Note that x, t and w are dimensionless variables in Questions F1 and F2. They have no physical significance in those questions.

In practice, derivatives are usually found by combining well-known 'standard derivatives' in a variety of ways, but both the standard derivatives and the rules for combining them are based on Equation 1.

Strictly speaking, *position*, *velocity* and *acceleration* are *vector quantities* in that each require three *scalar components* for their complete specification in three dimensions. However, in the case of linear motion each of these quantities is effectively specified by a single scalar component. Thus, in the context of linear motion we will refer to x, v_x and a_x as position, velocity and acceleration, respectively, even though each is, in reality, only a single component of the corresponding vector.

Mention of *Newton's laws*, *mass* and *force* takes us beyond *kinematics* and into the realm of *dynamics*. These topics are fully discussed in the physics strand of *FLAP*, see the *Glossary* for details.

For reasons that will become clear later, *inverse differentiation* is usually referred to as *indefinite integration*.

Any other function of the form $F(x) = x^2 + C$, where C is any constant would be equally acceptable. C is then called an *integration constant* or a *constant of integration*.

The constant K here represents the difference between the integration constants C_1 and C_2 associated with the inverse derivatives $F_1(x)$ and $F_2(x)$,

i.e. $K = C_1 - C_2$.

Note that in this case we have used additional information about the problem, the condition $v_x(0) = 0$, to determine the relevant value of C. This is often necessary in physical problems.

In fact, *g* varies with distance from the centre of the Earth, and takes slightly different values at different places on the Earth’s surface. However, provided we consider a reasonably narrow range of vertical heights and restrict our attention to one locality it is reasonable to treat *g* as a constant.

Note the distinction between *displacement* and *distance*: in one dimension displacement may be positive or negative according to direction, but distance must always be positive.

If x is a positive quantity then $|\,x\,| = x$ and $|\,{-x}\,| = x$.

The physical significance of *simple harmonic motion* is discussed in detail in the physics strand of *FLAP*.

ω is the Greek letter omega.

You can confirm that this is correct by showing that its derivative is $-A\omega^2 \sin(\omega t)$.

By combining Equations 10 and 12 it is possible to characterize simple harmonic motion by the requirement that $a_x = -\omega^2 x$. This is the usual starting point for the analysis of simple harmonic motion and is considered elsewhere in *FLAP*. See the *Glossary* for details.

The shaded area in Figure 1b is a trapezium, the area of which is generally given by

$$\frac{1}{2}(\text{height}_1 + \text{height}_2) \times \text{base}$$

Note that whichever inverse derivative of $f(x)$ we choose to use when evaluating the area under the graph, the associated constant of integration C will play no part in the final answer since it will cancel when we calculate the *difference* $F(b) - F(a)$.

The symbol ≈ means ‘approximately equal to’.

Here we are using the definition of the derivative to justify saying

$$\lim_{\Delta x \to 0}\left(\frac{\Delta A}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\frac{A(x + \Delta x) - A(x)}{\Delta x}\right]$$

$$= \frac{dA}{dx}$$

Remember, the *distance* between two points is equal to the *magnitude* of the *displacement* from one point to the other.

The symbol ≤ should be read as ‘is less than or equal to’.

Note that we have used the *summation symbol* (Σ) as a shorthand way of indicating the sum.

Notice that, while it is not essential that the points dividing the interval are equally spaced, it is very convenient if they are.

In this case a *very* crude approximation since the actual value of the integral is 24.

The extension x increases from 0 to L, and we imagine this interval to be divided into a large number of subintervals by points $x_1, x_2, \dots, x_{n+1}$ where $x_1 = 0$ and $x_{n+1} = L$. The sum

$$\sum_{i=1}^{n} F_x(x_i)\,\Delta x_i$$

is thus a good approximation to the work done if the Δx_i are small. With practice it is unnecessary to write down this intermediate step, and we go straight to the integral as in this example.

As you perform more and more integrals this result will become very familiar. Nonetheless it is worth pausing to note how remarkable it is that the limit of a sum represented by the symbol on the left can be evaluated with the aid of (inverse) differentiation as shown by the difference on the right. The link between summation and differentiation is far from obvious to most physicists.

Notice that $\int_1^3 t^4\, dt$ is *defined* to be a limit of a sum — we can *evaluate* it using inverse differentiation.

You can confirm this by differentiating.

This notation arises from the fundamental theorem of calculus, from which it is evident that definite integrals are closely related to inverse derivatives.

The density is defined as the mass per unit volume.

This force acts perpendicular to the dam wall.

The angular brackets $\langle\ \rangle$ mean ‘the average of’ the term enclosed.

If you imagine the arrow in Figure 10a rotating through 360° and carrying the satellite with it, then you can see that the satellite must also rotate through 360°.

You might have anticipated this result since in Figure 10b the section of the satellite to the left of centre is brought nearer to the Earth and so will be pulled further in this direction — not compensated sufficiently by the other half of the satellite, on which the force is weakened, as it is further from the Earth.

This statement is true for the functions treated in this module, but the definition of the definite integral as the limit of a sum allows it to be applied in cases where simple graphical interpretation is not possible.

