Mathematical Methods and Applications

CONTENTS

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- Ordinary Differential Equations
  - Review of 1^{st} and 2^{nd} order linear odes
  - Higher order linear odes

slides only have selected topics from the Handout

- Partial Differential Equations
  - Arbitrary functions
  - Similarities with the solution of ode's
  - Separation of variables
    * Finding a solution.
    * Superposition to get the required solution

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Definition of a differential equation

– an equation involving derivatives or differentials ...

e.g. 1 $\quad (y'')^2 + 3x = 2(y')^3$ where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$

e.g. 2 $\quad \frac{dy}{dx} + \frac{y}{x} = y^2$

1.–4. have ONE independent variable and are ordinary differential equations

(o.d.e's)

e.g. 3 $\quad \frac{d^2Q}{dt^2} - 3\frac{dQ}{dt} + 2Q = 4\sin 2t$

e.g. 4 $\quad \frac{dy}{dx} = \frac{x+y}{x-y}$ or equivalently $(x+y)\,dx + (y-x)\,dy = 0$

e.g. 5 $\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

has more than one independent variable (x and y) and is a partial differential equation

(p.d.e.)

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Definition of a differential equation

– an equation involving derivatives or differentials ...

e.g. 1 $\quad (y'')^2 + 3x = 2(y')^3 \quad$ where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ — second order

e.g. 2 $\quad \frac{dy}{dx} + \frac{y}{x} = y^2$ — first order

nonlinear (in the dept. variable)

e.g. 3 $\quad \frac{d^2Q}{dt^2} - 3\frac{dQ}{dt} + 2Q = 4\sin 2t$ — second order

e.g. 4 $\quad \frac{dy}{dx} = \frac{x+y}{x-y}$ or equivalently $(x+y)\,dx + (y-x)\,dy = 0$ — first order

e.g. 5 $\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ — second order

- The highest order of derivative defines the order of the differential equation.

Solutions of differential equations

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— a relation between the variables which is free of derivatives and which satisfies the differential equation

eg. $\boxed{\dfrac{dy}{dx} = 3x^2}$ has general solution $\boxed{y = x^3 + c}$

where c is an arbitrary constant. The general solution of an n^{th} order differential equation has n arbitrary constants.

$\dfrac{dy}{dx} = 3x^2$ has a particular solution $\boxed{y = x^3 + 1}$.

This can be found by assigning a value to the arbitrary constant c.

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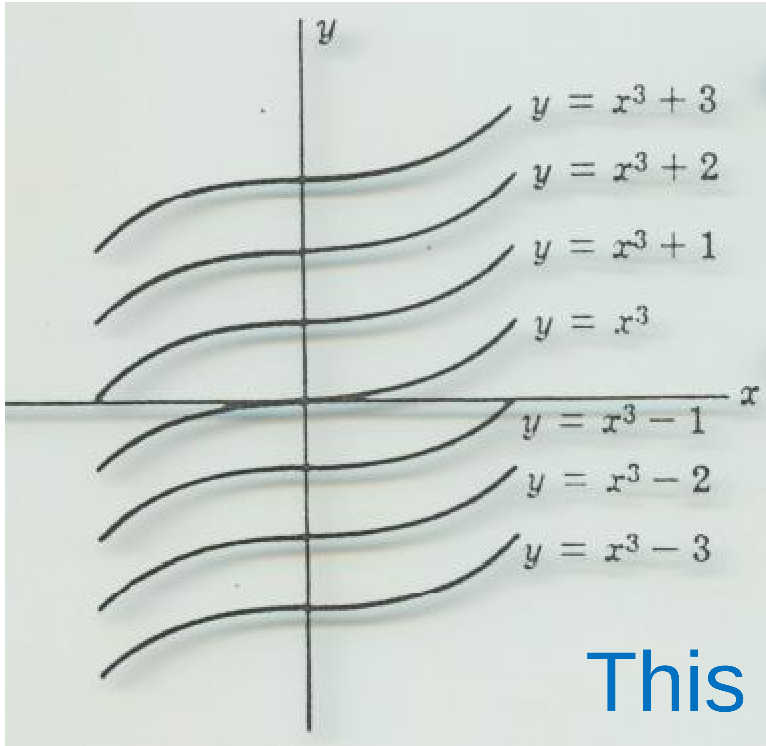
e.g. $\frac{dy}{dx} = 3x^2$ has general solution $y = x^3 + c$

$\frac{dy}{dx} = 3x^2$ has a particular solution $y = x^3 + 1$.



- General solution gives a family of curves.
- Particular solution is one of those curves.

