Mathematical Methods and Applications

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HANDOUT 2

— VECTOR CALCULUS (continued)

- Recap : vector field
  line integral
  conservative fields

- cross product of vectors
  (including matrix determinants)

- triple products of vectors

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Let's recap ...

We considered a force that varies in space

i.e. $\underset{\sim}{F}(x,y,z)$ .... vector field

We can specify <u>how</u> this varies in space

$$\underset{\sim}{F} = F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k}$$

i.e. $\underset{\sim}{F} = (F_x, F_y, F_z)$

where its components F_x, F_y, F_z are functions of position.

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The example we took was (page 27)

$$\underset{\sim}{F} = (y^2, x^2, 0).$$

Work done by a constant force along a straight line is just

$$W = \underset{\sim}{F} \cdot \underset{\sim}{d}$$

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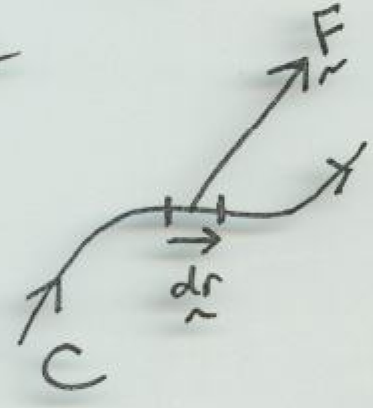
But when the force varies in space and we move along a curve C, we need to sum up the contributions to W along this curve



$$\delta W = \underset{\sim}{F} \cdot \underset{\sim}{dr}$$

$$\rightarrow \quad W = \int_C \underset{\sim}{F} \cdot \underset{\sim}{dr}$$

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Since $\underset{\sim}{dr} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$,

$$W = \int_C F_x\,dx + F_y\,dy + F_z\,dz$$

i.e. $$W = \int_{\substack{\text{along}\\ x}} F_x\,dx + \int_{\substack{\text{along}\\ y}} F_y\,dy + \int_{\substack{\text{along}\\ z}} F_z\,dz$$

we are now summing up individual contributions along the x, y and z axes.

Returning to the specific example

$$\underset{\sim}{F} = (y^2, x^2, 0) \,,$$

$F_x = y^2$, $F_y = x^2$, $F_z = 0$. This tells us how the field varies in space and

$$W = \int_{\text{along } x} F_x \, dx + \int_{\text{along } y} F_y \, dy \qquad (F_z = 0)$$

i.e. $$W = \int_{\text{along } x} y^2 \, dx + \int_{\text{along } y} x^2 \, dy$$

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$$W = \int_{\text{along } x} y^2\,dx + \int_{\text{along } y} x^2\,dy$$

where …

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The values of y^2 along x and x^2 along y are given by the curve along which we are integrating,

e.g. $y = x^2 \rightarrow y^2 = x^4$

$y = x^2 \rightarrow x^2 = y$

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i.e.
$$\boxed{W = \int_{\text{along } x} x^4\,dx + \int_{\text{along } y} y\,dy}$$

and we have reduced the line integral to two integrals that are easily worked out.

We showed that for certain fields the work done depends on the curve i.e. the path taken.

Path independence of the integral?

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Mathematically, this hinges on whether we can write

$$dW = \underset{\sim}{F} . d\underset{\sim}{r} .$$

Because then

$$\int_A^B \underset{\sim}{F} . d\underset{\sim}{r} = \int_A^B dW = \left[W \right]_A^B$$

$$= W_B - W_A$$

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i.e. $dW = \underset{\sim}{F} \cdot d\underset{\sim}{r}$ gives work done going from A to B as simply the difference in potential between A and B i.e. $W_B - W_A$.

e.g. if we only have a gravitational force then the work done moving an object from A to B depends only on the difference in gravitational potential between the two points. It is independent of the path we take.

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So what if we go from A to B and then back to A again?

