Basic Mathematics

Straight Line Graphs

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence at sketching graphs of linear functions.

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1. Straight Line Graphs (Introduction)

A general linear function has the form $y = mx + c$, where m, c are constants.

Example 1 If x is the temperature in $^\circ C$ and y the temperature in $^\circ F$ then there is a simple rule relating the values of x and y. The table illustrates this rule for various values of x and y.

$x(^\circ C)$	$y(^\circ F)$	
0	32	freezing point of water
10	50	temperature on a cold day
25	77	temperature on a warm day
37	98.6	blood temperature
100	212	boiling point of water

The general rule is

$$y = \frac{9}{5}x + 32\,,$$

so that $m = 9/5$ and $c = 32$ in this case. A graph of this relationship is shown on the next page.

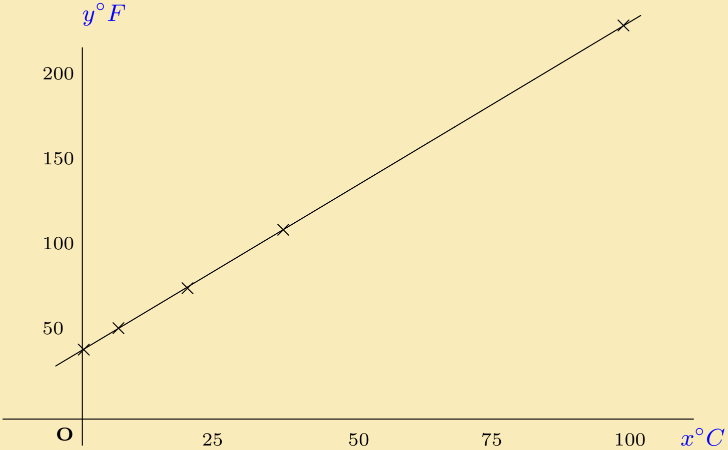


Diagram 1 Sketch of $y = \frac{9}{5}x + 32$

Example 2 A straight line passes through the two points $P(x, y)$ and $Q(x, y)$ with coordinates $P(0, 2)$ and $Q(1, 5)$. Find the equation of this straight line.

Solution The general equation of a straight line is $y = mx + c$. Since the line passes through the points P, with coordinates $x = 0$, $y = 2$, and Q, with coordinates $x = 1, y = 5$, these coordinates must satisfy this equation, i.e.

$$\begin{aligned} 2 &= m \times 0 + c \\ 1 &= m \times 5 + c \end{aligned}$$

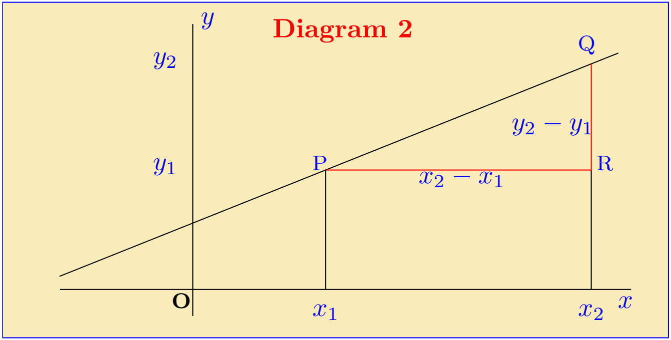
(See the package on **simultaneous equations**.) Solving these equations gives $c = 2$ and $m = 3$, i.e. the line is $y = 3x + 2$.

Exercise 1. In each of the following find the equation of the straight line through the given pairs of points. (Click on the green letters for solution.)

(a) The points $P(0, -3)$ and $Q(2, 1)$.
(b) The points $P(0, 4)$ and $Q(1, 3)$.

2. Gradient of a Straight Line

The gradient of a straight line is defined as follows. Suppose that two points P,Q, on the line have coordinates $P(x_1, y_1)$ and $Q(x_2, y_2)$.