This first order o.d.e. has one arbitrary constant, C

The **general solution** of an ***n**th* order o.d.e. has ***n*** arbitrary constants that can take any values.

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In an **initial value problem**, one solves an ***n**th* order o.d.e. to find the general solution and then applies ***n*** **boundary conditions** ("initial values/conditions") to find a **particular solution** that does not have any arbitrary constants.

Solving O.D.E.'s

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- $$\boxed{\frac{dy}{dx} = f(x)} \longrightarrow y = \int f(x)\, dx$$

by "**direct integration**"

Ex $$\boxed{\frac{d^2x}{dt^2} = -\sin\omega t}\,, \quad \omega = \text{constant}.$$

$$\Rightarrow \quad x = \frac{1}{\omega^2}\sin\omega t + At + B$$

i.e. also works for higher-order o.d.e's. Here, <u>second</u> order needs <u>two</u> integrations ... giving <u>two</u> arbitrary constants (*A* and *B*).

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- $$\boxed{\frac{dy}{dx} = f(x)g(y)} \rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

by "**separation of variables**"

Ex

$$\boxed{\frac{dx}{dt} + \frac{x}{\tau} = 0} \Rightarrow \frac{dx}{x} = -\frac{dt}{\tau}$$

$$\Rightarrow \int_{x(0)}^{x(t)} \frac{dx}{x} = -\frac{1}{\tau}\int_{t=0}^{t} dt \Rightarrow \boxed{x(t) = x(0)\,e^{-t/\tau}}$$

, after some manipulation

arbitrary constant (initial condition)

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

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First order linear o.d.e. – use the **integrating factor method**

Multiply the equation by integrating factor $\boxed{IF = e^{\int P(x)\,dx}}$

to give $\boxed{\frac{d}{dx}(IF\ y) = IF\ Q(x)}$.

Then integrate both sides with respect to x,

giving $\boxed{IF\ y = \int IF\ Q(x)\,dx}$.

Finally, divide by IF to get y.

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

$$\boxed{IF = e^{\int P(x)\,dx}}$$

$$\boxed{\frac{d}{dx}(IF\ y) = IF\ Q(x).}$$

$$\boxed{IF\ y = \int IF\ Q(x)\,dx.}$$

divide by IF

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Ex $\boxed{x\dfrac{dy}{dx} - y = x^2}$ subject to $\boxed{y(1) = 3}$

Ans $\dfrac{dy}{dx} - \left(\dfrac{1}{x}\right)y = x$ i.e. $P(x) = -\dfrac{1}{x}$, $Q(x) = x$

$$IF = e^{\int P(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x}$$

$$= e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}.$$

Multiply equation: $\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x^2}y = 1$ i.e. $\dfrac{d}{dx}\left[\dfrac{1}{x}y\right] = 1$

Integrate: $\dfrac{1}{x}y = x + C$ i.e. $\boxed{y = x^2 + Cx}$. (general solution)

$\boxed{x\dfrac{dy}{dx} - y = x^2}$ subject to $\boxed{y(1) = 3}$

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$\boxed{y = x^2 + Cx \quad \text{(general solution)}}$

Particular solution with $y(1) = 3$

i.e. $y = 3$ when $x = 1$

$\therefore \quad 3 = 1^2 + C.1$

i.e. $3 = 1 + C$ i.e. $C = 2$.

$\therefore$ Particular solution is $\boxed{y = x^2 + 2x}$.

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0}$$

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Second order linear o.d.e. with **constant coefficients** a,b,c

It is called a **homogeneous equation** because the RHS = 0.

Setting $y = A\, e^{mx}$

gives $$\boxed{am^2+bm+c=0}$$

(the "auxiliary equation")

Then $$m=\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$$

gives three different cases ...

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$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0}$$

$$y = A\,e^{mx} \Rightarrow \boxed{am^2+bm+c=0}$$

gives

$$\Rightarrow m=\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$$

i) real different roots m_1 , m_2 and $\boxed{y=A\,e^{m_1x}+B\,e^{m_2x}}$,

OR

ii) real equal roots $m_1 = m_2$ and $\boxed{y=(A+Bx)e^{m_1x}}$,

OR

iii) complex roots

$m_{1,2}=p\pm i\,q$ and $\boxed{y=e^{px}(A\cos qx+B\sin qx)}$.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$am^2+bm+c=0$$

i) real different roots m_1 , m_2

$$\boxed{y=Ae^{m_1x}+Be^{m_2x}}$$

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Ex

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$$

Ans

$$m^2+5m+6=0$$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \left\{\begin{array}{l} a=1 \\ b=5 \\ c=6 \end{array}\right.$$

