Handout 1
1st page

2nd year (Level 5) Physics courses

Mathematical Methods and Applications

20 Credits

Semester 1
Mathematical Methods (formally called ‘Theoretical Physics I’)
Dr Graham S McDonald

Semester 2
Applications in Electricty, Magnetism and AC Circuit Theory
Dr Tiehan Shen

Handout 1
pages 1 & 2

Assessment

January Test 40%

May Exam 60%

Core Books

Semester 1 - K.A Stroud, ‘Advanced Engineering Mathematics’ 4^{th} Ed, Macmillan Press 2003

Semester 2 - University Physics with Modern Physics, by Young and Freedman 13^{th} Edition (2011), Pearson

Syllabus outline

Handout 1
pages 1 & 2

<u>Semester 1 – Dr Graham S McDonald</u>

- Vector calculus, including: gradient, divergence, flux and curl, the divergence theorem and Stokes' theorem.

- Matrices, determinants, eigenvalues and eigenvectors. Applications of matrices.

- Partial differential equations and methods of solution, e.g. separation of variables.

<u>Semester 2 – Dr Tiehan Shen</u>

The magnetic field. Biot and Savart law and Ampere's law.
Electromagnetic induction. Magnetic flux; Faraday's and Lenz's law
Transients in LR, RC and LCR circuits;
AC Theory and complex analysis: reactance, impedance and resonance

Semester 1 topics in more detail …

Handout 1
pages 1 & 2

Syllabus:

Theory

- Vector Calculus

  – Review of fundamental concepts. Scalar, vector and conservative fields. Grad, divergence, flux and curl. The divergence theorem and Stoke's theorem. The Laplacian and curvilinear coordinates

  – Examples from electrostatics, magnetism, fluid dynamics, mechanics, heat flow

- Determinants and Matrices

  – Basic definitions and operations. Cramer's rule and Laplace expansion. Rank, linear independence, elementary row operations and matrices in echelon form. Properties of determinants. Special matrices and matrix inversion. Eigenvalues and eigenvectors

  – Applications in electrical circuits, rotation of co-ordinates, transmission through single and cascaded linear systems (such as in optics and electronics)

- Differential Equations

  – Review of ordinary differential equations. Important partial differential equations (PDEs). Solution of PDEs and the role of arbitrary functions. Separation of variables.

  – Examples drawn from a broad range of physics

H1
p4
top

Books

- KA Stroud, "Further Engineering Mathematics", (Macmillan) – reasonable coverage but not particularly deep
- Mary L Boas, "Mathematical Methods in Physical Sciences", (John Wiley) - good physical interpretations but many folk find it a bit too difficult and fast.

H1
p4
bot

- Erwin Kreyszig, "Advanced Engineering Mathematics", (Wiley) – massive indepth text at a massive price

- MR Spiegel, "Schaum's Outline Series. Theory and problems of Advanced Mathematics for Engineers & Scientists", (MacGraw-Hill) – not a bad book, that covers a lot of maths, but mostly gives a mathematical viewpoint.

• Review of fundamental concepts

H1
P5,6

... molto rapida!

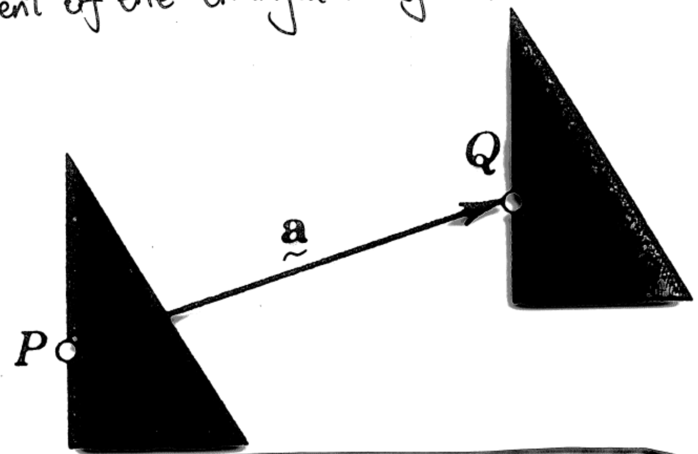
In science and engineering, there are basically two types of quantity:

SCALAR QUANTITIES that are determined by their magnitude i.e. a number of units on an appropriate scale. For example, mass, volume, density, temperature, potential, charge and distance.

AND

VECTOR QUANTITIES that have both magnitude and direction. These can be represented by arrows that point in the appropriate direction and whose length give the magnitude.

Displacement is a vector quantity. In the example below, the arrow points in the direction of displacement of the triangular object.



The length of this vector (the distance between the points P and Q) gives the magnitude of displacement.

If we call this vector $\underset{\sim}{a}$ then the length of $\underset{\sim}{a}$ is denoted $|\underset{\sim}{a}|$.

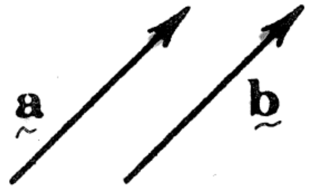
If $\underset{\sim}{a}$ is a UNIT VECTOR then the length will be one (i.e. UNITY) and $|\underset{\sim}{a}| = 1$.

Other examples of vector quantities are force, velocity, acceleration, stress, electric field, magnetic induction.

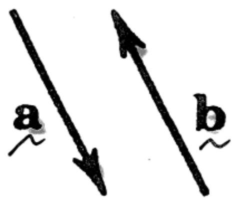
EQUALITY OF VECTORS

$\underset{\sim}{a} = \underset{\sim}{b}$ when the vectors have the same length and the same direction.

However, they do not need to have the same starting point. For example, while resolving forces on an object, one can translate each vector to any convenient position to work out the total force.

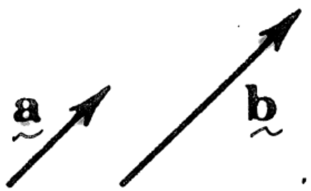


Equal vectors, $(\underset{\sim}{a} = \underset{\sim}{b})$



Vectors having the same length but different direction

$(\underset{\sim}{a} \neq \underset{\sim}{b})$



Vectors having the same direction but different length.

COMPONENTS OF A VECTOR

Choosing an xyz Cartesian coordinate system (rectangular and with the same scale on each axis), the vector $\underset{\sim}{a}$ from P at (x_1, y_1, z_1) to Q at (x_2, y_2, z_2) ....