IF

$$\underset{\sim}{F} \cdot d\underset{\sim}{r} = dW$$

THEN

$$\text{work done} = \int_A^B \underset{\sim}{F} \cdot d\underset{\sim}{r} + \int_B^A \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

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AND $W = \int_A^B dW + \int_B^A dW$

$$= \left[W\right]_A^B + \left[W\right]_B^A$$

$$= (W_B - W_A) + (W_A - W_B) = 0$$

"i.e. there has been no energy gained or lost in our travels".

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For example, we didn't encounter any dissipation or friction and energy has been CONSERVED

— hence the name CONSERVATIVE FIELD

Nonconservative fields are also called

DISSIPATIVE FIELDS.

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But what properties of the vector field allows us to write

$$dW = \underset{\sim}{F} \cdot d\underset{\sim}{r} \quad ?$$

i.e. to write

$$F_x\,dx + F_y\,dy + F_z\,dz = dW$$

(an exact differential).

For a function $W(x,y,z)$, the exact differential dW is defined as

$$dW = \left(\frac{\partial W}{\partial x}\right)dx + \left(\frac{\partial W}{\partial y}\right)dy + \left(\frac{\partial W}{\partial z}\right)dz .$$

From …

$$F_x\,dx + F_y\,dy + F_z\,dz = dW$$

$$dW = \left(\frac{\partial W}{\partial x}\right)dx + \left(\frac{\partial W}{\partial y}\right)dy + \left(\frac{\partial W}{\partial z}\right)dz$$

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So the property of the field that makes it conservative is that we have a function $W(x,y,z)$, the "scalar potential" such that

$$\left\{ F_x = \frac{\partial W}{\partial x}, \quad F_y = \frac{\partial W}{\partial y}, \quad F_z = \frac{\partial W}{\partial z} \right\}.$$

$$F_x = \frac{\partial W}{\partial x}, \quad F_y = \frac{\partial W}{\partial y}, \quad F_z = \frac{\partial W}{\partial z}$$

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But that's not much use if we don't know what $W(x,y,z)$ is!

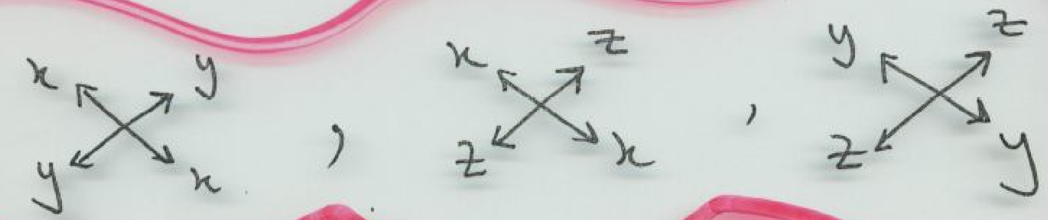
In terms of only $\underset{\sim}{F}(x,y,z)$ we can use the fact that

$$\boxed{\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)}, \text{ etc},$$

to re-express the conditions for the field to be to re-express the conditions for the field to be conservative as...

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \quad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}.$$

Notice that pattern here...



So, given a field $\underset{\sim}{F}(x, y, z)$, we can test whether it is conservative by seeing if the above relationships hold.

We then tested a general (radial) inverse square law field

$$\underset{\sim}{V}(\underset{\sim}{r}) = \frac{\eta\,\hat{\underset{\sim}{r}}}{r^2} = \frac{\eta\,\underset{\sim}{r}}{r^3}.$$

Note that if $\eta = -GMm$

then we have Newton's law of gravitation

(OR) if $\eta = \dfrac{Qq}{4\pi\varepsilon_0}$

then we have Coulomb's law,

→ these are just examples of a much more general mathematical and physical result.

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Note We are given $\underset{\sim}{V}(\underset{\sim}{r})$ and not $\underset{\sim}{V}(x,y,z)$ here

so how do we work out $\frac{\partial V_x}{\partial y}$, for example?