$\Rightarrow$ $m=-2$ or $m=-3$

$\Rightarrow$

$$y=Ae^{m_1x}+Be^{m_2x}$$

$$\text{i.e. } y=Ae^{-2x}+Be^{-3x}.$$

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

$$am^2+bm+c=0$$

ii) real equal roots $m_1 = m_2$

$$\boxed{y=(A+Bx)e^{m_1 x}}$$

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Ex

$$\boxed{\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0}$$

Ans

$$m^2-6m+9=0 \quad \Rightarrow \quad m=\frac{+6\pm\sqrt{36-36}}{2}$$

$$\text{i.e. } m=\frac{6\pm 0}{2}=3$$

real equal roots, $m_1=m_2=3$

$$y=(A+Bx)e^{m_1 x}$$

$$\therefore \quad \boxed{y=(A+Bx)e^{3x}}.$$

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0}$$

$$\boxed{am^2+bm+c=0}$$

iii) complex roots $m_{1,2}=p\pm iq$

$$\boxed{y=e^{px}(A\cos qx+B\sin qx)}$$

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Ex $\boxed{\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0}$

Ans $m^2+m+1=0$ ⇒ $m=\frac{-1\pm\sqrt{1-4}}{2}$

$=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

complex roots $m_{1,2}=p\pm iq$, where $p=-\frac{1}{2}$

$(\ i=\sqrt{-1}\ )$ $q=\frac{\sqrt{3}}{2}$

$\therefore\ y=e^{px}(A\cos qx+B\sin qx)$

i.e. $\boxed{y=e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)}$.

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)}$$

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Second order linear o.d.e. with **constant coefficients** a,b,c

It is **not homogeneous** since RHS is not zero.

Step One

Solve the corresponding homogeneous equation to get $\boxed{y=y_{CF}}$

This is called the "complementary function".

Step Two

The general solution of the full equation is $\boxed{y=y_{CF}+y_{PS}}$.

Where y_{PS} is a particular solution of the full equation.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

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general solution $y = y_{CF} + y_{PS}$

Find y_{PS} by substituting a trial form into the full equation and equate the coefficients of the functions involved

(e.g. e^{2x}, x^2, $\cos x$, etc.).

$f(x)$	**Trial form of y_{PS}**
k	C
kx …	$Cx+D$
kx^2 …	Cx^2+Dx+E
$k\cos ax$ OR $k\sin ax$	$C\cos ax + D\sin ax$
ke^{ax}	Ce^{ax}
Sum/product of the above	Sum/product of the above
(k, a are given constants)	(C, D, E are constants to be determined)

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

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general solution $y = y_{CF} + y_{PS}$

ALSO

Trial form of y_{PS}
C
$Cx+D$
$Cx^2 + Dx + E$
$C\cos ax + D\sin ax$
Ce^{ax}
Sum/product of the above
(C, D, E are constants to be determined)

If the suggested form of y_{PS} already appears in y_{CF} then multiply the trial form of y_{PS} by x until it does not

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \Rightarrow \boxed{y = y_{CF} + y_{PS}}$$

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$$\boxed{\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=e^{-3t}\ ; \quad x=\tfrac{1}{2} \text{ and } \frac{dx}{dt}=-2 \text{ at } t=0}$$

A.E. is $m^2+4m+3=0$ i.e. $(m+3)(m+1)=0$ i.e. $m=-3$ or $m=-1$

$$x_{CF} = Ae^{-3t}+Be^{-t}.$$

Try $x_{PS} = Ce^{-3t}$? $\Rightarrow$ No. Already in x_{CF}.

Try $x_{PS} = Cte^{-3t}$ $\Rightarrow$ $\frac{dx_{PS}}{dt} = Ce^{-3t}-3tCe^{-3t}$

$$= (1-3t)Ce^{-3t}.$$

$$\frac{d^2x_{PS}}{dt^2} = -3.Ce^{-3t}-3(1-3t)Ce^{-3t}$$

$$= (9t-6)Ce^{-3t}.$$

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-3t}.$$

general solution is $x = x_{CF} + x_{PS}$

$$x_{CF} = Ae^{-3t} + Be^{-t}$$

$$x_{PS} = Cte^{-3t}$$

$$\frac{dx_{PS}}{dt} = (1-3t)Ce^{-3t}$$

$$\frac{d^2x_{PS}}{dt^2} = (9t-6)Ce^{-3t}$$

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Substitute: $(9t-6)Ce^{-3t} + 4(1-3t)Ce^{-3t} + 3Cte^{-3t} = e^{-3t}$

Coeff. e^{-3t}: $\cancel{9tC} - 6C + 4C - \cancel{12tC} + \cancel{3Ct} = 1$

i.e. $C = -\frac{1}{2}$

$x_{ps} = -\frac{1}{2}te^{-3t}$,

general solution is $\boxed{x = x_{CF} + x_{PS} = Ae^{-3t} + Be^{-t} - \frac{1}{2}te^{-3t}}$.