... has components

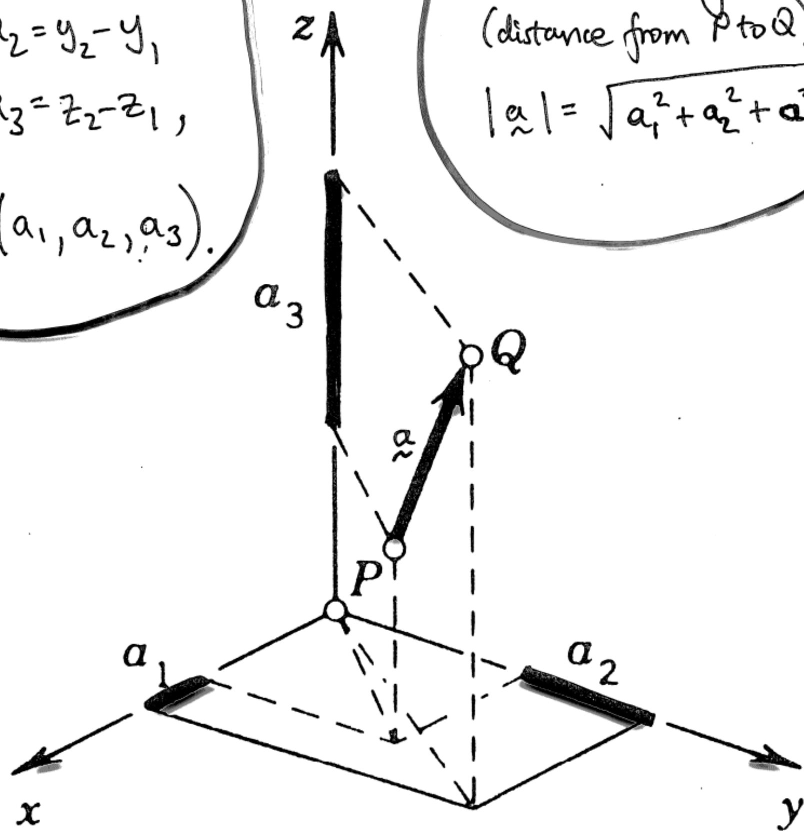
$$a_1 = x_2 - x_1$$
$$a_2 = y_2 - y_1$$
$$a_3 = z_2 - z_1,$$

i.e.

$$\underset{\sim}{a} = (a_1, a_2, a_3).$$

... has length (distance from P to Q)

$$|\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



Ex The vector $\underset{\sim}{a}$ has initial point P at $(4,0,2)$ and terminal point Q at $(6,-1,2)$.
What are the components of $\underset{\sim}{a}$?
What is the length of $\underset{\sim}{a}$?
What is the unit vector along the same direction as $\underset{\sim}{a}$?

Ans Denoting P as (x_1, y_1, z_1) and Q as (x_2, y_2, z_2), components of $\underset{\sim}{a}$ are

$$a_1 = x_2 - x_1 = 6-4 = 2$$
$$a_2 = y_2 - y_1 = -1-0 = -1$$
$$a_3 = z_2 - z_1 = 2-2 = 0$$

i.e. $\underset{\sim}{a} = (a_1, a_2, a_3)$

i.e. $\underset{\sim}{a} = (2, -1, 0)$

Length of $\underset{\sim}{a}$ is $|\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$$= \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

Unit vector along $\underset{\sim}{a}$ is $\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

i.e. $\dfrac{(2,-1,0)}{\sqrt{5}}$

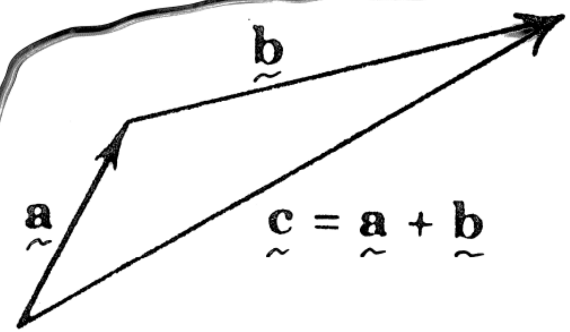
i.e. $\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right)$

(9) ADDITION OF VECTORS

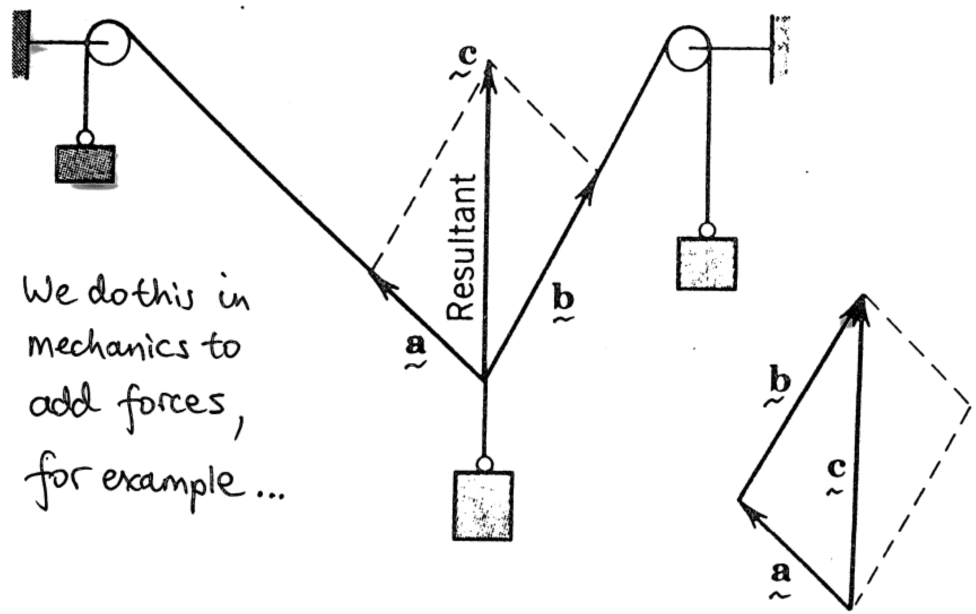
The sum of two vectors $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$ is

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1+b_1,\ a_2+b_2,\ a_3+b_3)$$

Geometrically, we can place $\underset{\sim}{b}$ to start at the 'tip' of $\underset{\sim}{a}$ ...



Physical example



We do this in mechanics to add forces, for example ...

