also given, derive the appropriate particular solution too.

Click on Exercise links for full worked solutions (there are 16 exercises in total).

[Notation: $y'' = \frac{d^2y}{dx^2}, \quad y' = \frac{dy}{dx}$]

Exercise 1. $2y'' + 3y' - 2y = 0$

Exercise 2. $y'' - 2y' + 2y = 0$

Exercise 3. $y'' - 2y' + y = 0$

Exercise 4. $y'' = -4y$

Exercise 5. $y'' = 4y$

Exercise 6. $36y'' - 36y' + 13y = 0$

Exercise 7. $3y'' + 2y' = 0$

Exercise 8. $16y'' - 8y' + y = 0$

Exercise 9. $y'' + 4y' + 5y = 0\,; \quad y(0) = 0 \quad \text{and} \quad y'(0) = 2$

Exercise 10. $y'' + 6y' + 13y = 0\,; \; y(0) = 2 \quad \text{and} \quad y'(0) = 1$

Exercise 11. $y'' + 6y' + 9y = 0\,; \quad y(0) = 1 \quad \text{and} \quad y'(0) = 2$

Exercise 12. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

Exercise 13. $\dfrac{d^2y}{d\tau^2} - 6\dfrac{dy}{d\tau} + 9y = 0$

Exercise 14. $\dfrac{d^2y}{d\tau^2} + 7\dfrac{dy}{d\tau} + 12y = 0$

Exercise 15. $\dfrac{d^2x}{d\tau^2} + 5\dfrac{dx}{d\tau} + 6x = 0$

Exercise 16. $4\dfrac{d^2x}{d\tau^2} + 8\dfrac{dx}{d\tau} + 3x = 0$

3. Answers

1. $y = Ae^{\frac{1}{2}x} + Be^{-2x}$,

2. $y = e^x(A\cos x + B\sin x)$,

3. $y = (A + Bx)e^x$,

4. $y = A\cos 2x + B\sin 2x$,

5. $y = Ae^{2x} + Be^{-2x}$,

6. $y = e^{\frac{1}{2}x}(A\cos\frac{x}{3} + B\sin\frac{x}{2})$,

7. $y = A + Be^{-\frac{2}{3}x}$,

8. $y = (A + Bx)e^{\frac{1}{4}x}$,

9. $y = e^{-2x}(A\cos x + B\sin x)$; $\quad y = 2e^{-2x}\sin x$,

10. $y = e^{-3x}(A\cos 2x + B\sin 2x)$; $\quad y = \frac{1}{2}e^{-3x}(4\cos 2x + 7\sin 2x)$,

11. $y = (A + Bx)e^{-3x}$; $\quad y = (1 + 5x)e^{-3x}$,

12. $y = e^{-\frac{1}{2}x}(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}}{2}x)$,

13. $y = (A + B\tau)e^{3\tau}$,

14. $y = Ae^{-4\tau} + Be^{-3\tau}$,

15. $x = Ae^{-2\tau} + Be^{-3\tau}$,

16. $x = Ae^{-\frac{3}{2}\tau} + Be^{-\frac{1}{2}\tau}$.

4. Solving quadratics

To solve the quadratic equation:

$$am^2 + bm + c = 0, \quad \text{where } a, b, c \text{ are constants,}$$

one can sometimes identify simple linear factors that multiply together to give the left-hand-side of the equation.

Alternatively, one can always use the quadratic formula:

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

to find the values of m (denoted m_1 and m_2) that satisfy the quadratic equation.

5. Tips on using solutions

● When looking at the THEORY, ANSWERS, SOLVING QUADRATICS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1. $\boxed{2y'' + 3y' - 2y = 0}$

$$\text{Set} \quad y = Ae^{mx} \rightarrow \frac{dy}{dx} = mAe^{mx} = my$$
$$\frac{d^2y}{dx^2} = m^2Ae^{mx} = m^2y$$

i.e. $2m^2y + 3my - 2y = 0$

i.e. $2m^2 + 3m - 2 = 0$: Auxiliary Equation (A.E.)

i.e. $(2m-1)(m+2) = 0$

i.e. $m_1 = \frac{1}{2}$ and $m_2 = -2$: Two Different Real Roots

i.e. $y_1 = Ae^{\frac{1}{2}x}$ and $y_2 = Be^{-2x}$: Two Independent Solutions

General solution is $y = y_1 + y_2 = Ae^{\frac{1}{2}x} + Be^{-2x}$, ($A$, B are arbitrary constants).