Boundary conditions: $\boxed{x=\frac{1}{2} \text{ and } \frac{dx}{dt}=-2 \text{ at } t=0}$

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general solution is $\boxed{x=x_{cf}+x_{ps}=Ae^{-3t}+Be^{-t}-\frac{1}{2}te^{-3t}}$

Boundary conditions $x=\frac{1}{2}$ when $t=0$: $\boxed{\frac{1}{2}=A+B \quad \text{(i)}}$

$$\frac{dx}{dt}=-3Ae^{-3t}-Be^{-t}+\frac{3}{2}te^{-3t}-\frac{1}{2}e^{-3t}$$

$\frac{dx}{dt}=-2$ when $t=0$: $-2=-3A-B-\frac{1}{2}$

i.e. $\boxed{-\frac{3}{2}=-3A-B \quad \text{(ii)}}$

Solve (i) and (ii) for A and B: (i)+(ii) gives $\frac{1}{2}-\frac{3}{2}=A-3A$ i.e. $A=\frac{1}{2}$.

Then, (i) gives $B=0$.

Particular solution is $\boxed{x=\frac{1}{2}e^{-3t}-\frac{1}{2}te^{-3t}=\frac{1}{2}(1-t)e^{-3t}}$.

Solution of higher order linear differential equations

(with constant coefficients)

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• HOMOGENEOUS EQUATIONS (RHS = 0)

i.e. $$\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = 0}$$

Set $y = e^{mx}$,

$\longrightarrow$ $a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$ characteristic / auxiliary equation

i.e. $a_0 (m - m_1)(m - m_2) \dots (m - m_n) = 0$

with roots $m_1, m_2, \dots, m_n$ $\rightarrow$ 3 cases ...

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = 0$$

$$a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

roots $m_1, m_2, \dots, m_n$

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3 cases (i) Roots all real and distinct:

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}}$$

(ii) Repeated roots (k times)

If m_1 has multiplicity k then its contribution to the solution is

$$\boxed{y = (c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{m_1 x}}$$

(iii) Complex roots always appear as conjugate pairs

Each pair of complex roots $p \pm iq$ contributes to the solution: $\boxed{y = e^{px}(A\cos qx + B\sin qx)}$

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Continuing with *n*th order, linear o.d.e's with constant coefficients ...

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- INHOMOGENEOUS EQUATIONS (RHS $\neq 0$)

General solution

$$\boxed{y = Y_{CF}(x) + Y_{PS}(x)}$$

where $Y_{CF}(x)$ = solution of the homogeneous equation

$Y_{PS}(x)$ = a particular solution of the full equation

To find $Y_{PS}(x)$

- Substitute a trial solution involving unknown constants $C, D, E, \ldots$
- Guess the trial solution from the form of the RHS (as before)

Ex Show that the general solution of the <u>inhomogeneous</u> system $\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = R(x)}$

is $y = Y_c(x) + Y_p(x)$ where $Y_c(x)$ = complementary solution
$Y_p(x)$ = particular solution

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Ans Write system as $\boxed{\phi(D)y = R(x)}$

where $D = \frac{d}{dx}$ and $\phi(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n$.

General complementary solution $Y_c(x)$ satisfies $\phi(D)y = 0$

and has the required n arbitrary constants. i.e $\boxed{\phi(D)Y_c = 0}$ ①

$Y_p(x)$ is a particular solution of $\phi(D)y = R(x)$

ie $\boxed{\phi(D)Y_p = R(x)}$ ②

Is $\boxed{y = Y_c(x) + Y_p(x)}$ the general solution of: $\boxed{\phi(D)y = R(x)}$?

We have ... $\boxed{\phi(D)Y_c = 0}$ ①

$\boxed{\phi(D)Y_p = R(x)}$ ②

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Add equations ① and ② to get

$$\phi(D)Y_c(x) + \phi(D)Y_p(x) = R(x)$$

i.e. $$\phi(D)\left[Y_c(x) + Y_p(x)\right] = R(x)$$

(since $\phi(D)$ is a linear differential operator)

$\therefore$ $y = Y_c(x) + Y_p(x)$ is a solution of $\phi(D)y = R(x)$

with n arbitrary constants i.e. the general solution.