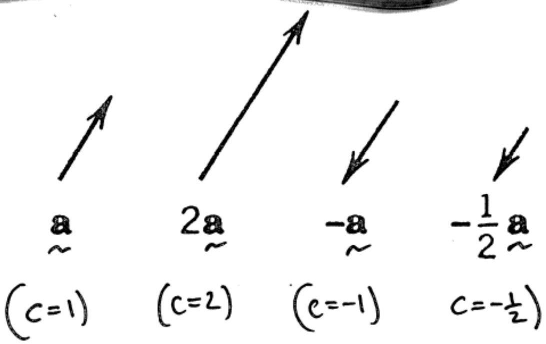
MULTIPLYING VECTORS BY A NUMBER

(i.e. by a scalar)

If we multiply the vector $\mathbf{a} = (a_1, a_2, a_3)$ by any scalar c (i.e. a real number c), then

$$c\mathbf{a} = (ca_1, ca_2, ca_3).$$

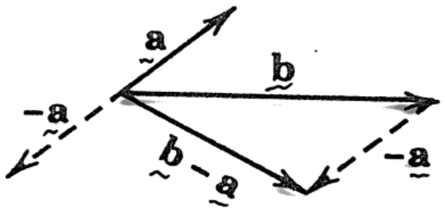
The result has the same or opposite direction depending on whether c is positive or negative.



To form the difference of vectors $\mathbf{b}$ and $\mathbf{a}$, we add $(-1)\mathbf{a} = -\mathbf{a}$

i.e. $\mathbf{b} + (-\mathbf{a})$

i.e. $\mathbf{b} - \mathbf{a}$



Ex. With respect to a given coordinate system, let

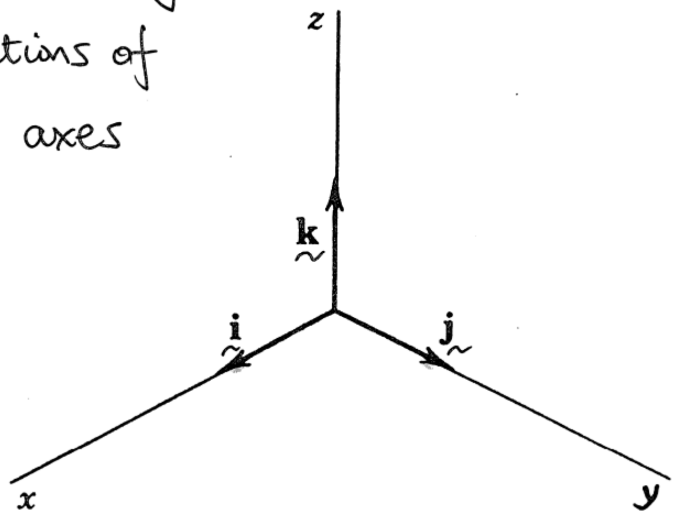
$\mathbf{a} = (4, 0, 1)$ and $\mathbf{b} = (2, -5, \frac{1}{3})$.

What is the vector $2(\mathbf{a} - \mathbf{b})$?

Ans.
$$2(\mathbf{a}-\mathbf{b}) = 2\left[(4,0,1) - (2,-5,\tfrac{1}{3})\right]$$
$$= 2\,(4-2,\ 0+5,\ 1-\tfrac{1}{3})$$
$$= 2\,(2,\ 5,\ \tfrac{2}{3}) = (4, 10, \tfrac{4}{3}).$$

UNIT VECTORS $\mathbf{i}, \mathbf{j}, \mathbf{k}$

In Cartesian representation, the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have magnitude 1 and are in the positive directions of the x, y, z axes



$\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ can thus be expressed as

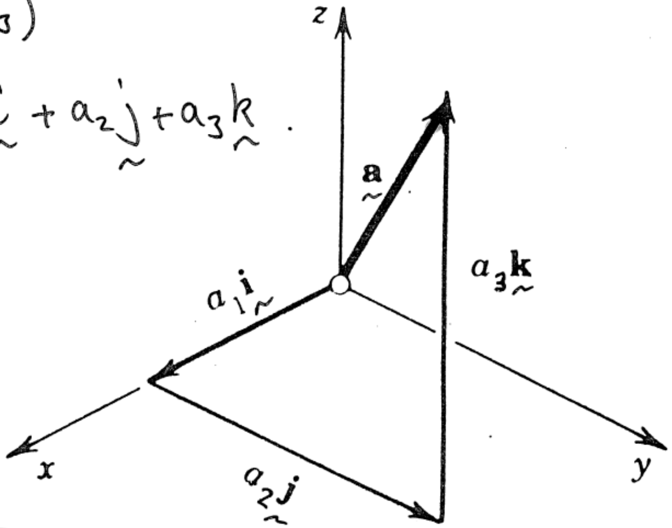
$$\underset{\sim}{i} = (1,0,0) \quad , \quad \underset{\sim}{j} = (0,1,0) \quad , \quad \underset{\sim}{k} = (0,0,1)$$

Then, for example,

$$\boxed{\begin{aligned} a_1 \underset{\sim}{i} &= (a_1, 0, 0) \\ a_2 \underset{\sim}{j} &= (0, a_2, 0) \\ a_3 \underset{\sim}{k} &= (0, 0, a_3). \end{aligned}}$$

If $\underset{\sim}{a} = (a_1, a_2, a_3)$

then $\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$.



Ex If $\underset{\sim}{a} = (4,0,1)$ and $\underset{\sim}{b} = (2,-5,\frac{1}{3})$

then $\underset{\sim}{a} = 4\underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 5\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k}$.

DOT PRODUCT OF VECTORS

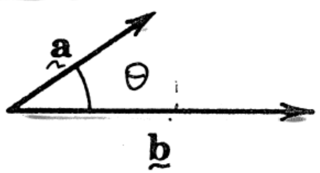
The dot product of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta \; ,$$

where θ is the angle between the vectors.

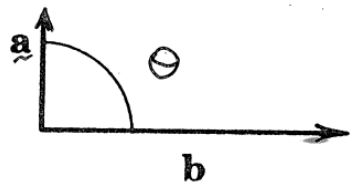
Everything on the RHS, i.e. $|\underset{\sim}{a}|$, $|\underset{\sim}{b}|$ and $\cos\theta$, are just numbers (scalars). So the dot product of two vectors is just a number. It is also called the SCALAR PRODUCT.