Return to Exercise 1

Exercise 2. $\boxed{y'' - 2y' + 2y = 0}$

$$\text{Set} \quad y = Ae^{mx}, \ \frac{dy}{dx} = my, \ \frac{d^2y}{dx^2} = m^2 y$$

$\rightarrow \quad m^2 - 2m + 2 = 0 \qquad$ <u>Auxiliary Equation</u> (A.E.)

i.e. $\quad am^2 + bm + c = 0 \quad$ with solutions $m = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$.

Since $\quad b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4 < 0\,,$

expect <u>Complex (Conjugate) Solutions</u>.

In fact, $\quad m = \frac{1}{2}\left(2 \pm \sqrt{-4}\right) = \frac{1}{2}\left(2 \pm 2\sqrt{-1}\right) = 1 \pm i\,.$

For $m = p \pm iq\,, \quad y = e^{px}(A\cos qx + B\sin qx)$

i.e. $\quad y = e^{x}(A\cos x + B\sin x), \quad$ since $p = 1$ and $q = 1$.

Return to Exercise 2

Exercise 3. $\boxed{y'' - 2y' + y = 0}$

$$m^2 - 2m + 1 = 0 \quad \text{(A.E.)}$$

$$\text{i.e.} \quad (m-1)^2 = 0 \qquad \text{i.e.} \quad m = 1 \ \text{(twice)}$$

$$\text{i.e.} \quad \underline{\text{EQUAL REAL ROOTS}}$$

Multiply one solution by x, to get two independent solutions

$$\text{i.e.} \quad y = y_1 + xy_2 = Ae^x + xBe^x = (A + Bx)e^x \ .$$

Return to Exercise 3

Exercise 4. $\boxed{y'' = -4y}$

$$m^2 = -4 \quad \text{i.e.} \quad m = \sqrt{-1} \cdot \sqrt{4} = i \cdot (\pm 2) = \pm 2i$$

(A.E.) i.e. complex conjugate solutions of form $p \pm iq$
with $p = 0,\ q = 2$.

General solution, $y = e^{px}\,(A\cos qx + B\sin qx)$

i.e. $y = e^{0}\,(A\cos 2x + B\sin 2x)$

i.e. $y = A\cos 2x + B\sin 2x$.

Return to Exercise 4

Exercise 5. $\boxed{y'' = 4y}$

$$
\begin{aligned}
\text{A.E. is} \quad & m^2 = 4 \\
\text{i.e.} \quad & m = \pm 2 \\
\text{i.e.} \quad & \text{real different roots} \\
\therefore \quad & y = y_1 + y_2 = Ae^{2x} + Be^{-2x} \ .
\end{aligned}
$$

Return to Exercise 5

Exercise 6. $\boxed{3y'' - 36y' + 13y = 0}$

$$\text{A.E. is} \quad 36m^2 - 36m + 13 = 0$$

$$\text{i.e.} \quad m = \tfrac{1}{2\cdot 36}\left(36 \pm \sqrt{(-36)^2 - 4\cdot 36\cdot 13}\right)$$

$$= \tfrac{1}{2}\left(1 \pm \sqrt{1 - \tfrac{4\cdot 13}{36}}\right)$$

$$= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{-16}{36}} = \tfrac{1}{2} \pm \tfrac{i}{2}\cdot\tfrac{4}{6}$$

$$\text{i.e.} \quad m = \tfrac{1}{2} \pm \tfrac{1}{3}i$$

$$\therefore \quad y = e^{\frac{1}{2}x}\left[A\cos\left(\tfrac{1}{3}x\right) + B\sin\left(\tfrac{1}{3}x\right)\right].$$

Return to Exercise 6

Exercise 7. $\boxed{3y'' + 2y' = 0}$

$$\text{A.E. is} \quad 3m^2 + 2m = 0 \quad \text{i.e.} \quad m(3m+2) = 0$$

$$\text{i.e.} \quad m_1 = 0 \quad \text{and} \quad m_2 = -\tfrac{2}{3}$$

$$\text{Real different roots} \quad : \quad y = Ae^{0\cdot x} + Be^{-\frac{2}{3}x}$$

$$\text{i.e.} \quad y = A + Be^{-\frac{2}{3}x} \ .$$

Return to Exercise 7

Exercise 8. $\boxed{16y'' - 8y' + y = 0}$

A.E. is $\quad 16m^2 - 8m + 1 = 0 \quad$ i.e. $\quad (4m-1)^2 = 0$

i.e. $\quad m = \frac{1}{4} \quad$ (twice)

Real equal roots: $\quad y = (A + Bx)e^{\frac{1}{4}x}$.