The **gradient** of the line is (see diagram above)

$$\text{gradient} = \frac{\text{RQ}}{\text{PR}} = \frac{y_2 - y_1}{x_2 - x_1}\,.$$

Example 3 From the table given in **example 1**, find the gradient of the line giving the relationship between $x°C$ and $y°F$.

Solution The boiling point, Q, of water is $100°C$ or $212°F$, i.e. $Q(100, 212)$. The freezing point of water is $0°C$ or $32°F$, i.e. $P(0, 32)$. The gradient is therefore

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}\,.$$

The equation was $y = (9/5)x + 32$. Comparing this with the general equation $y = mx + c$ shows that m **is the value of the gradient**.

EXERCISE 2. Find the gradient of the line through the points P, Q with the following coordinates. (Click on the green letters for solution.)

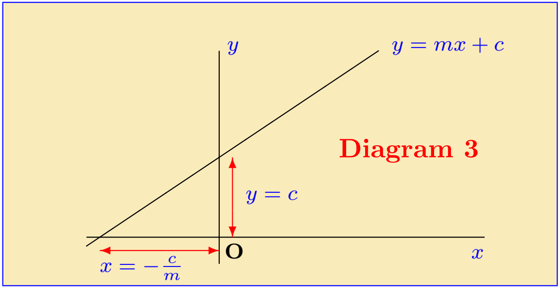
(a) $P(3, 9)$, $Q(2, 3)$ (b) $P(-1, 2)$, $Q(2, -1)$ (c) $P(1, 2)$, $Q(4, 3)$

3. Intercepts of a Straight Line

By putting $x = 0$ into the equation $y = mx + c$, the point where the straight line crosses the y axis is found to be $y = c$. This is known as the *intercept on the y axis*. The *intercept on the x axis*, i.e. when $y = 0$, is at

$$\begin{aligned} 0 &= mx + c \\ -c &= mx \\ -c/m &= x\,. \end{aligned}$$

The x and y intercepts.



Example 4 By rearranging the equation $2y-3x-5=0$, show that it is a straight line and find its gradient and intercept. Sketch the line.

Solution Rearranging the equation,

$$\begin{aligned} 3y-2x-5 &= 0 \\ 3y &= 2x+5 \\ y &= \left(\frac{2}{3}\right)x+\left(\frac{5}{3}\right) \end{aligned}$$

(Equation of a straight line with $m=2/3$ and $c=5/3$.)

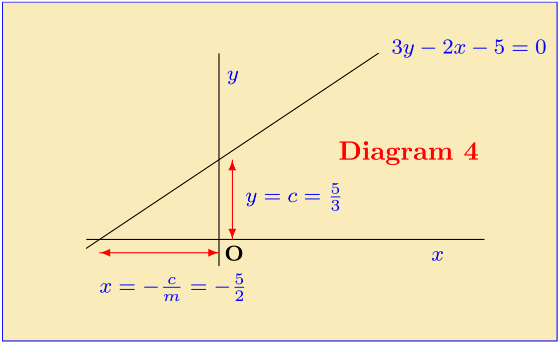


Diagram 4

Exercise 3. Each of the following equations represent straight lines. By rearranging each of them find their gradient and the intercepts on the y and x axis. Sketch the straight lines they represent. (Click on the green letters for solution.)

(a) $2y - 2x + 3 = 0$, (b) $3y - 5x + 6 = 0$, (c) $2y + 4x + 3 = 0$.

Quiz A straight line has the equation $3x + y + 3 = 0$. If P is the point where the line crosses the x axis and Q is the point where it crosses the y axis, which of the following pair is P, Q?

(a) $P(3, 0)$, $Q(0, -1)$, (b) $P(-1, 0)$, $Q(0, 3)$,
(c) $P(-1, 0)$, $Q(0, -3)$, (d) $P(-3, 0)$, $Q(0, -1)$.

Quiz If the straight line $6x + 2y = 3$ is written in the form $y = mx + c$, which of the following is the correct set of values for m and c?