PARTIAL DIFFERENTIAL EQUATIONS

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- Some important p.d.e.'s
- Role of arbitrary functions
- Connections with solution of o.d.e's
  - direct integration
  - homogeneous systems
  - inhomogeneous systems
- Separation of variables

very important p.d.e. technique

  - finding a solution
  - superposition to get the required solution (Fourier analysis)

Some important partial differential equations

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Laplace's equation: $\boxed{\nabla^2 u = 0}$

u represents a potential in absence of sources/sinks

e.g. gravitational potential (where there is no matter)
electrostatic potential (where there are no charges)
temperature (where no sources of heat) : steady-state
velocity potential of incompressible fluid (when there are no vortices/sources/sinks)

Poisson's equation $\boxed{\nabla^2 u = f(x,y,z)}$

as above but $f(x,y,z)$ is the source density

e.g. electric charge density

Diffusion or heat flow equation

$$\boxed{\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}}$$

no sources but diffusion in time

e.g. non-steady-state temperature evolution

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there is no p311 !

Wave equation

$$\boxed{\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}}$$

v = wave speed

u = displacement of vibrating string
or amplitude of wave in gas, liquid, etc.
or electric/magnetic field

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Role of arbitrary functions

A partial differential equation involves two or more independent variables and partial derivatives with respect to these variables.

As with o.d.e.'s, the order of the equation is the order of the highest derivative present.

e.g. $\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = 2x - y}$: second order p.d.e. independent variables x, y

The arbitrary constants of general solutions of o.d.e.'s become arbitrary functions in the general solution of p.d.e.'s. Particular solutions then have a particular choice of arbitrary function.

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Why arbitrary functions?

Think about what a partial differential means.

For the equation $\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = 2x - y}$,

there is the general solution $u = x^2 y - \frac{1}{2} x y^2 + \underbrace{F(x) + G(y)}_{\text{arbitrary functions.}}$

$$\frac{\partial u}{\partial y} = x^2 - xy + G'(y)$$

i.e. the whole function of x, $F(x)$, is treated as a constant in the operation $\frac{\partial}{\partial y}$. So if we integrated $\frac{\partial u}{\partial y}$ with respect to y, we would generally have to introduce a function of x (rather than just an integration constant).

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = 2x - y}$$

general solution $u = x^2 y - \frac{1}{2}xy^2 + F(x) + G(y)$

$$\Rightarrow \frac{\partial u}{\partial y} = x^2 - xy + G'(y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2 - xy + G'(y)\right) = 2x - y, \text{ as required.}$$

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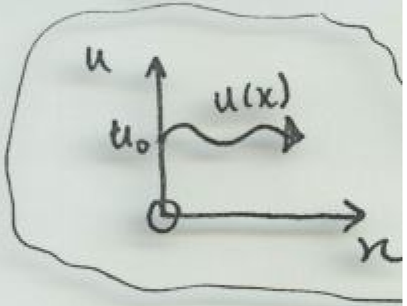
Another difference with odes is that initial-value problems

e.g. $\frac{du}{dx} = f(x,y)$ with $\underbrace{u(x=0) = u_0}_{\text{initial value}}$



tend to become boundary-value problems:

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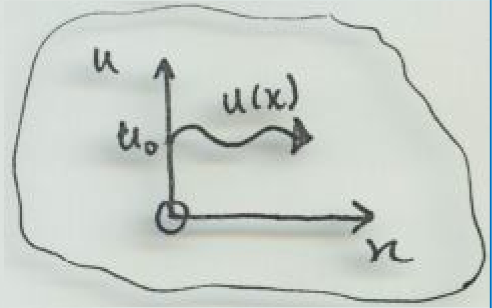
ode's

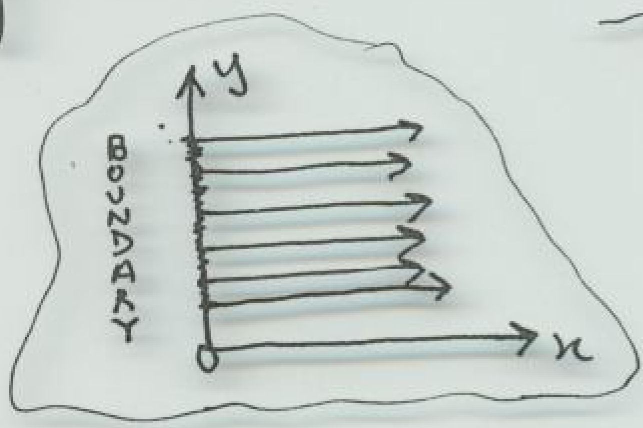
e.g. $\boxed{\dfrac{du}{dx} = f(x,y)}$

with $\underbrace{u(x=0) = u_0}_{\text{initial value}}$



p.d.e's

e.g. $\dfrac{\partial^2 u}{\partial x \partial y} = f(x,y)$ with $\underbrace{u(x=0, y) = g(y)}_{\text{boundary}}$



i.e. because we have more than one independent variable, boundary conditions are not specified at a point.