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$r^2 = x^2 + y^2 + z^2$

Differentiate both sides with respect to y (partially)

$$\to \quad 2r\frac{\partial r}{\partial y} = 0 + 2y + 0$$

i.e. $2r\frac{\partial r}{\partial y} = 2y$ and $\frac{\partial r}{\partial y} = \frac{y}{r}$.

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Then,

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\eta x}{r^3}\right)$$

$$= \eta x \frac{\partial}{\partial y}\left(\frac{1}{r^3}\right)$$

$$= \eta x . \left(\frac{-3}{r^4}\right)\left(\frac{\partial r}{\partial y}\right)$$

$$= \frac{-3\eta x}{r^4}\left(\frac{y}{r}\right) \quad \text{, from above,}$$

$$= \frac{-3\eta x y}{r^5} .$$

Chain Rule ...

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y}$$

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$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

REVISION TIME AGAIN …

End of recap, let's move on …

ancora più rapidamente! ☺

CROSS PRODUCT OF VECTORS

The cross product of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is written as

$$\underset{\sim}{a} \times \underset{\sim}{b}$$

where the MAGNITUDE of the cross product is given by

$$|\underset{\wedge}{a} \times \underset{\wedge}{b}| = |\underset{\wedge}{a}||\underset{\wedge}{b}| \sin\theta ,$$

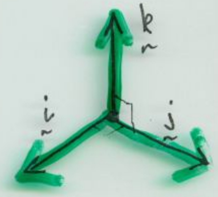
where θ is the angle between the vectors.

But the cross product is also known as the VECTOR PRODUCT, i.e. $\underset{\wedge}{a} \times \underset{\sim}{b}$ is a vector, and we also define a direction that is the result of the cross product.

Let's look at the unit basis vectors $\underset{\sim}{i}, \underset{\wedge}{j}, \underset{\wedge}{k}$ and consider what the definition of the magnitude implies.

By definition, the unit vectors have

$$|\underset{\sim}{i}| = |\underset{\sim}{j}| = |\underset{\wedge}{k}| = 1 .$$



So, $|\underset{\wedge}{i} \times \underset{\sim}{i}| = |\underset{\wedge}{i}||\underset{\sim}{i}| \sin\theta$

$= 1 . 1 . 0 = 0$

(since $\theta = 0^\circ$)

Similarly, $|\underset{\wedge}{j} \times \underset{\wedge}{j}| = |\underset{\wedge}{k} \times \underset{\wedge}{k}| = 0 .$

Whereas, $|\underset{\wedge}{i} \times \underset{\wedge}{j}| = |\underset{\wedge}{i}||\underset{\sim}{j}| \sin\theta$

$= 1 . 1 . 1 = 1$

(since $\theta = 90^\circ$)

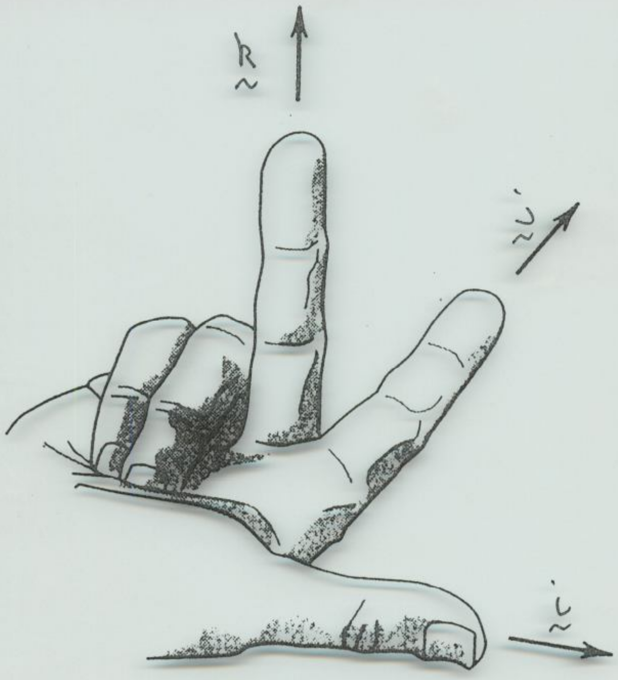
Similarly, $|\underset{\wedge}{j} \times \underset{\wedge}{k}| = |\underset{\wedge}{k} \times \underset{\wedge}{i}| = 1$