(a) $m = 3$, $c = 3/2$, (b) $m = 3$, $c = -3/2$,
(c) $m = -3$, $c = -3/2$, (d) $m = -3$, $c = 3/2$.

4. Positive and Negative Gradients

If a line has gradient $m = 1$ then, *providing that the scales are the same for both axes*, it makes an angle of $45°$ with the positive x-axis. If $m > 1$ then the gradient is steeper. If $0 < m < 1$ then the line makes an angle between $0°$ and $45°$ with the positive x-axis.

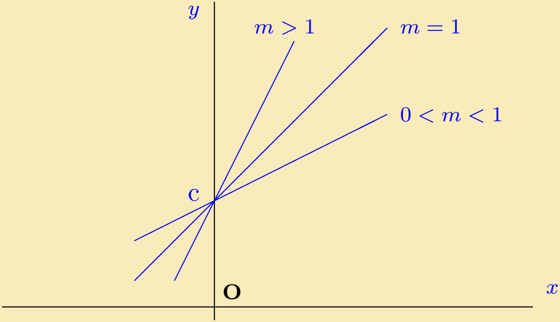


Diagram 5 Lines with *positive* gradients.

The diagram below illustrates lines similar to those of **diagram 5** except with **negative** gradients. They are the mirror images of the straight lines which are shown in **diagram 5**, with the y axis acting as the mirror.

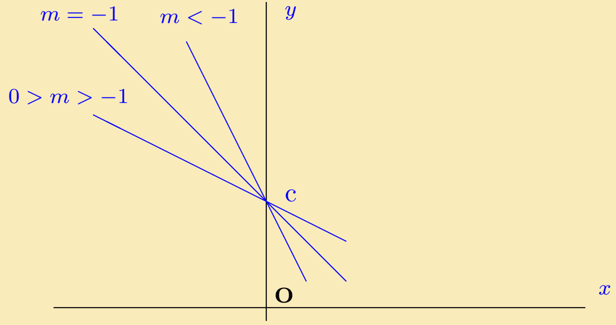


Diagram 6 Lines with *negative* gradients.

EXERCISE 4. In each of the following either the coordinates of two points, P, Q are given, or the coordinates of a single point R and a gradient m. In each case, find the equation of the line.

(a) $P(1,1)$, $Q(2,-1)$, (b) $R(1,2)$, $m = 2$, (c) $P(-1,2)$, $Q(1,-3)$.
(d) $R(-2,1)$, $m = 4$. (e) $P(-1,2), Q(-3,3)$ (f) $P(1,2), Q(-4,7)$

Example 5 Two lines are described as follows: the first has gradient -1 and passes through the point R(2,1); the second passes through the two points with coordinates P(2,0) and Q(0,4). Find the equation of both lines and find the coordinates of their point of intersection.

Solution The first line has gradient $m = -1$ so it must be $y = (-1)x + c$, i.e. $y = -x + c$, for some c. Since the line passes through the point $R(2,1)$ these values of x, y must satisfy the equation. Thus $2 = -(1) + c$ so $c = 3$. The first line therefore has equation $y = -x + 3$. For the second case both points lie on the line and so satisfy the equation. If the equation is $y = mx + c$ then putting these values into the equation gives

$$\begin{aligned} y &= mx + c \\ 0 &= 2m + c \qquad \text{(using the coordinates of } P\text{)} \\ 4 &= c \qquad \text{(using the coordinates of } Q\text{)} \end{aligned}$$

These equations yield $m = -2$ and $c = 4$. The second line thus has the equation $y = -2x + 4$. The equations of the two lines can now be rewritten as

$$\begin{aligned} y + x &= 3 \qquad (1) \\ y + 2x &= 4 \qquad (2) \end{aligned}$$

which is a pair of simultaneous equations. Subtracting equation (1) from equation (2) gives $x = 1$ and substituting this into the first equation then yields $y = 2$. The point of intersection thus has coordinates $x = 1$, $y = 2$. (By substituting these coordinates into equation (2) and verifying that they satisfy the equation, it can be checked that this is also a point on the second line.)