THREE CASES ARISE:



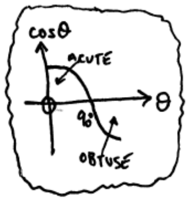
$\underset{\sim}{a} \cdot \underset{\sim}{b} > 0$

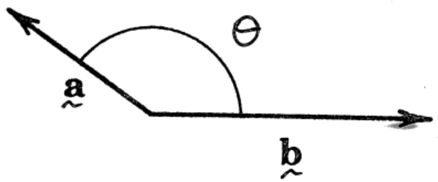
Acute angle : $\cos\theta > 0$



$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

Right angle : $\cos\theta = 0$





$\underset{\sim}{a} \cdot \underset{\sim}{b} < 0$

Obtuse angle : $\cos\theta < 0$

When $\theta = 0$ i.e. two vectors are parallel

then $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$.

For example, $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}||\underset{\sim}{a}| = |\underset{\sim}{a}|^2$.

Since $|\underset{\sim}{i}| = |\underset{\sim}{j}| = |\underset{\sim}{k}| = 1$ i.e. unit vectors have unit magnitude,

then
$$\underset{\sim}{i} \cdot \underset{\sim}{i} = |\underset{\sim}{i}|^2 = 1$$
$$\underset{\sim}{j} \cdot \underset{\sim}{j} = |\underset{\sim}{j}|^2 = 1$$
$$\underset{\sim}{k} \cdot \underset{\sim}{k} = |\underset{\sim}{k}|^2 = 1.$$

When $\theta = 90^\circ$, i.e. two vectors are perpendicular

then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

For example,
$$\underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{i} = 0$$
$$\underset{\sim}{i} \cdot \underset{\sim}{k} = \underset{\sim}{k} \cdot \underset{\sim}{i} = 0$$
$$\underset{\sim}{j} \cdot \underset{\sim}{k} = \underset{\sim}{k} \cdot \underset{\sim}{j} = 0.$$

DOT PRODUCT IN TERMS OF VECTOR COMPONENTS

If $\underset{\sim}{a} = (a_1, a_2, a_3) = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

and $\underset{\sim}{b} = (b_1, b_2, b_3) = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$,

then $\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}) \cdot (b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k})$

i.e.
$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= a_1 b_1 \underset{\sim}{i} \cdot \underset{\sim}{i} + a_1 b_2 \underset{\sim}{i} \cdot \underset{\sim}{j} + a_1 b_3 \underset{\sim}{i} \cdot \underset{\sim}{k} \\ &+ a_2 b_1 \underset{\sim}{j} \cdot \underset{\sim}{i} + a_2 b_2 \underset{\sim}{j} \cdot \underset{\sim}{j} + a_2 b_3 \underset{\sim}{j} \cdot \underset{\sim}{k} \\ &+ a_3 b_1 \underset{\sim}{k} \cdot \underset{\sim}{i} + a_3 b_2 \underset{\sim}{k} \cdot \underset{\sim}{j} + a_3 b_3 \underset{\sim}{k} \cdot \underset{\sim}{k} \end{aligned}$$

i.e. $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Since $\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$ and all the other dot products are between perpendicular vectors and thus give zero.

ANGLE BETWEEN TWO VECTORS

(17)

The way to work out the angle between two vectors is (in my opinion) most easily remembered by recalling the definition of the dot product.

Ex Find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, where $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$.

Ans $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, where θ is the angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$\therefore \cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underset{\sim}{a}||\underset{\sim}{b}|},$$

where $\underset{\sim}{a} = (a_1, a_2, a_3) = (2, 2, -1)$,

$\underset{\sim}{b} = (b_1, b_2, b_3) = (6, -3, 2)$,

$|\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{4+4+1} = 3$

$|\underset{\sim}{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{36+9+4} = 7$

$$\therefore \cos\theta = \frac{2.6 + 2.(-3) + (-1).2}{3.7} = \frac{12-6-2}{21} = \frac{4}{21}$$

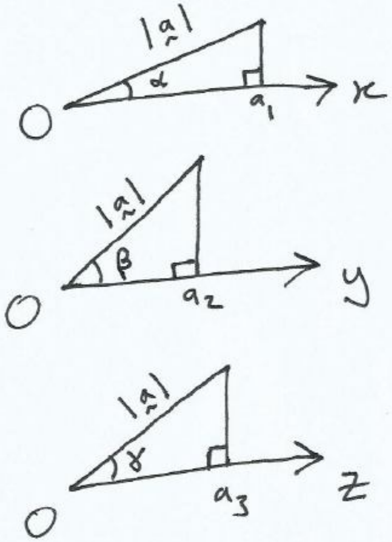
$$\therefore \theta = \cos^{-1}\left(\frac{4}{21}\right).$$

"17b"

Something to copy into the space around the middle of page and to right of (18)...

Specifying direction of a vector (length-independent):

'DIRECTION COSINES' appear in solid-state theory and express angles made with each Cartesian axis... for $\underset{\sim}{a} = (a_1, a_2, a_3)$,



$\cos\alpha = l$

$\cos\beta = m$

$\cos\gamma = n$

H1
p18
top

Using the dot product to find the angle between two vectors is <u>formally equivalent</u> to working out the direction cosines $[l, m, n]$ and $[l', m', n']$ of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, respectively, and using the result that

$$\boxed{\cos\theta = ll' + mm' + nn'.}$$

Since $l = \dfrac{a_1}{|\underset{\sim}{a}|}$, $m = \dfrac{a_2}{|\underset{\sim}{a}|}$, $n = \dfrac{a_3}{|\underset{\sim}{a}|}$

and $l' = \dfrac{b_1}{|\underset{\sim}{b}|}$, $m' = \dfrac{b_2}{|\underset{\sim}{b}|}$, $n' = \dfrac{b_3}{|\underset{\sim}{b}|}$,

H1
p18
bot

$$\boxed{\cos\theta = ll' + mm' + nn'}$$

$$l = \frac{a_1}{|\underset{\sim}{a}|}, \quad m = \frac{a_2}{|\underset{\sim}{a}|}, \quad n = \frac{a_3}{|\underset{\sim}{a}|}$$

$$l' = \frac{b_1}{|\underset{\sim}{b}|}, \quad m' = \frac{b_2}{|\underset{\sim}{b}|}, \quad n' = \frac{b_3}{|\underset{\sim}{b}|}$$

$$\cos\theta = \frac{a_1 b_1}{|\underset{\sim}{a}||\underset{\sim}{b}|} + \frac{a_2 b_2}{|\underset{\sim}{a}||\underset{\sim}{b}|} + \frac{a_3 b_3}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}.$$

Note: We have already seen that the dot product of two vectors tells us whether the angle between two vectors is acute or obtuse (the sign of the result is different).