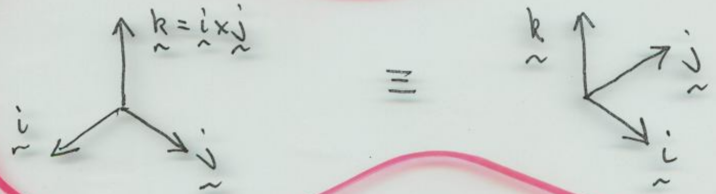
and $|\underset{\wedge}{k} \times \underset{\wedge}{j}| = |\underset{\sim}{j} \times \underset{\wedge}{k}| = 1 .$

So $|\underset{\sim}{i} \times \underset{\sim}{j}| = 1$, i.e. it's a unit vector, but what direction does this unit vector point in?

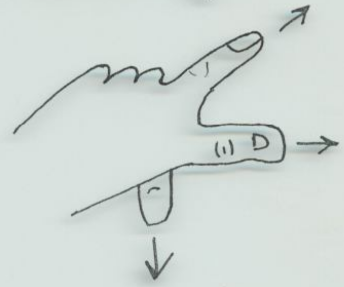
→ We define a RIGHT-HANDED SYSTEM such that $\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$

"THE RIGHT HAND RULE"





A left-handed system would have $\underset{\sim}{k}$ pointing in the opposite direction ...



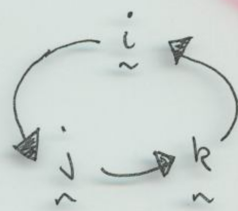
We then have that

$$\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$$

$$\underset{\sim}{j} \times \underset{\sim}{k} = \underset{\sim}{i}$$

$$\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j}$$

i.e. there is a cyclic pattern:



We already know that ...

$$\underset{\sim}{i} \times \underset{\sim}{i} = \underset{\sim}{0}$$

$$\underset{\sim}{j} \times \underset{\sim}{j} = \underset{\sim}{0}$$

$$\underset{\sim}{k} \times \underset{\sim}{k} = \underset{\sim}{0}$$

If we apply the right hand rule to $\mathbf{j} \times \mathbf{i}$

then we see that $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

i.e.

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

So when we go against the arrows in the cyclic pattern

we get a minus sign.

One key result here is that $\boxed{\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i})}$.

This is unlike the dot product

where we had $\boxed{\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}}$

Having defined the magnitude and direction of the cross product in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we can use these definitions to work out $\mathbf{a} \times \mathbf{b}$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}) \\
&= a_1 b_1 (\mathbf{i} \times \mathbf{i}) + a_1 b_2 (\mathbf{i} \times \mathbf{j}) + a_1 b_3 (\mathbf{i} \times \mathbf{k}) \\
&\quad + a_2 b_1 (\mathbf{j} \times \mathbf{i}) + a_2 b_2 (\mathbf{j} \times \mathbf{j}) + a_2 b_3 (\mathbf{j} \times \mathbf{k}) \\
&\quad + a_3 b_1 (\mathbf{k} \times \mathbf{i}) + a_3 b_2 (\mathbf{k} \times \mathbf{j}) + a_3 b_3 (\mathbf{k} \times \mathbf{k}) \\
&= 0 + a_1 b_1 \mathbf{k} + a_1 b_3 (-\mathbf{j}) \\
&\quad + a_2 b_1 (-\mathbf{k}) + 0 + a_2 b_3 \mathbf{i} \\
&\quad + a_3 b_1 (\mathbf{j}) + a_3 b_2 (-\mathbf{i}) + 0
\end{aligned}$$

Collecting together terms in $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ ...