5. Some Useful Facts

- Parallel lines have the same gradient. Thus, for example, the lines with equations $y = 3x + 7$ and $y = 3x - 2$ are parallel.
- Lines parallel to the x-axis have equations of the form $y = k$, for some constant, k.
- Lines parallel to the y-axis (when $m = 0$) have equations of the form $x = k$, for some constant, k.
- The larger the *absolute* value of m, the 'steeper' the slope of the line.
- If two lines intersect at right angles then the product of their gradients is -1. The lines $y = -7x+4$ and $y = (1/7)x + 5$, for example, intersect each other at right angles.

6. Quiz on Straight Lines

Begin Quiz In each of the following, choose the solution from the options given.

1. The straight line through $P(-5, 4)$ and $Q(2, -3)$.
(a) $2x - y = -14$ (b) $-x + 2y = 17$
(c) $-x + y - 1 = 0$ (d) $x + y + 1 = 0$

2. The gradient m and intercept c of $-2x + 3y + 6 = 0$.
(a) $m = 2/3,\ c = -2$ (b) $m = -2/3,\ c = 2$
(c) $m = 3/2,\ c = -3$ (d) $m = -3/2,\ c = 3$

3. The straight line with gradient $m = -3$ passing through $R(-1, 3)$.
(a) $-3x + y = 0$ (b) $2y - 6x = 4$
(c) $y - 3x = 1$ (d) $y + 3x = 0$

4. The point of intersection of the lines $2x + y = 1$ and $3x - 2y = 5$.
(a) $(-1,\ 1)$ (b) $(1,\ -1)$ (c) $(3,\ -5)$ (d) $(2,\ -3)$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The general equation of a straight line is $y = mx + c$. Since the line passes through the points P and Q, the coordinates of both points must satisfy this equation. The point P has coordinates $x = 0, y = -3$ and the point Q has coordinates $x = 2, y = 1$. These satisfy the pair of simultaneous equations

$$\begin{aligned} -3 &= m \times 0 + c \\ 1 &= m \times 2 + c \end{aligned}$$

Solving these equations gives $c = -3$ and $m = 2$, i.e. the line is $y = 2x - 3$.

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Exercise 1(b) The general equation of a straight line is $y = mx + c$. Since the line passes through the points P and Q, the coordinates of both points must satisfy this equation. The point P has coordinates $x = 0, y = 4$ and the point Q has coordinates $x = 1, y = 3$. These satisfy the pair of simultaneous equations

$$\begin{aligned} 4 &= m \times 0 + c \\ 3 &= m \times 1 + c \end{aligned}$$

Solving these equations gives $c = 4$ and $m = -3$, i.e. the line is $y = -3x + 4$.

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Exercise 2(a) For $P(3,9)$, $Q(2,3)$, the gradient is given by

$$\begin{aligned}\text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-9}{2-3} \\ &= \frac{-6}{-1} \\ &= 6\,,\end{aligned}$$

so that $m = 6$ in this case.

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Exercise 2(b) For $P(-1, 2)$, $Q(2, -1)$ the gradient is given by

$$\begin{aligned} \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 2}{2 - (-1)} \\ &= -1\,, \end{aligned}$$

so that $m = -1$ in this case. We shall interpret the negative gradient later in this package.

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Exercise 2(c) For $P(1,2)$, $Q(4,3)$ the gradient is given by

$$\begin{aligned} \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-2}{4-1} \\ &= \frac{1}{3}\,, \end{aligned}$$

so that $m = 1/3$ in this case.