MEANING OF THE DOT PRODUCT

To understand what the dot product means, ask yourself what is the result of calculating $\underset{\sim}{a} \cdot \hat{\imath}$, $\underset{\sim}{a} \cdot \hat{\jmath}$ and $\underset{\sim}{a} \cdot \hat{k}$.

If $\underset{\sim}{a} = (a_1, a_2, a_3)$ then

$$\underset{\sim}{a} \cdot \hat{\imath} = (a_1, a_2, a_3) \cdot (1,0,0) = a_1$$
$$\underset{\sim}{a} \cdot \hat{\jmath} = (a_1, a_2, a_3) \cdot (0,1,0) = a_2$$
$$\underset{\sim}{a} \cdot \hat{k} = (a_1, a_2, a_3) \cdot (0,0,1) = a_3 .$$

The dot product gives the component of $\underset{\sim}{a}$ along that particular direction here. In other words, $\underset{\sim}{a} \cdot \hat{\imath}$ gives the projection of $\underset{\sim}{a}$ along the $\hat{\imath}$ direction, i.e. onto the x-axis, and similarly for $\underset{\sim}{a} \cdot \hat{\jmath}$ and $\underset{\sim}{a} \cdot \hat{k}$.

This is a general result.

If $\underset{\sim}{\hat{n}}$ is the unit vector in any direction

then $\boxed{\underset{\sim}{a} \cdot \underset{\sim}{\hat{n}} \text{ gives the projection of } \underset{\sim}{a} \text{ along that direction.}}$

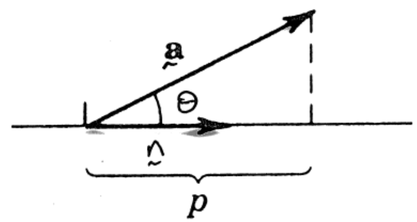
If $\underset{\sim}{\hat{n}}$ is a unit vector then $|\underset{\sim}{\hat{n}}| = 1$.

So $\underset{\sim}{a} \cdot \underset{\sim}{\hat{n}} = |\underset{\sim}{a}||\underset{\sim}{\hat{n}}| \cos\theta = |\underset{\sim}{a}| \cos\theta$.

If we let $p = |\underset{\sim}{a}| \cos\theta$ then

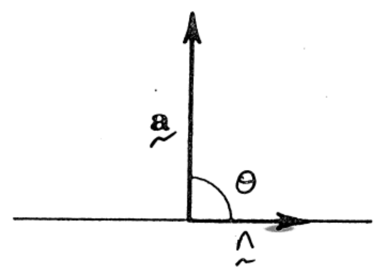
THREE CASES ARISE:

Acute angle: $\underset{\sim}{a}$ simply projected onto $\underset{\sim}{\hat{n}}$



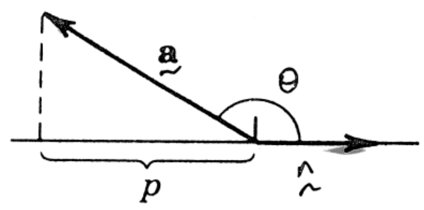
$(p > 0)$

Right angle: there is no projection of $\underset{\sim}{a}$ onto $\underset{\sim}{\hat{n}}$



$(p = 0)$

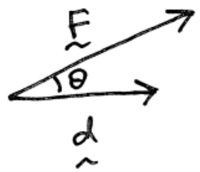
Obtuse angle: the projection is negative to denote that $\underset{\sim}{a}$ and $\underset{\sim}{\hat{n}}$ are in different directions



$(p < 0)$

THE USE OF THE DOT PRODUCT IN PHYSICS

Imagine a force $\underset{\sim}{F}$ that drags an object through a displacement $\underset{\sim}{d}$.



To calculate the work done by the force, we resolve the force along the direction of the displacement.

— component of $\underset{\sim}{F}$ along $\underset{\sim}{d}$ is $|\underset{\sim}{F}|\cos\theta$

— work done is "force × distance along which it acts"

i.e. work done is $\left(|\underset{\sim}{F}|\cos\theta\right) . |\underset{\sim}{d}|$

$$= |\underset{\sim}{F}||\underset{\sim}{d}|\cos\theta$$

$\therefore$ Work done $= \underset{\sim}{F} \cdot \underset{\sim}{d}$

The above implicitly assumes that both vectors $\underset{\sim}{F}$ and $\underset{\sim}{d}$ are constant in magnitude and direction during the displacement. But, for example, the force may be a function of space i.e. $\underset{\sim}{F}(x,y,z)$

This brings us to the notion of a FIELD.

A field is basically a region of space where quantities, such as force, may assume different values depending upon where one is within this space.

H1
p22
top

More technically, a ‘field’ is actually the physical quantity itself and the ‘space’ can include time.

space.

There are two types of field:

- Scalar field — where a scalar quantity can have different values in different places. If the scalar quantity is denoted ϕ then $\phi(x,y,z)$.

- vector field — where a vector quantity may have different values in different places. If the vector quantity is denoted $\underset{\sim}{V}$ then $\underset{\sim}{V}(x,y,z)$.