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We will deal here with linear partial differential equations that have constant coefficients.

e.g. $$\boxed{a_0 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial x \partial y} + a_2 \frac{\partial^2 u}{\partial y^2} + a_3 \frac{\partial u}{\partial x} + a_4 \frac{\partial u}{\partial y} + a_5 u = f(x,y)}$$

is second order, linear in u and $a_1, a_2, \ldots, a_5$ are constants

If $f(x,y) = 0$ then the equation is homogeneous.

Solution by direct integration

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Ex Starting with $\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = 2xy}$ derive the general solution.

Soln Consider the left-hand side as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ and integrate

with respect to x ... i.e. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 2xy$

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gives $\dfrac{\partial u}{\partial y} = x^2 y + F(y)$

Then, integrate with respect to y ... $\Rightarrow$ $u = \dfrac{x^2 y^2}{2} + \int F(y)\,dy + g(x)$

i.e. $\boxed{u = \dfrac{x^2 y^2}{2} + H(y) + g(x)}$

, where $H(y) = \int F(y)\,dy$

NB
General solution of pde
of **order 2** has **2 arbitrary functions**.

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Homogeneous systems

Recall that for ode's one finds the solution of a homogeneous equation by setting $\boxed{y = e^{mx}}$ and then seeking the roots of the resulting characteristic equation, where x is the independent variable.

Now we may have two independent variables, x and t

for example:
$$\boxed{\frac{\partial y}{\partial x} + \frac{1}{c}\frac{\partial y}{\partial t} = 0}$$

Set $\boxed{y = e^{ax+bt}}$ i.e. $a.e^{ax+bt} + \frac{1}{c}.b.e^{ax+bt} = 0$

i.e. $\left(a + \frac{b}{c}\right)e^{ax+bt} = 0$ i.e. $a + \frac{b}{c} = 0$ i.e. $b = -ac$

$y = e^{ax+bt}$ $\Rightarrow$ $b = -ac$

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A solution of $\boxed{\dfrac{\partial y}{\partial x} + \dfrac{1}{c}\dfrac{\partial y}{\partial t} = 0}$

is then $y = e^{ax+bt} = e^{ax-act} = e^{a(x-ct)}$,

for any a.

This is not the arbitrary function but it suggests an arbitrary function $\boxed{y = F(x-ct)}$.

Let's show that this is a solution...

$\boxed{y = F(x-ct)}$ solution of $\boxed{\dfrac{\partial y}{\partial x} + \dfrac{1}{c}\dfrac{\partial y}{\partial t} = 0}$??

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Let $u = x - ct$ i.e. $y = F(u)$.

$$\frac{\partial y}{\partial x} = \underbrace{\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}}_{\text{(CHAIN RULE)}} = \frac{\partial F}{\partial u} \quad ; \quad \frac{\partial y}{\partial t} = \underbrace{\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}}_{\text{(CHAIN RULE)}} = \frac{\partial F}{\partial u}\cdot(-c)$$

$$\therefore \quad \frac{\partial y}{\partial x} + \frac{1}{c}\frac{\partial y}{\partial t} = \frac{\partial F}{\partial u} + \frac{1}{c}\cdot(-c)\frac{\partial F}{\partial u} = 0.$$

i.e. this arbitrary function **is** a solution.

- This technique can allow one to quickly determine the general solution of homogeneous partial differential equations.

The wave equation in one space dimension

Another example ...

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i.e. $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

where the Laplacian, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \rightarrow \frac{\partial^2}{\partial x^2}$, i.e one space dimension only

i.e. $\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0}$: homogeneous equation

system: linear & homogeneous

"*superposition principle*"

Set $u = e^{ax+bt}$, $a^2 - \frac{1}{v^2}b^2 = 0$ i.e. $b = \pm av$

$\therefore$ $u = e^{ax \pm avt}$

$u = e^{a(x \pm vt)}$, for any a.

... *two solutions*.

General solution of the 1D wave equation is

$$\boxed{u = F(x+vt) + G(x-vt)},$$

where F and G are arbitrary functions.