$\exp(a)$ is an alternative way of writing e^a.

✧

$$v_x(t) = \frac{dx}{dt} = 3at^2 + b$$

and $$a_x(t) = \frac{dv_x}{dt} = 6at$$

When $t = 0$, $x(0) = c$, so c represents the *initial position* of the moving object.

Similarly, $v_x(0) = b$, so b represents the *initial velocity* of the moving object. ❑

✧

Three suitable functions would be

$F(x) = x^2 - 2, \quad F(x) = x^2$

and $\quad F(x) = x^2 + 1$ ☞

since in each case $\dfrac{dF}{dx} = f(x)$. ❑

✧

In each of the following C represents an arbitrary constant:

(a) $F(x) = \dfrac{x^3}{3} + C$ since

$$\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = \frac{3x^2}{3} = x^2$$

(b) $F(x) = x^3 - x^2 + C$ since

$$\frac{d}{dx}(x^3 - x^2 + C) = 3x^2 - 2x$$

(c) $\dfrac{ax^2}{4} + C$ since

$$\frac{d}{dx}\left(\frac{ax^2}{4} + C\right) = \frac{2ax}{4} = \frac{ax}{2}$$ ❑

✧

In each of the following C represents an arbitrary constant:

(a) $F(t) = \dfrac{-t^{-2}}{2} + C$ since

$$\frac{d}{dt}\left(\frac{-t^{-2}}{2} + C\right) = \frac{-(-2)t^{-3}}{2} = t^{-3}$$

(b) $F(t) = 6t^{-0.5} + 2t^{2.5} + C$ since

$$\frac{d}{dx}(6t^{-0.5} + 2t^{2.5} + C) = -3t^{-1.5} + 5t^{1.5}$$

❏

✧

We know that $a_x(t) = \dfrac{dv_x}{dt}$ and, using inverse differentiation, we can see that $v_x(t) = At^4 + C$, where C is an arbitrary constant, since differentiating $At^4 + C$ produces $4At^3$.

But we also know that $v_x = 0$ when $t = 0$, and since $v_x(0) = C$, it follows that $C = 0$ in this case. Hence $v_x(t) = At^4$ and consequently

$v_x(3\,\text{s}) = (1\,\text{m}\,\text{s}^{-5})(3\,\text{s})^4 = 81\,\text{m}\,\text{s}^{-1}$. ❏ ☞

✧

The hatched area is given by

$$\tfrac{1}{2}(50\,\mathrm{m\,s^{-1}} + 100\,\mathrm{m\,s^{-1}}) \times (10\,\mathrm{s} - 5\,\mathrm{s})$$
$$= 375\,\mathrm{m}$$

Consequently, $x(t_2) - x(t_1) = 375\,\mathrm{m}$ ❑

✧

The areas under the graph, A_1, A_2 and A_3, shown in Figure 4 are

$$A_1 = (10\,\mathrm{m\,s^{-1}}) \times (4\,\mathrm{s}) = 40\,\mathrm{m}$$

$$A_2 = \tfrac{1}{2}(10\,\mathrm{m\,s^{-1}}) \times (2\,\mathrm{s}) = 10\,\mathrm{m}$$

$$A_3 = \tfrac{1}{2}(-5\,\mathrm{m\,s^{-1}}) \times (1\,\mathrm{s}) = -2.5\,\mathrm{m}$$

The displacement s_x from the initial position after 7 s is therefore

$$\begin{aligned} s_x(7\,\mathrm{s}) &= x(7\,\mathrm{s}) - x(0) \\ &= (40 + 10 - 2.5)\,\mathrm{m} - 0\,\mathrm{m} = 47.5\,\mathrm{m} \end{aligned}$$

During the final second of its journey s_x changes by –2.5 m. The distance travelled during that final second is therefore 2.5 m. ❑

✧

Using inverse differentiation, we see that $F(x) = x^3 + C$, where C is a constant, is an inverse derivative of $f(x) = 3x^2$

since $\dfrac{dF}{dx} = f(x)$.

It follows that the required area under the graph is

$$\left[F(x)\right]_1^3 = \left[x^3 + C\right]_1^3 = (27 + C) - (1 + C)$$
$$= 26$$

❏

✧

The graph of $y = 9 - x^2$ is shown in

Figure 6.

In this case we note that an inverse derivative of $y = 9 - x^2$ is

$$Y(x) = 9x - \frac{x^3}{3} + C$$

where C is a constant

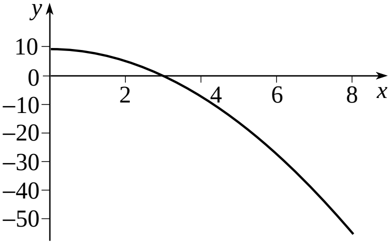

Figure 6 The graph of $y = 9 - x^2$.

Choosing $C = 0$, it follows that the area under the curve between $x = 4$ and $x = 7$ is

$$[Y(x)]_4^7 = \left[9x - \frac{x^3}{3}\right]_4^7$$

$$= \left(63 - \frac{343}{3}\right) - \left(36 - \frac{64}{3}\right) = -66$$

However, the question asks for the *magnitude* of the area under the curve, so the answer to the question is +66. ❑

✧

(a) distance travelled = VT.