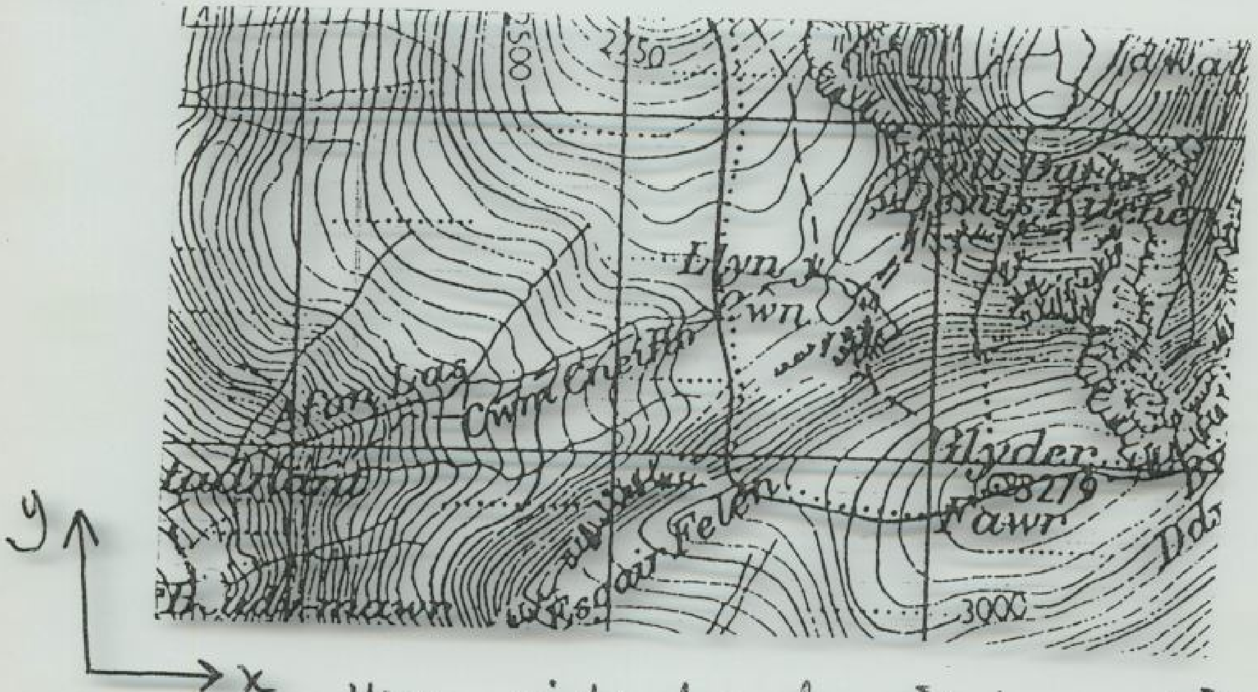
H1
p22
bot

SCALAR FIELD

At each point in a region of space (x, y, z) one associates a number/scalar through, say,

$\phi(x, y, z)$ = scalar field.

e.g. a 2D scalar field = height(x, y)



Here, points of equal magnitude are linked with curves i.e. contours.

H1
p23
bot

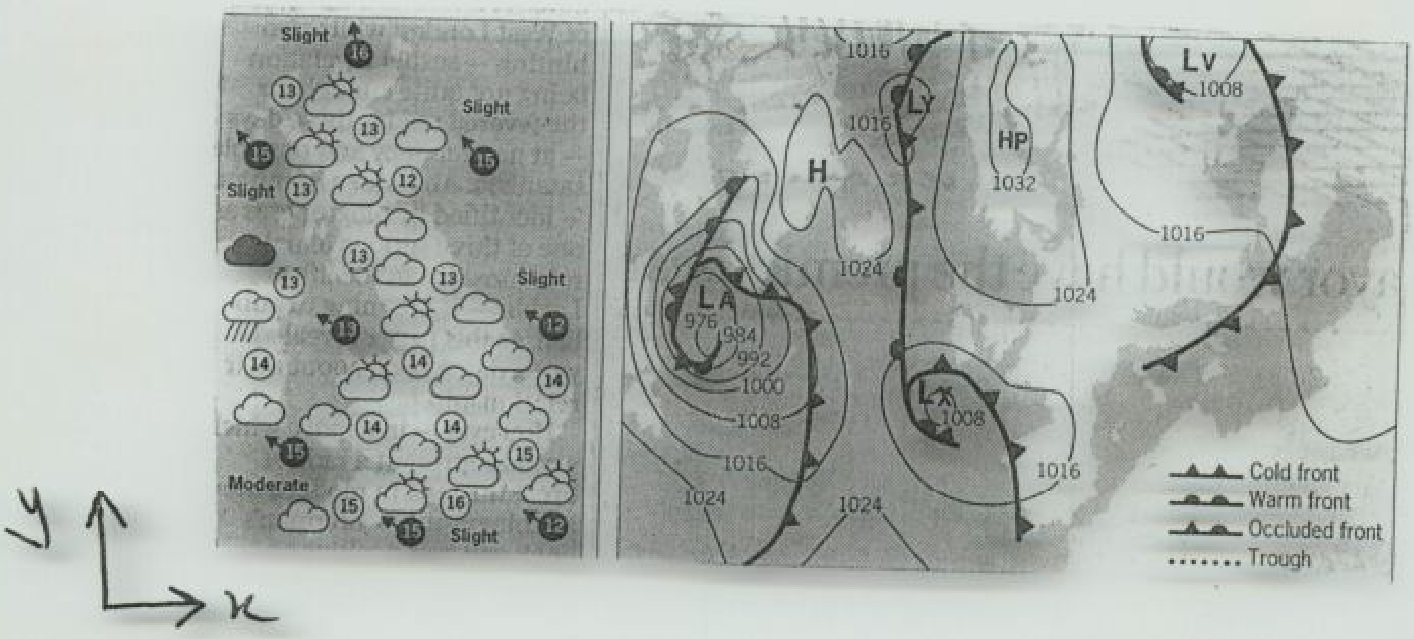
One can define a scalar field in terms of any scalar quantities, e.g. temperature, gravitational potential, density, or not even specify the quantity and just consider magnitude

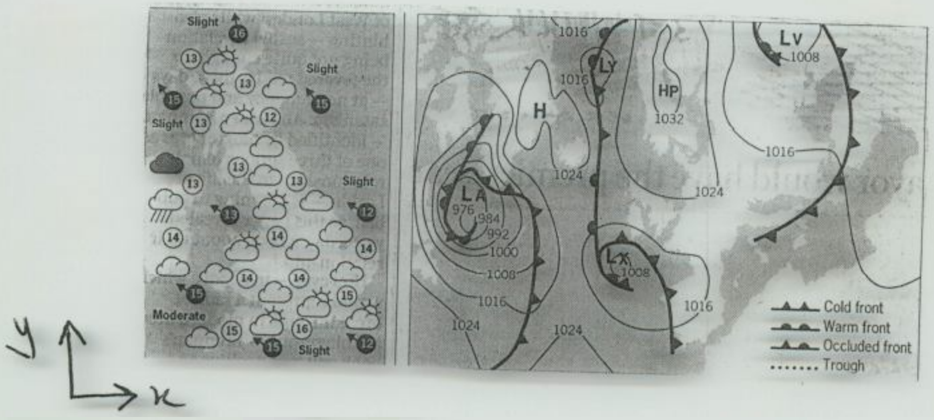
e.g. $\phi(x,y,z) = x^3 y - z^2$ defines a scalar field.

