Intermediate Mathematics

Introduction to Partial Differentiation

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of partial differentiation.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Partial Differentiation (Introduction)

In the package on **introductory differentiation**, rates of change of functions were shown to be measured by the *derivative*. Many applications require functions with more than one variable: the ideal gas law, for example, is

$$pV = kT$$

where p is the pressure, V the volume, T the absolute temperature of the gas, and k is a constant. Rearranging this equation as

$$p = \frac{kT}{V}$$

shows that p is a function of T and V. If one of the variables, say T, is kept fixed and V changes, then the derivative of p with respect to V measures the *rate of change* of *pressure* with respect to *volume*. In this case, it is called *the partial derivative of p with respect to V* and written as

$$\frac{\partial p}{\partial V}\,.$$

Example 1 If $p = \dfrac{kT}{V}$, find the partial derivatives of p:

(a) with respect to T, (b) with respect to V.

Solution

(a) This part of the example proceeds as follows:

$$\begin{aligned} p &= \frac{kT}{V}, \\ \therefore \ \frac{\partial p}{\partial T} &= \frac{k}{V}, \end{aligned}$$

where V is treated as a constant for this calculation.

(b) For this part, T is treated as a constant. Thus

$$\begin{aligned} p &= kT\frac{1}{V} = kTV^{-1}, \\ \therefore \frac{\partial p}{\partial V} &= -kTV^{-2} = -\frac{kT}{V^2}. \end{aligned}$$

The symbol ∂ is used whenever a function with more than one variable is being differentiated but the techniques of *partial* differentiation are exactly the same as for (*ordinary*) differentiation.

Example 2 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the function $z = x^2y^3$.

Solution

$$\begin{aligned} z &= x^2y^3 \\ \therefore \frac{\partial z}{\partial x} &= 2xy^3\,, \\ \text{and } \frac{\partial z}{\partial y} &= x^2 3y^2\,, \\ &= 3x^2y^2\,. \end{aligned}$$

For the first part y^3 is treated as a constant and the derivative of x^2 *with respect to* x is $2x$.

For the second part x^2 is treated as a constant and the derivative of y^3 *with respect to* y is $3y^2$.

EXERCISE 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions. (Click on the green letters for solutions.)

(a) $z = x^2y^4$, (b) $z = (x^4 + x^2)y^3$, (c) $z = y^{\frac{1}{2}}\sin(x)$.

2. The Rules of Partial Differentiation

Since *partial differentiation* is essentially the same as *ordinary differentiation*, the *product*, *quotient* and *chain* rules may be applied.

Example 3 Find $\dfrac{\partial z}{\partial x}$ for each of the following functions.

$$\text{(a) } z = xy\cos(xy)\,, \quad \text{(b) } z = \frac{x-y}{x+y}\,, \quad \text{(c) } z = (3x+y)^2\,.$$

Solution

(a) Here $z = uv$, where $u = xy$ and $v = \cos(xy)$ so the *product rule* applies (see the package on the Product and Quotient Rules).

$$\begin{array}{rclcrcl} u &=& xy & \text{and} & v &=& \cos(xy) \\ \therefore \dfrac{\partial u}{\partial x} &=& y & \text{and} & \dfrac{\partial v}{\partial x} &=& -y\sin(xy)\,. \end{array}$$

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} = y\cos(xy) - xy^2\sin(xy)\,.$$

(b) Here $z = u/v$, where $u = x - y$ and $v = x + y$ so the *quotient rule* applies (see the package on the Product and Quotient Rules).

$$\begin{aligned} u &= x - y \quad \text{and} \quad v = x + y \\ \therefore \frac{\partial u}{\partial x} &= 1 \qquad \text{and} \quad \frac{\partial v}{\partial x} = 1 \,. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2} \\ &= \frac{(x+y) - (x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} \,. \end{aligned}$$

(c) In this case $z = (3x + y)^2$ so $z = u^2$ where $u = 3x + y$, and the *chain rule* applies (see the package on the Chain Rule).

$$\begin{aligned} z &= u^2 \quad \text{and} \quad u = 3x + y \\ \therefore \frac{\partial z}{\partial u} &= 2u \quad \text{and} \quad \frac{\partial u}{\partial x} = 3 \,. \end{aligned}$$

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = 2(3x + y)3 = 6(3x + y) \,.$$

Exercise 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions. (Click on the green letters for solutions.)

(a) $z = (x^2+3x)\sin(y)$, (b) $z = \frac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x)\cos(xy)$, (e) $z = e^{(x^2+y^2)}$, (f) $z = \sin(x^2+y)$.

Quiz If $z = \cos(xy)$, which of the following statements is true?