(b) If the interval t_2 to t_3 is subdivided into n short intervals of duration Δt_i then the distance travelled in any one of those intervals will be approximately

$$v(t_i)\,\Delta t_i$$

and the total distance travelled will be approximately,

$$\sum_{i=1}^{n} v(t_i)\,\Delta t_i$$

The accurate value of the total distance travelled will be given by the limit of the above sum as the number of intervals increases and the duration of the longest approaches zero, i.e. as $\Delta t \to 0$, but that is just a definite integral, so the total distance travelled between t_2 and t_3 is just

$$\int_{t_2}^{t_3} v(t)\,dt = \int_{t_2}^{t_3} (at - bt^2)\,dt \quad ❏$$

✧

If the height of the column, from $h = 0$ to $h = H$, is divided into n subintervals of height Δh_i we can approximate the mass of the column by

$$\sum_{i=1}^{n} \rho(h_i) A \Delta h_i$$

In the limit as the width of the largest slab tends to zero, i.e. as $\Delta h \to 0$, this sum becomes the definite integral

$$\int_0^H \rho(h) A\, dh = \int_0^H \rho_0 e^{-h/\lambda} A\, dh \quad ❏$$

✧

The work done will be

$$\int_0^L F_x(x)\,dx = \int_0^L \frac{\lambda x}{L}\,dx$$

(Note that although the length of the string is increased from L to $2L$, the extension x only increases from 0 to L, so these are the limits of integration, with respect to x, in this case.) ❑

✧

(a) In this case the integrand is $at - bt^2$.

An inverse derivative of this function is

$$\frac{at^2}{2} - \frac{bt^3}{3} + C$$

☞

Choosing $C = 0$, it follows from the fundamental theorem of calculus that

$$\int_{t_2}^{t_3} (at - bt^2)\,dt = \left[\frac{at^2}{2} - \frac{bt^3}{3}\right]_{t_2}^{t_3}$$

$$= \left[\frac{at_3^2}{2} - \frac{bt_3^3}{3}\right] - \left[\frac{at_2^2}{2} - \frac{bt_2^3}{3}\right]$$

i.e.

$$\int_{t_2}^{t_3} (at - bt^2)\,dt = \frac{a}{2}(t_3^2 - t_2^2) - \frac{b}{3}(t_3^3 - t_2^3)$$

(b) In this case the integrand is $A\rho_0 e^{-h/\lambda}$.

An inverse derivative of this function is $-A\rho_0\lambda e^{-h/\lambda} + C$.

Choosing $C = 0$, it follows from the fundamental theorem of calculus that

$$\int_0^H \rho_0 e^{-h/\lambda} A\,dh = \left[-A\rho_0\lambda e^{-h/\lambda}\right]_0^H$$

$$= (-A\rho_0\lambda e^{-H/\lambda}) - (-A\rho_0\lambda)$$

i.e. $$\int_0^H \rho_0 e^{-h/\lambda} A\,dh = A\rho_0\lambda(1 - e^{-H/\lambda})$$

(c) In this case the integrand is $\dfrac{\lambda x}{L}$.

An inverse derivative of this function is

$$\frac{\lambda x^2}{2L} + C.$$

Choosing $C = 0$, it follows from the fundamental theorem of calculus that

$$\int_0^L \frac{\lambda x}{L}\,dx = \left[\frac{\lambda x^2}{2L}\right]_0^L = \frac{\lambda L^2}{2L} - 0 = \frac{\lambda L}{2}$$ ❏

Section 1 contains the various *Opening items* that are a common feature of all *FLAP* modules. These are designed to give you a clear view of the module's content and the prior knowledge that it assumes, so that you can assess for yourself the extent to which you need to study the module and the degree to which you are prepared for such a study. Sections 2 and 3 contain the basic teaching material of the module, dealing with mathematical *expressions* and with relations between expressions, particularly *equality* and *inequality*. Section 2 starts with a review of the basic *operations* (addition, subtraction, multiplication and division) together with their symbolic representation and order of priority in written expressions. This topic is taken further in Subsection 2.2 which concentrates on *brackets* and their use in simplifying and expanding expressions. The manipulation of algebraic and arithmetic *fractions* (a common source of errors) is discussed in Subsection 2.3, and the topic of *powers*, *roots* and *reciprocals* is covered in Subsection 2.4. The writing and rearrangement of simple *equations* is dealt with in Subsections 3.1 and 3.2, while Subsection 3.3 looks at *proportionality*. (Both *direct* and *inverse* proportionality are considered, together with the equations that reflect such relationships.) Subsection 3.4 deals with the subject of *inequalities* and reviews the rules for their manipulation. As is usual in *FLAP*, the module ends with a section of *Closing items* (Section 4) that includes a summary, a list of the things that you should be able to do on completing the module, and an *Exit text* designed to let you assess your achievements and alert you to any remaining difficulties. The answers to all the numbered questions included in the module can be found at the end of the module in Section 5.