Return to Exercise 8

Exercise 9. $\boxed{y'' + 4y' + 5y = 0 \quad ; \quad y(0) = 0 \quad \text{and} \quad y'(0) = 2}$

A.E. $\quad m^2 + 4m + 5 = 0, \quad m = \frac{1}{2}\left(-4 \pm \sqrt{16 - 20}\right)$

$$= -2 \pm \tfrac{2}{2} i = -2 \pm i$$

General solution is $\quad y = e^{-2x}(A\cos x + B\sin x)\,.$

Particular solution <u>has</u> $\quad y = 0$ when $x = 0$

i.e. $\quad 0 = e^0(A\cos(0) + B\sin(0))$

$$= 1 \cdot (A + 0) = A$$

i.e. $\quad A = 0\,.$

<u>And</u> $\quad \frac{dy}{dx} = 2$ when $x = 0$

$$
\begin{aligned}
A = 0 \text{ gives} \qquad y &= e^{-2x} B \sin x \\
\frac{dy}{dx} &= -2e^{-2x} B \sin x + e^{-2x} B \cos x \\
&= Be^{-2x}(\cos x - 2\sin x) \\
\text{i.e.} \qquad 2 &= B \cdot e^{0}[\cos(0) - 2\sin(0)] \\
&= B \cdot 1[1-0] \\
\text{i.e.} \qquad B &= 2\,.
\end{aligned}
$$

$\therefore$ particular solution is $\quad y = 2e^{-2x}\sin x\ .$

Return to Exercise 9

Exercise 10. $\boxed{y'' + 6y' + 13y = 0 \quad ; \quad y(0) = 2 \quad \text{and} \quad y'(0) = 1}$

$$\begin{aligned} \text{A.E.} \quad m^2 + 6m + 13 = 0 \quad \text{i.e.} \quad m &= \tfrac{1}{2}\left(-6 \pm \sqrt{36 - 52}\right) \\ &= -3 \pm \tfrac{1}{2}\sqrt{-16} \\ \text{i.e.} \quad m &= -3 \pm 2i \end{aligned}$$

General solution is $\quad y = e^{-3x}(A\cos 2x + B\sin 2x)\,.$

Particular solution has $\quad y = 2$ when $x = 0$

i.e. $\quad 2 = e^0(A\cos(0) + B\sin(0))$

i.e. $\quad A = 2\,.$

$$
\begin{aligned}
\frac{dy}{dx} &= -3e^{-3x}(A\cos 2x + B\sin 2x) + e^{-3x}(-2A\sin 2x + 2B\cos 2x) \\
&= e^{-3x}\left[(2B-3A)\cos 2x - (3B+2A)\sin 2x\right] \\
1 &= e^{0}\left[(2B-3A)\cos(0) - (3B+2A)\sin(0)\right]
\end{aligned}
$$

$\left(\text{i.e. } \frac{dy}{dx} = 1 \text{ when } x = 0\right)$

i.e. $1 = 2B - 3A$

i.e. $1 = 2B - 6 \quad (\text{using } A = 2)$

i.e. $\frac{7}{2} = B.$

$\therefore$ Particular solution is $\quad y = \frac{1}{2}e^{-3x}(4\cos 2x + 7\sin 2x)$.

Return to Exercise 10

Exercise 11. $\boxed{y'' + 6y' + 9y = 0 \quad ; \quad y(0) = 1 \quad \text{and} \quad y'(0) = 2}$

A.E. is $\quad m^2 + 6m + 9 = 0 \quad$ i.e. $\quad (m+3)^2 = 0$

i.e. $\quad m = -3 \quad$ (twice)

General solution : $\quad y = (A + Bx)e^{-3x}$.