Inhomogeneous systems (linear & constant coefficients)

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To solve an inhomogeneous ode for the general solution, we added the general solution of the homogeneous ode to a particular solution of the full equation. $\Rightarrow$ $\boxed{y = Y_{cf}(x) + Y_{ps}(x)}$

We can do the same for partial differential equations.

Ex $\boxed{\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = e^{2x+y}}$

Ans Set $u = e^{ax+by}$ in $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = 0$

The general solution of the homogeneous equation can be written as $\boxed{u = F(2x+y) + G(2x-y)}$.

$$\boxed{\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} = e^{2x+y}}$$

$$\boxed{Y_{cf}(x) = F(2x+y) + G(2x-y)}$$

$$\boxed{Y_{ps}(x) = \,?}$$

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Try $\boxed{u = Ce^{2x+y}}$ as a particular solution and determine C? No. We already have $F(2x+y)$ in the complementary solution. Try $\boxed{u = Cxe^{2x+y} \quad (\text{or } u = Cye^{2x+y})}$.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 2Ce^{2x+y} + 2Ce^{2x+y} + 4Cxe^{2x+y} \; ; \; -4\frac{\partial^2 u}{\partial y^2} = -4Cxe^{2x+y}$$

$$\therefore \; 4Ce^{2x+y} = e^{2x+y} \quad \text{and} \quad C = \frac{1}{4}.$$

$\therefore$ General solution is $\boxed{u = F(2x+y) + G(2x-y) + \frac{1}{4}xe^{2x+y}}$.

Separation of variables

(***the p.d.e. technique***!)

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Here we assume that the solution can be expressed as a product of unknown functions of each of the independent variables

$$\boxed{\text{e.g.} \quad u(x,y) = X(x)Y(y)}$$

How do we know that the solution is of this form?

Generally, the solution we seek is not of this form!

But we can combine separable solutions together to get the desired solution.

Let's start with some simple examples...

Ex Solve the boundary-value problem

$$\boxed{\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \quad u(0,y) = 8e^{-3y}}$$

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Ans Set $\boxed{u = X(x)Y(y)}$ and substitute to find...

$X_x Y = 4 X Y_y$, where the subscript denotes partial derivative.

$$\therefore \boxed{\frac{X_x}{4X} = \frac{Y_y}{Y}}$$

Left-hand side only depends on x but is true for all x.

Right-hand side only depends on y but is true for all y.

This implies that each side of the equation equals a constant since x and y are independent.

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$$\boxed{\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}}$$ with $$\boxed{u = X(x)Y(y)}$$ ...

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This implies that each side of the equation equals a constant since x and y are independent.

i.e. $$\boxed{\frac{X_x}{4X} = c = \frac{Y_y}{Y}}$$, $c =$ "separation constant"

We can now write this as two ordinary differential equations

i.e. $$\frac{dX}{dx} = c4X \quad \text{and} \quad \frac{dY}{dy} = cY$$

with solutions $X = Ae^{4cx}$ and $Y = Be^{cy}$

$$\boxed{u = XY = Ke^{c(4x+y)}}$$, $(K = AB)$

$$\boxed{\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}}$$ with $\boxed{u = X(x)Y(y)}$ $\Rightarrow$ $\boxed{u = XY = Ke^{c(4x+y)}}$

but boundary-value problem also has ... $\boxed{u(0,y) = 8e^{-3y}}$

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Now apply the boundary condition $u(0,y) = 8e^{-3y}$

i.e. $Ke^{c(4x+y)} \xrightarrow{x=0} Ke^{cy} = 8e^{-3y}$

i.e. $K=8$, $c=-3$.

Required solution is

$$\boxed{u = 8e^{-3(4x+y)}}.$$

Note This is a solution that is separable.

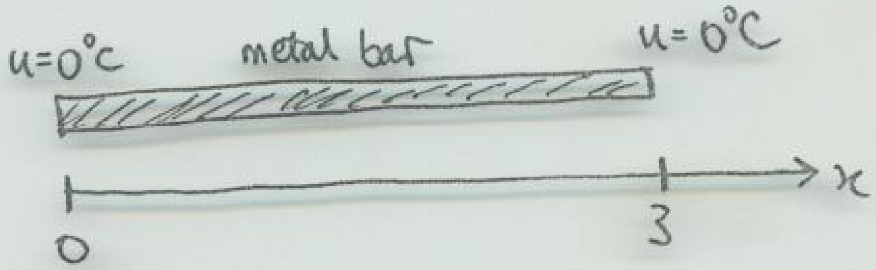
The following example results in a final solution that is not separable ...