H1
p24
top

VECTOR FIELD

At each point in a region of space (x,y,z) one associates a vector e.g. a force field such as electric field, magnetic field, gravitational field or a field defined through velocity.





H1
p24
bot

In the above example,

temperature, $T(x,y)$ = scalar field

wind velocity, $\underset{\sim}{v}(x,y)$ = vector field

pressure magnitude, $P(x,y)$ = scalar field.

Also,

$$\underset{\sim}{V}(x,y,z) = xy^2\,\underset{\sim}{i} - 2yz^3\,\underset{\sim}{j} + x^2z\,\underset{\sim}{k}$$

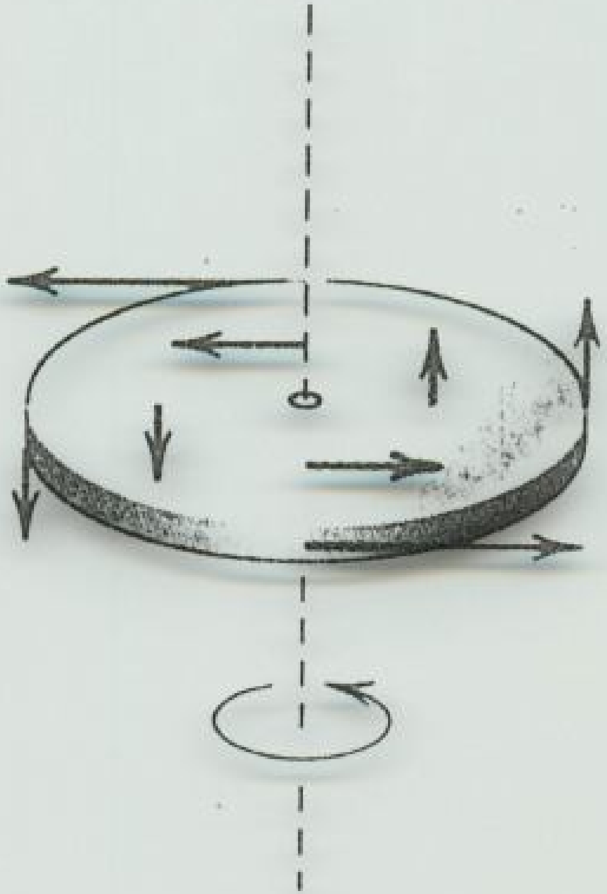
defines a vector field at any point (x,y,z),

H1
p25
top

Other examples of vector fields...

velocity

$\underset{\sim}{v}(x,y,z)$



Velocity field
of a rotating body

H1
p25
bot

acceleration

$\underset{\sim}{g}(x,y,z)$

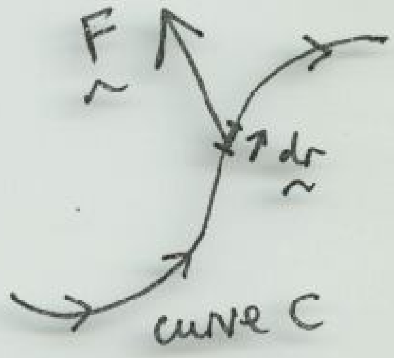
Gravitational field

H1
p26
top

Return to the example of work done as a scalar product and allow the force to vary in space.

We can also let the displacement be more general while the force is acting

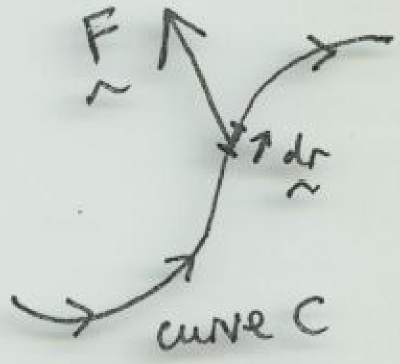
If the force $\underset{\sim}{F}(x,y,z)$ has components (F_x, F_y, F_z) and the displacement is along some curve C, then the total work done is found by adding up all the contributions made from individual line segments $d\underset{\sim}{r}$ along curve C ...



This segment can be considered so short that the force is constant over this

H1
p26
bot

along curve C ...



This segment can be considered so short that the force is constant over this segment

i.e. work done (over segment $d\underset{\sim}{r}$) $= \underset{\sim}{F} \cdot d\underset{\sim}{r}$

~~then~~ work done (over curve C), $W = \int_C \hat{F} \cdot d\hat{r}$ (LINE INTEGRAL)

i.e. $W = \int_C (F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$

i.e. $W = \int_C F_x dx + F_y dy + F_z dz$.

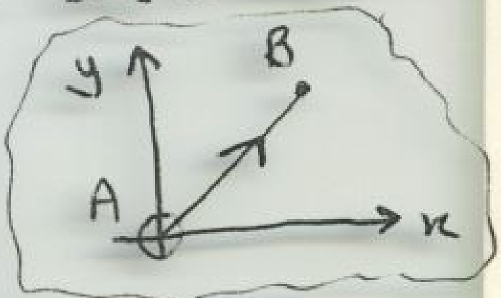
H1
p27
top

$\underset{\sim}{F}(x,y,z)$ defines a vector field, but are there different types of vector field?

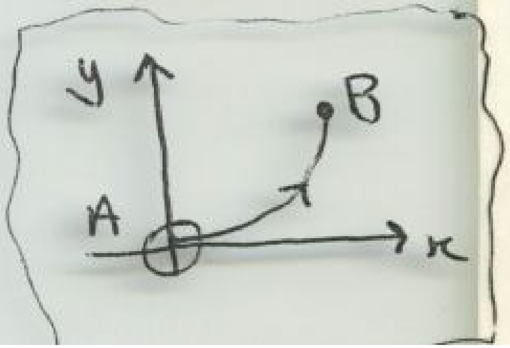
Ex Particle moving in the x-y plane under the influence of force $\underset{\sim}{F} = (y^2, x^2, 0)$. What is the work done in going from A at $(x,y) = (0,0)$ to B at $(x,y) = (1,1)$?

Ans What route do we take? Where is curve C? Does it matter?

Let's try two different routes (a) along $y = x$



(b) along $y = x^2$



force $\underset{\sim}{F} = (y^2, x^2, 0)$. What is the work done in going from A at $(x,y) = (0,0)$ to B at $(x,y) = (1,1)$?