Particular solution has $\quad y = 1$ when $x = 0$

i.e. $\quad 1 = (A + 0)e^0$

i.e. $\quad A = 1$.

$$\begin{aligned} \frac{dy}{dx} &= Be^{-3x} + (A+Bx)\cdot(-3)e^{-3x} \\ &= e^{-3x}\left[B - 3(A+Bx)\right] \end{aligned}$$

$$\begin{array}{lll} \text{i.e.} & 2 = e^0[B-3(A+0)], & \text{since } \frac{dy}{dx} = 2 \text{ when } x = 0 \\ \text{i.e.} & 2 = B - 3A & \\ \text{i.e.} & B = 5 & (\text{using } A = 1)\,. \end{array}$$

$\therefore$ Particular solution is $y = (1+5x)e^{-3x}$.

Return to Exercise 11

Exercise 12. $\boxed{\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0}$

$$\text{Set} \quad y = Ae^{mx}\,, \quad \frac{dy}{dx} = my\,, \quad \frac{d^2y}{dx^2} = m^2y$$

A.E. $\quad m^2 + m + 1 = 0$

i.e. $\quad m = \frac{1}{2}[-1 \pm \sqrt{1-4}] = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad$ i.e. $p \pm iq$ with $\quad p = -\frac{1}{2}$

$q = \frac{\sqrt{3}}{2}$

General solution: $\quad y = e^{px}(A\cos qx + B\sin qx)$

$$\text{i.e.} \quad y = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right).$$

Return to Exercise 12

Exercise 13. $\boxed{\frac{d^2y}{d\tau^2} - 6\frac{dy}{d\tau} + 9y = 0}$

A.E. $m^2 - 6m + 9 = 0$

i.e. $(m-3)^2 = 0$

i.e. $m = 3$ (twice)

and $y = Ae^{3\tau}$ (twice)

To get two independent solutions, multiply one by τ

i.e. $y = (A + B\tau)e^{3\tau}$.

Return to Exercise 13

Exercise 14. $\boxed{\frac{d^2y}{d\tau^2} + 7\frac{dy}{d\tau} + 12y = 0}$

A.E. $\quad m^2 + 7m + 12 = 0$

i.e. $\quad (m+4)(m+3) = 0$

i.e. $\quad m_1 = -4 \quad$ and $\quad m_2 = -3$

i.e. two different real roots giving two independent solutions

i.e. $\quad y_1 = Ae^{-4\tau} \quad$ and $\quad y_2 = Be^{-3\tau}$

general solution is $y = y_1 + y_2 = Ae^{-4\tau} + Be^{-3\tau}$.

Return to Exercise 14

Exercise 15. $\boxed{\frac{d^2x}{d\tau^2} + 5\frac{dx}{d\tau} + 6x = 0}$

$$\text{Set} \quad x = Ae^{m\tau}, \quad \text{i.e.} \quad \frac{dx}{d\tau} = mx, \quad \text{and} \quad \frac{d^2x}{d\tau^2} = m^2x$$

A.E. $m^2 + 5m + 6 = 0$

i.e. $(m+2)(m+3) = 0$

i.e. $m_1 = -2$ and $m_2 = -3$ (two different real roots)

These give two independent solutions $x(\tau)$

i.e. $x_1 = Ae^{-2\tau}$ and $x_2 = Be^{-3\tau}$

general solution is $x(\tau) = Ae^{-2\tau} + Be^{-3\tau}$.

Return to Exercise 15

Exercise 16. $\boxed{4\frac{d^2x}{d\tau^2} + 8\frac{dx}{d\tau} + 3x = 0}$

$$\begin{array}{ll} \text{A.E.} & 4m^2 + 8m + 3 = 0 \\ \text{i.e.} & (2m+3)(2m+1) = 0 \\ \text{i.e.} & m_1 = -\frac{3}{2} \quad \text{and} \quad m_2 = -\frac{1}{2} \quad \text{(different real roots)} \\ \text{i.e.} & x_1(\tau) = Ae^{-\frac{3}{2}\tau} \quad \text{and} \quad x_2(\tau) = Be^{-\frac{1}{2}\tau} \quad \text{(independent solutions)} \end{array}$$

$\therefore$ general solution is $x(\tau) = Ae^{-\frac{3}{2}\tau} + Be^{-\frac{1}{2}\tau}$.

Return to Exercise 16