(a) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (b) $\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$,

(c) $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (d) $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$.

3. Higher Order Partial Derivatives

Derivatives of order two and higher were introduced in the package on **Maxima and Minima**. Finding higher order derivatives of functions of more than one variable is similar to ordinary differentiation.

Example 4 Find $\frac{\partial^2 z}{\partial x^2}$ if $z = \mathrm{e}^{(x^3+y^2)}$.

Solution First differentiate z with respect to x, keeping y constant, then differentiate this function with respect to x, again keeping y constant.

$$
\begin{aligned}
z &= \mathrm{e}^{(x^3+y^2)} \\
\therefore\ \frac{\partial z}{\partial x} &= 3x^2\mathrm{e}^{(x^3+y^2)} \text{ using the chain rule} \\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial(3x^2)}{\partial x}\mathrm{e}^{(x^3+y^2)} + 3x^2\frac{\partial(\mathrm{e}^{(x^3+y^2)})}{\partial x} \text{ using the product rule} \\
\frac{\partial^2 z}{\partial x^2} &= 6x\mathrm{e}^{(x^3+y^2)} + 3x^2(3x^2\mathrm{e}^{(x^3+y^2)}) \\
&= (9x^4+6x)\mathrm{e}^{(x^3+y^2)}
\end{aligned}
$$

In addition to both $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$, when there are two variables there is also the possibility of a *mixed second order derivative*.

Example 5 Find $\frac{\partial^2 z}{\partial x \partial y}$ if $z = \mathrm{e}^{(x^3+y^2)}$.

Solution The symbol $\frac{\partial^2 z}{\partial x \partial y}$ is interpreted as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$; in words, first differentiate z with respect to y, keeping x constant, then differentiate this function with respect to x, keeping y constant. (It is this differentiation, first with respect to x and then with respect to y, that leads to the name of *mixed derivative*.)

$$
\begin{array}{lrcl}
\text{First with } x \text{ constant} & \frac{\partial z}{\partial y} & = & 2y\mathrm{e}^{(x^3+y^2)} \text{ (using the chain rule.)} \\
\text{Second with } y \text{ constant} & \frac{\partial^2 z}{\partial x \partial y} & = & \frac{\partial}{\partial x}\left(2y\mathrm{e}^{(x^3+y^2)}\right) \\
& & = & 3x^2 2y\mathrm{e}^{(x^3+y^2)} \\
& & = & 6x^2 y\mathrm{e}^{(x^3+y^2)} .
\end{array}
$$

The obvious question now to be answered is: what happens if the order of differentiation is reversed?

Example 6 Find $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$ if $z = \mathrm{e}^{(x^3+y^2)}$.

Solution

First with y constant $\qquad \dfrac{\partial z}{\partial x} = 3x^2\mathrm{e}^{(x^3+y^2)}$ (using the chain rule).

Second with x constant

$$\begin{aligned}\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(3x^2\mathrm{e}^{(x^3+y^2)}\right)\\ &= 2y3x^2\mathrm{e}^{(x^3+y^2)}\\ &= 6x^2y\mathrm{e}^{(x^3+y^2)} = \frac{\partial^2 z}{\partial x \partial y}\,.\end{aligned}$$

As a general rule, when calculating *mixed derivatives* the order of differentiation may be reversed without affecting the final result.

EXERCISE 3. Confirm the statement on the previous page by finding both $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ for each of the following functions, whose first order partial derivatives have already been found in exercise 2. (Click on the green letters for solutions.)

(a) $z = (x^2+3x)\sin(y)$, (b) $z = \frac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x)\cos(xy)$, (e) $z = \mathrm{e}^{(x^2+y^2)}$, (f) $z = \sin(x^2+y)$.

Notation For first and second order partial derivatives there is a compact notation. Thus $\frac{\partial f}{\partial x}$ can be written as f_x and $\frac{\partial f}{\partial y}$ as f_y. Similarly $\frac{\partial^2 f}{\partial x^2}$ is written f_{xx} while $\frac{\partial^2 f}{\partial x \partial y}$ is written f_{xy}.

Quiz If $z = \mathrm{e}^{-y}\sin(x)$, which of the following is $z_{xx} + z_{yy}$?

(a) $\mathrm{e}^{-y}\sin(x)$, (b) 0, (c) $-\mathrm{e}^{-y}\sin(x)$, (d) $-\mathrm{e}^{-y}\cos(x)$.

4. Quiz on Partial Derivatives

Choose the correct option for each of the following.