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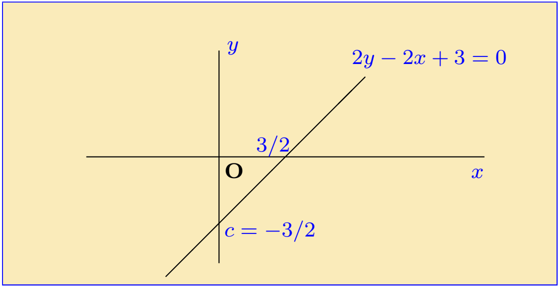
Exercise 3(a) For the equation $2y - 2x + 3 = 0$,

$$\begin{aligned} 2y - 2x + 3 &= 0 \\ 2y &= 2x - 3 \\ y &= x - \frac{3}{2} \end{aligned}$$

so that $m = 1$ and $c = -3/2$.
The intercept on the x axis is $-c/m = -(-3/2)/1 = 3/2$.



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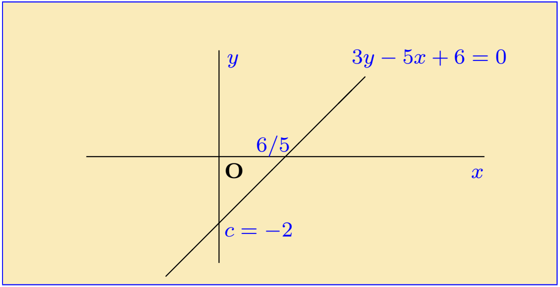
Exercise 3(b) For the equation $3y - 5x + 6 = 0$,

$$\begin{aligned} 3y - 5x + 6 &= 0 \\ 3y &= 5x - 6 \\ y &= \frac{5}{3}x - 2 \end{aligned}$$

so that $m = 5/3$ and $c = -2$.
The intercept on the x axis is $-c/m = -(-2)/(5/3) = 6/5$.



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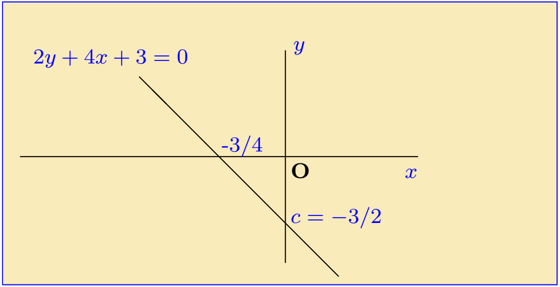
Exercise 3(c) For the equation $2y + 4x + 3 = 0$,

$$\begin{aligned} 2y + 4x + 3 &= 0 \\ 2y &= -4x - 3 \\ y &= -2x - \frac{3}{2} \end{aligned}$$

so that $m = -2$ and $c = -3/2$.
The intercept on the x axis is $-c/m = -(-3/2)/(-2) = -3/4$.



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Exercise 4(a) Let the line be $y = mx + c$. Since both $P(1,1)$ and $Q(2,-1)$ lie on the line, both sets of coordinates must satisfy the equation. Thus we have

$$\begin{aligned} 1 &= m \times 1 + c && \textit{using the coordinates of } P \\ -1 &= m \times 2 + c && \textit{using the coordinates of } Q \\ \text{or} \quad m + c &= 1 \\ 2m + c &= -1\,. \end{aligned}$$

This is a set of simultaneous equations which can be solved to give $m = -2$ and $c = 3$. (See the package on **simultaneous equations** for the technique for solving them.) The required equation is thus

$$y = -2x + 3\,.$$

Substituting the coordinates for P and then Q into this equation will confirm that this line passes through both of these points.