H1
p27
bot

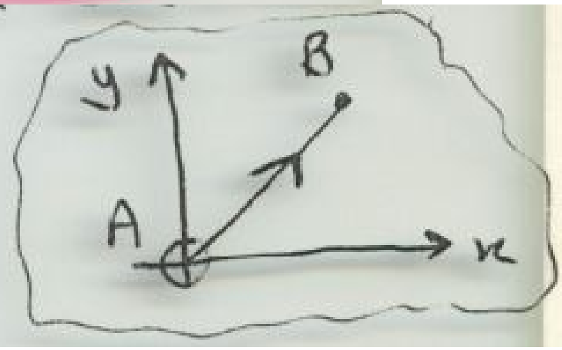
In each case, we have

$$W = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C (F_x dx + F_y dy) = \int_{\text{along } x} F_x dx + \int_{\text{along } y} F_y dy$$

(a) $$W = \int_{x=0}^{x=1} y^2 dx + \int_{y=0}^{y=1} x^2 dy$$

$$= \int_0^1 x^2 dx + \int_0^1 y^2 dy \quad (\text{since we go via } y = x)$$

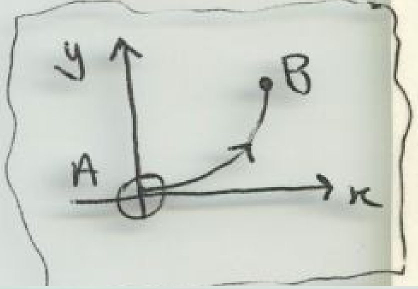
$$= \left[\tfrac{1}{3}x^3\right]_0^1 + \left[\tfrac{1}{3}y^3\right]_0^1 = \frac{2}{3}.$$



(a) gave W = 2/3

(b) along $y = x^2$

$\underset{\sim}{F} = (y^2, x^2, 0)$



H1
p28
top

(b) $$W = \int_{x=0}^{x=1} y^2\,dx + \int_{y=0}^{y=1} x^2\,dy$$

$$= \int_0^1 x^4\,dx + \int_0^1 y\,dy \quad \left(\text{since we go via } y = x^2\right)$$

$$= \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$$

So for this force the work done clearly depends on the path.

→ "NON-CONSERVATIVE FORCE" ≡ work done depends on path

H1
p28
bot

Let's try another field ...

Ex $\underset{\sim}{F} = (xy^2, yx^2, 0)$ and go from A to B by the two routes

Ans $W_{A(y=x)B} = \int_0^1 xy^2\,dx + \int_0^1 yx^2\,dy = \int_0^1 x^3\,dx + \int_0^1 y^3\,dy$

$= \left[\frac{1}{4}x^4\right]_0^1 + \left[\frac{1}{4}y^4\right]_0^1 = \frac{1}{2}$.

$W_{A(y=x^2)B} = \int_0^1 x^5\,dx + \int_0^1 y^2\,dy = \left[\frac{1}{6}x^6\right]_0^1 + \left[\frac{1}{3}y^3\right]_0^1 = \frac{1}{2}$!

In this case, the work done was the same for both paths.

Is this a fluke?

H1
p29
top

No! For this field, the work done by the force is always independent of the path – a "CONSERVATIVE FORCE"

In other words, the work done only depends on the start and end points.

One can write this as

$$\boxed{\int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^B dW = W_B - W_A}$$

H1
p29
bot

Since $\underset{\sim}{F}.d\underset{\sim}{r} = F_x dx + F_y dy + F_z dz$ then this is equivalent to

$$\boxed{F_x dx + F_y dy + F_z dz = dW = \text{PERFECT/EXACT DIFFERENTIAL}}$$

$$\int_A^B \underset{\sim}{F}.d\underset{\sim}{r} = \int_A^B dW = W_B - W_A$$

Also, $\left(\frac{\partial W}{\partial x}\right)dx + \left(\frac{\partial W}{\partial y}\right)dy + \left(\frac{\partial W}{\partial z}\right)dz = dW$

ie. $F_x = \frac{\partial W}{\partial x}$, $F_y = \frac{\partial W}{\partial y}$, $F_z = \frac{\partial W}{\partial z}$

but, $\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right) \Rightarrow \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$

$\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial z}\right) \Rightarrow \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$

$\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial z}\right) \Rightarrow \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$

H1
p30
top

But these are just mathematical properties of the vector field.
we could actually be dealing with a vector field in mechanics,
hydrodynamics, gravitation or electromagnetism, for example.

so let's be general and cover all these cases...

A vector field $\underset{\sim}{V}(x,y,z)$ is conservative when their exists
a "scalar potential" $\phi(x,y,z)$ such that

$$\boxed{\int_A^B \underset{\sim}{V} \cdot d\underset{\sim}{r} = \int_A^B d\phi = \phi_B - \phi_A}$$

PATH
INDEPENDENT

H1
p30
bot

or equivalently,

$$\boxed{d\phi = V_x\,dx + V_y\,dy + V_z\,dz}$$

EXACT DIFFERENTIAL, $d\phi$

or equivalently,

$$\boxed{\begin{gathered}\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}\\[2ex] \frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}\\[2ex] \frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}\end{gathered}}$$

H1
p31
top

Ex A vector field following an "inverse square law"

i.e. $$\boxed{\underset{\sim}{V}(\underset{\sim}{r}) = \frac{\eta\,\hat{\underset{\sim}{r}}}{r^2} = \frac{\eta\,\underset{\sim}{r}}{r^3}}$$

radial field

$\underset{\sim}{r}$ position vector

where $\hat{\underset{\sim}{r}}$ = unit vector i.e. $\underset{\sim}{r} = r\,\hat{\underset{\sim}{r}}$

η = constant independent of r, depending on the physical example.

$$\underset{\sim}{V}(\underset{\sim}{r}) = \frac{\eta\,(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})}{r^3} = \left(\frac{\eta x}{r^3}, \frac{\eta y}{r^3}, \frac{\eta z}{r^3}\right)$$

H1
p31
bot

$$r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{\partial r}{\partial y} = 2y \quad \therefore \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

then $$\frac{\partial V_x}{\partial y} = -3\eta\frac{xy}{r^5} \quad ; \quad \frac{\partial V_y}{\partial x} = -3\eta\frac{xy}{r^5}$$

... and the same applies to all the other coordinate pairings

i.e. x, z and y, z

$\therefore$ Any radial vector field following an inverse square law is a CONSERVATIVE FIELD.