Begin Quiz

1. If $z = x^2 + 3xy + y^3$ then $\dfrac{\partial z}{\partial x}$ is

(a) $2x + 3y + 3y^2$, (b) $2x + 3x + 3y^2$,
(c) $2x + 3x$, (d) $2x + 3y$.

2. If $w = 1/r$, where $r^2 = x^2 + y^2 + z^2$, then $xw_x + yw_y + zw_z$ is

(a) $-1/r$, (b) $1/r$, (c) $-1/r^2$, (d) $1/r^2$.

3. If $u = \sqrt{\dfrac{x}{y}}$ then u_{xx} is

(a) $-\dfrac{1}{4\sqrt{y^3x^3}}$, (b) $-\dfrac{1}{4\sqrt{yx^3}}$, (c) $-\dfrac{1}{8\sqrt{y^3x^3}}$, (d) $-\dfrac{1}{8\sqrt{yx^3}}$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) To calculate the partial derivative $\frac{\partial z}{\partial x}$ of the function $z = x^2 y^4$, the factor y^4 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2 y^4\right) = \frac{\partial}{\partial x}\left(x^2\right) \times y^4 = 2x^{(2-1)} \times y^4 = 2xy^4 \,.$$

Similarly, to find the partial derivative $\frac{\partial z}{\partial y}$, the factor x^2 is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y^4\right) = x^2 \times \frac{\partial}{\partial y}\left(y^4\right) = x^2 \times 4y^{(4-1)} = 4x^2 y^3 \,.$$

Click on the green square to return

Exercise 1(b) To calculate $\dfrac{\partial z}{\partial x}$ for the function $z = (x^4 + x^2)y^3$, the factor y^3 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left((x^4 + x^2)y^3\right) = \frac{\partial}{\partial x}\left(x^4 + x^2\right) \times y^3 = (4x^3 + 2x)y^3 \,.$$

To find the partial derivative $\dfrac{\partial z}{\partial y}$ the factor $(x^4 + x^2)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left((x^4 + x^2)y^3\right) = (x^4 + x^2) \times \frac{\partial}{\partial y} y^3 = 3(x^4 + x^2)y^2 \,.$$

Click on the green square to return

Exercise 1(c) If $z = y^{\frac{1}{2}} \sin(x)$ then to calculate $\dfrac{\partial z}{\partial x}$ the $y^{\frac{1}{2}}$ factor is kept constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(y^{\frac{1}{2}} \sin(x)\right) = y^{\frac{1}{2}} \times \frac{\partial}{\partial x}(\sin(x)) = y^{\frac{1}{2}} \cos(x) \,.$$

Similarly, to evaluate the partial derivative $\dfrac{\partial z}{\partial y}$ the factor $\sin(x)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(y^{\frac{1}{2}} \sin(x)\right) = \frac{\partial}{\partial y} y^{\frac{1}{2}} \times \sin(x) = \frac{1}{2} y^{-\frac{1}{2}} \sin(x) \,.$$

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Exercise 2(a) The function $z = (x^2 + 3x)\sin(y)$ can be written as $z = uv$, where $u = (x^2 + 3x)$ and $v = \sin(y)$. The partial derivatives of u and v with respect to the variable x are

$$\frac{\partial u}{\partial x} = 2x + 3\,, \qquad \frac{\partial v}{\partial x} = 0\,,$$

while the partial derivatives with respect to y are

$$\frac{\partial u}{\partial y} = 0\,, \qquad \frac{\partial v}{\partial y} = \cos(y)\,.$$

Applying the *product rule*

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} = (2x + 3)\sin(y)\,.$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y} = (x^2 + 3x)\cos(y)\,.$$

Click on the green square to return

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Exercise 2(b)

The function $z = \dfrac{\cos(x)}{y^5}$ can be written as $z = \cos(x)y^{-5}$.

Treating the factor y^{-5} as a constant and differentiating with respect to x:

$$\frac{\partial z}{\partial x} = -\sin(x)y^{-5} = -\frac{\sin(x)}{y^5} \, .$$

Treating $\cos(x)$ as a constant and differentiating with respect to y:

$$\frac{\partial v}{\partial y} = \cos(x)(-5y^{-6}) = -5\frac{\cos(x)}{y^6} \, .$$

Click on the green square to return

Exercise 2(c) The function $z = \ln(xy)$ can be rewritten as (see the package on logarithms)

$$z = \ln(xy) = \ln(x) + \ln(y)\,.$$

Thus the partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\ln(x) + \ln(y)) = \frac{\partial}{\partial x}\ln(x) = \frac{1}{x}\,.$$

Similarly the partial derivative of z with respect to y is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\ln(x) + \ln(y)) = \frac{\partial}{\partial y}\ln(y) = \frac{1}{y}\,.$$