Ex Solve the heat flow equation $\boxed{\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}}$

for $0 < x < 3$, $t > 0$, given that $\boxed{u(0,t) = u(3,t) = 0}$ end points

and $\boxed{u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x}$. $t = 0$

Ans If u = temperature then we could be describing the following ...



i.e. a bar of length 3 units whose temperature is kept at 0°C at each end.

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$$\boxed{\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}}$$ with $\boxed{u(0,t) = u(3,t) = 0}$ and

$$\boxed{u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x}$$

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Initially, at $t=0$, the distribution of temperature along the bar is given by $u(x,0)$.

We wish to know how the temperature evolves with time i.e. $u(x,t)$.

Set $\boxed{u = X(x)T(t)}$ in the pde : $XT_t = 2X_{xx}T$

i.e. $$\boxed{\frac{X_{xx}}{X} = \frac{T_t}{2T} = -\lambda^2}$$ (separation constant)

We use $-\lambda^2$ to avoid unphysical solutions that result if $+\lambda^2$ is taken.

$\boxed{\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2}}$ and $\boxed{u = X(x)\,T(t)}$ gave $\boxed{\dfrac{X_{xx}}{X} = \dfrac{T_t}{aT} = -\lambda^2}$

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This gives two ode's

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + a\lambda^2 T = 0$$

Simple harmonic oscillator

$$\boxed{X = A_1\cos\lambda x + B_1\sin\lambda x} \quad ; \quad \boxed{T = c_1 e^{-a\lambda^2 t}}$$

i.e. a solution is $u = XT = c_1 e^{-a\lambda^2 t}\left(A_1\cos\lambda x + B_1\sin\lambda x\right)$

or simply, $\boxed{u = e^{-a\lambda^2 t}\left(A\cos\lambda x + B\sin\lambda x\right)}$

$u = e^{-2\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$ subject to: $u(0,t) = u(3,t) = 0$

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and $u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$.

Apply boundary conditions at $x = 0, 3$

$x=0$, $\quad u = 0 = e^{-2\lambda^2 t}(A + 0)$ i.e. $\boxed{A = 0}$

then $\quad u = e^{-2\lambda^2 t} B\sin\lambda x$

$x=3$, $\quad u=0 = e^{-2\lambda^2 t} B\sin\lambda 3$ i.e. $3\lambda = m\pi$, $\quad m = 0, \pm1, \pm2, \ldots$

i.e. $\boxed{\lambda = \frac{m\pi}{3}}$

Solution is now $\boxed{u = e^{-\frac{2m^2\pi^2}{9}t}\left(B\sin\frac{m\pi}{3}x\right)}$

$$u = e^{-2\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$ subject to: $u(0,t) = u(3,t) = 0$

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and $u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$.

$$u = e^{-\frac{2m^2\pi^2}{9}t}\left(B\sin\frac{m\pi x}{3}\right)$$

End point conditions imply:

solution = discrete spectrum of sine waves.

Initial excitation is a superposition of three such sine waves ...

Superposition principle: sum of such solutions is also a solution

$$\text{e.g. } u = e^{-\frac{2m_1^2\pi^2 t}{9}} B_1 \sin\left(\frac{m_1\pi x}{3}\right) + e^{-\frac{2m_2^2\pi^2 t}{9}} B_2 \sin\left(\frac{m_2\pi x}{3}\right)$$

$$+ e^{-\frac{2m_3^2\pi^2 t}{9}} B_3 \sin\left(\frac{m_3\pi x}{3}\right)$$

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$$u = e^{-\frac{2m_1^2\pi^2 t}{9}} B_1 \sin\left(\frac{m_1\pi x}{3}\right) + e^{-\frac{2m_2^2\pi^2 t}{9}} B_2 \sin\left(\frac{m_2\pi x}{3}\right) + e^{-\frac{2m_3^2\pi^2 t}{9}} B_3 \sin\left(\frac{m_3\pi x}{3}\right)$$

Final boundary condition is

$$u(x,0) = 5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x)$$

where $u(x,0) = B_1 \sin\left(\frac{m_1\pi x}{3}\right) + B_2 \sin\left(\frac{m_2\pi x}{3}\right) + B_3 \sin\left(\frac{m_3\pi x}{3}\right)$

$t = 0$

i.e. $B_1 = 5,\ B_2 = -3,\ B_3 = 2$ and $m_1 = 12,\ m_2 = 24,\ m_3 = 30$

$\therefore$ Required solution is

$$\boxed{u(x,t) = 5e^{-32\pi^2 t}\sin(4\pi x) - 3e^{-128\pi^2 t}\sin(8\pi x) + 2e^{-200\pi^2 t}\sin(10\pi x)}\ .$$