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Exercise 4(b) Since $m = 2$, the equation must have the form $y = 2x + c$ and only the value of c remains to be found. The line passes through $R(1, 2)$ so the coordinates of this point must satisfy the equation. Thus

$$\begin{aligned} y &= 2x + c \\ 2 &= 2 \times 1 + c \qquad \textit{using the coordinates of } R \end{aligned}$$

giving $c = 0$. The equation of the line is now

$$y = 2x\,.$$

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Exercise 4(c) Let the line be $y = mx + c$. Since both $P(-1, 2)$ and $Q(1, -3)$ lie on the line, both sets of coordinates must satisfy the equation. Thus we have

$$\begin{array}{lrcll} & 2 & = & m \times (-1) + c & \quad \textit{using the coordinates of } P \\ & -3 & = & m \times 1 + c & \quad \textit{using the coordinates of } Q \\ \text{or} & -m + c & = & 2 & \\ & m + c & = & -3\,. & \end{array}$$

This is a set of simultaneous equations which can be solved to give $m = -5/2$ and $c = -1/2$. (See the package on **simultaneous equations** for the technique for solving them.) The required equation is thus

$$y = -\frac{5}{2}x - \frac{1}{2}\,.$$

Substituting the coordinates for P and then Q into this equation will confirm that this line passes through both of these points.

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Exercise 4(d) Since $m = 4$, the equation must have the form $y = 4x + c$ and only c remains to be found. The line passes through $R(-2, 1)$ so the coordinates of this point must satisfy the equation. Thus

$$\begin{aligned} y &= 4x + c \\ 1 &= 4\times(-2)+c \qquad \textit{using the coordinates of } R \\ \text{or } -8 + c &= 1 \\ c &= 9\,, \end{aligned}$$

and the equation of the line is

$$y = 4x + 9\,.$$

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Exercise 4(e) Let the line be $y = mx + c$. The coordinates $P(-1, 2)$ and $Q(-3, 3)$ both lie on the line so both sets of coordinates must satisfy the equation. We have

$$\begin{aligned} 2 &= m\times(-1) + c && \textit{using the coordinates of } P \\ 3 &= m\times(-3) + c && \textit{using the coordinates of } Q \\ \text{or} \quad -m + c &= 2 \\ -3m + c &= 3\,. \end{aligned}$$

This set of simultaneous equations can be solved to give
$m = -1/2$ and $c = 3/2$. (See the package on **simultaneous equations** for the technique for solving them.) The required equation is thus

$$y = -\frac{1}{2}x + \frac{3}{2}\,.$$

Substituting the coordinates for P and then Q into this equation will confirm that this line passes through both of these points.
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Exercise 4(f) Let the line be $y = mx + c$. Both $P(1, 2)$ and $Q(-4, 7)$ lie on the line so both sets of coordinates must satisfy the equation. Thus

$$\begin{aligned} 2 &= m\times(1) + c \quad \textit{using the coordinates of } P \\ 7 &= m\times(-4) + c \quad \textit{using the coordinates of } Q \\ \text{or} \quad m + c &= 2 \\ -4m + c &= 7\,. \end{aligned}$$

The solution to this set of simultaneous equations is found to be $m = -1$ and $c = 3$. (See the package on **simultaneous equations** for the technique for solving them.) The equation of the line is thus

$$y = -x + 3\,.$$

Substituting the coordinates for P and then Q into this equation will confirm that this line passes through both of these points.

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Solutions to Quizzes

Solution to Quiz:

The line crosses the x axis when $y = 0$. Putting this into the equation of the line, $3x + y + 3 = 0$, gives

$$\begin{aligned} 3x + 0 + 3 &= 0 \\ 3x &= -3 \\ x &= -1\,. \end{aligned}$$

Thus $P(-1, 0)$ is the first point.

The line crosses the y axis when $x = 0$. Putting this into the equation of the line,

$$\begin{aligned} 3\times(0) + y + 3 &= 0 \\ y + 3 &= 0 \\ y &= -3\,. \end{aligned}$$

Thus $Q(0, -3)$ is the second point. End Quiz

Solution to Quiz:
The equation $6x + 2y = 3$ is rearranged as follows:

$$\begin{aligned} 6x + 2y &= 3 \\ 2y &= -6x + 3 \\ y &= -\frac{6}{2}x + \frac{3}{2} \\ &= -3x + \frac{3}{2} \end{aligned}$$

so $m = -3$ and $c = 3/2$. End Quiz