Click on the green square to return $\square$

Exercise 2(d) To calculate the partial derivatives of the function $z = \sin(x)\cos(xy)$ the *product rule* has to be applied

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \cos(xy)\frac{\partial}{\partial x}\sin(x) + \sin(x)\frac{\partial}{\partial x}\cos(xy)\,,\\
\frac{\partial z}{\partial y} &= \cos(xy)\frac{\partial}{\partial y}\sin(x) + \sin(x)\frac{\partial}{\partial y}\cos(xy)\,.
\end{aligned}$$

Using the *chain rule* with $u = xy$ for the partial derivatives of $\cos(xy)$

$$\begin{aligned}
\frac{\partial}{\partial x}\cos(xy) &= \frac{\partial\cos(u)}{\partial u}\,\frac{\partial u}{\partial x} = -\sin(u)y = -y\sin(xy)\,,\\
\frac{\partial}{\partial y}\cos(xy) &= \frac{\partial\cos(u)}{\partial u}\,\frac{\partial u}{\partial y} = -\sin(u)x = -x\sin(xy)\,.
\end{aligned}$$

Thus the partial derivatives of $z = \sin(x)\cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(xy)\cos(x) - y\sin(x)\sin(xy)\,, \quad \frac{\partial z}{\partial y} = -x\sin(x)\sin(xy)\,.$$

Click on the green square to return

Exercise 2(e) To calculate the partial derivatives of $z = \mathrm{e}^{(x^2+y^2)}$ the *chain rule* has to be applied with $u = (x^2 + y^2)$:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u}\left(\mathrm{e}^u\right)\frac{\partial u}{\partial x} = \mathrm{e}^u\,\frac{\partial u}{\partial x}\,,\\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u}\left(\mathrm{e}^u\right)\frac{\partial u}{\partial y} = \mathrm{e}^u\frac{\partial u}{\partial y}\,.\end{aligned}$$

The partial derivatives of $u = (x^2 + y^2)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial(x^2)}{\partial x} = 2x\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial(y^2)}{\partial y} = 2y\,.$$

Therefore the partial derivatives of the function $z = \mathrm{e}^{(x^2+y^2)}$ are

$$\begin{aligned}\frac{\partial z}{\partial x} &= \mathrm{e}^u\frac{\partial u}{\partial x} = 2x\,\mathrm{e}^{(x^2+y^2)}\,,\\ \frac{\partial z}{\partial x} &= \mathrm{e}^u\frac{\partial u}{\partial x} = 2y\,\mathrm{e}^{(x^2+y^2)}\,.\end{aligned}$$

Click on the green square to return

Exercise 2(f) Applying the *chain rule* with $u = x^2 + y$ the partial derivatives of the function $z = \sin(x^2 + y)$ can be written as

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial u}(\sin(u))\ \frac{\partial u}{\partial x} = \cos(u)\,\frac{\partial u}{\partial x}\,, \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u}(\sin(u))\ \frac{\partial u}{\partial y} = \cos(u)\frac{\partial u}{\partial y}\,. \end{aligned}$$

The partial derivatives of $u = x^2 + y$ are

$$\frac{\partial u}{\partial x} = \frac{\partial x^2}{\partial x} = 2x\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial y}{\partial y} = 1\,.$$

Thus the partial derivatives of the function $z = \sin(x^2 + y)$ are

$$\begin{aligned} \frac{\partial z}{\partial x} &= \cos(u)\frac{\partial u}{\partial x} = 2x\ \cos(x^2 + y)\,, \\ \frac{\partial z}{\partial y} &= \cos(u)\frac{\partial u}{\partial y} = \cos(x^2 + y)\,. \end{aligned}$$

Click on the green square to return

Exercise 3(a)

From **exercise 2(a)**, the first order partial derivatives of $z = (x^2 + 3x)\sin(y)$ are

$$\frac{\partial z}{\partial x} = (2x+3)\sin(y)\,, \qquad \frac{\partial z}{\partial y} = (x^2+3x)\cos(y)\,.$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left((x^2+3x)\cos(y)\right) = (2x+3)\cos(y)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left((2x+3)\sin(y)\right) = (2x+3)\cos(y)\,.
\end{aligned}$$

Click on the green square to return

Exercise 3(b)

From **exercise 2(b)**, the first order partial derivatives of $z = \dfrac{\cos(x)}{y^5}$ are

$$\frac{\partial z}{\partial x} = -\,\frac{\sin(x)}{y^5}\,, \qquad \frac{\partial z}{\partial y} = -\,5\,\frac{\cos(x)}{y^6}\,,$$

so the *mixed* second order derivatives are

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-5\frac{\cos(x)}{y^6}\right) = 5\,\frac{\sin(x)}{y^6}\,, \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{y^5}\right) = 5\,\frac{\sin(x)}{y^6}\,. \end{aligned}$$

Click on the green square to return

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Exercise 3(c)
From **exercise 2(c)**, the first order partial derivatives of $z = \ln(xy)$ are

$$\frac{\partial z}{\partial x} = \frac{1}{x}\,, \qquad \frac{\partial z}{\partial y} = \frac{1}{y}\,.$$

The *mixed* second order derivatives are

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) = 0\,,\\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0\,.\end{aligned}$$

Click on the green square to return

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Exercise 3(d) From **exercise 2(d)**, the first order partial derivatives of $z = \sin(x)\cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(x)\cos(xy) - y\sin(x)\sin(xy)\,, \qquad \frac{\partial z}{\partial y} = -\,x\sin(x)\sin(xy)\,.$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-x\sin(x)\sin(xy)\right) \\
&= -\sin(x)\sin(xy) - x\cos(x)\sin(xy) - xy\sin(x)\cos(xy)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos(x)\cos(xy) - y\sin(x)\sin(xy)\right) \\
&= -x\cos(x)\sin(xy) - \sin(x)\sin(xy) - xy\sin(x)\cos(xy)\,.
\end{aligned}$$

N.B. In the solution above a *product of three functions* has been differentiated. This can be done by using two applications of the *product rule*.

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Exercise 3(e) From **exercise 2(e)**, the first order partial derivatives of $z = \mathrm{e}^{(x^2+y^2)}$ are

$$\frac{\partial z}{\partial x} = 2x\mathrm{e}^{(x^2+y^2)}\,, \qquad \frac{\partial z}{\partial y} = 2y\mathrm{e}^{(x^2+y^2)}\,.$$

The *mixed* second order derivatives are thus

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y\mathrm{e}^{(x^2+y^2)}\right) = 4xy\mathrm{e}^{(x^2+y^2)}\,,\\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\mathrm{e}^{(x^2+y^2)}\right) = 4yx\mathrm{e}^{(x^2+y^2)}\,.
\end{aligned}$$

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Exercise 3(f) From **exercise 2(f)**, the first order partial derivatives of $z = \sin(x^2 + y)$ are

$$\frac{\partial z}{\partial x} = 2x\cos(x^2 + y)\,, \qquad \frac{\partial z}{\partial y} = \cos(x^2 + y)\,.$$

The *mixed* second order derivatives are thus

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\cos(x^2 + y)\right) = -2x\sin(x^2 + y)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\cos(x^2 + y)\right) = -2x\sin(x^2 + y)\,.
\end{aligned}$$

Click on the green square to return

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Solutions to Quizzes

Solution to Quiz:

To determine which of the options is correct, the partial derivatives of $z = \cos(xy)$ must be calculated. From the calculations of **exercise 2(d)** the partial derivatives of $z = \cos(xy)$ are

$$\begin{aligned}\frac{\partial}{\partial x}\cos(xy) &= \frac{\partial \cos(u)}{\partial u}\,\frac{\partial u}{\partial x} = -\sin(u)y = -y\sin(xy)\,,\\ \frac{\partial}{\partial y}\cos(xy) &= \frac{\partial \cos(u)}{\partial u}\,\frac{\partial u}{\partial y} = -\sin(u)x = -x\sin(xy)\,.\end{aligned}$$

Therefore

$$\frac{1}{y}\frac{\partial}{\partial x}\ \cos(xy) = -\sin(xy) = \frac{1}{x}\frac{\partial}{\partial y}\ \cos(xy)\,.$$

The other choices, if checked, will be found to be false. End Quiz

Solution to Quiz:

The first order derivatives of $z = \mathrm{e}^{-y}\sin(x)$ are

$$z_x = \mathrm{e}^{-y}\cos(x)\,, \qquad z_y = -\,\mathrm{e}^{-y}\sin(x)\,,$$

where e^{-y} is kept constant for the first differentiation and $\sin(x)$ for the second. Continuing in this way, the second order derivatives z_{xx} and z_{yy} are given by the expressions

$$\begin{aligned} z_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\mathrm{e}^{-y}\cos(x)\right) = -\mathrm{e}^{-y}\sin(x)\,, \\ z_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\mathrm{e}^{-y}\sin(x)\right) = \mathrm{e}^{-y}\sin(x)\,. \end{aligned}$$

Adding these two equations together gives

$$z_{xx} + z_{yy} = 0\,.$$

End Quiz