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2 b_3 - a_3 b_2)\,\underset{\sim}{i} + (a_3 b_1 - a_1 b_3)\,\underset{\sim}{j} + (a_1 b_2 - a_2 b_1)\,\underset{\sim}{k}.$$

The "easiest" way to remember this is in terms of matrix determinants. A matrix is just a table.

A 2x2 matrix may be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The determinant of this 2x2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{(a number and } \underline{\text{not}} \text{ a matrix)}$$

The pattern is

" $\begin{matrix} a & b \\ c & d \end{matrix}$ TIMES MINUS $\begin{matrix} a & b \\ c & d \end{matrix}$ TIMES "

A 3x3 matrix may be

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

and the determinant is defined as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

THINK OF ...

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \rightarrow \begin{matrix} b_2 & b_3 \\ c_2 & c_3 \end{matrix} \qquad \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \rightarrow \begin{matrix} b_1 & b_3 \\ c_1 & c_3 \end{matrix} \qquad \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \rightarrow \begin{matrix} b_1 & b_2 \\ c_1 & c_2 \end{matrix}$$

* DON'T FORGET THAT MINUS SIGN IN FRONT OF a_2 ! *

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 \left(b_2c_3 - b_3c_2\right) - a_2 \left(b_1c_3 - b_3c_1\right) + a_3 \left(b_1c_2 - b_2c_1\right),$$

by using the definition of the determinant of a 2x2 matrix.

A "nice" compact form for $\underset{\sim}{a} \times \underset{\sim}{b}$ is

$$\boxed{\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

where $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$.

i.e. $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

--- thinking of ---

$$\begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \qquad \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \qquad \begin{matrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$$

then
$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1b_3 - a_3b_1$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

and
$$\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{i}\left(a_2b_3 - a_3b_2\right) - \underset{\sim}{j}\left(a_1b_3 - a_3b_1\right) + \underset{\sim}{k}\left(a_1b_2 - a_2b_1\right)$$

DON'T FORGET THAT MINUS SIGN
* IN FRONT OF $\underset{\sim}{j}$! *

Ex If $\underset{\sim}{a} = (1, 1, 2)$ and $\underset{\sim}{b} = (-1, 2, -1)$
what is $\underset{\sim}{a} \times \underset{\sim}{b}$?

Ans Here, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$
$b_1 = -1$, $b_2 = 2$, $b_3 = -1$

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i.e. $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 1 & 2 \\ -1 & 2 & -1 \end{vmatrix}$

$$= \underset{\sim}{i}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= \underset{\sim}{i}\left[1.(-1) - 2.2\right] - \underset{\sim}{j}\left[1.(-1) - 2.(-1)\right] + \underset{\sim}{k}\left[1.2 - 1.(-1)\right]$$

$$= \underset{\sim}{i}\left[-1-4\right] - \underset{\sim}{j}\left[-1+2\right] + \underset{\sim}{k}\left[2+1\right]$$

$$= -5\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$

Ex If $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 6\underset{\sim}{j} - 3\underset{\sim}{k}$

what is $\underset{\sim}{a} \times \underset{\sim}{b}$?

Ans $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, where $\underset{\sim}{a} = (4, 3, -1)$,

$\underset{\sim}{b} = (2, -6, -3)$

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i.e. $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix}$

$$= \underset{\sim}{i}\begin{vmatrix} 3 & -1 \\ -6 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix}$$

$$= \underset{\sim}{i}\left[3.(-3) - (-1)(-6)\right] - \underset{\sim}{j}\left[4.(-3) - (-1).2\right]$$

$$+ \underset{\sim}{k}\left[4(-6) - 3.2\right]$$

$$= \underset{\sim}{i}\left[-9-6\right] - \underset{\sim}{j}\left[-12+2\right] + \underset{\sim}{k}\left[-24-6\right]$$

$$= -15\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\sim}{k}.$$

In each of the above examples, what is the direction of $\underset{\sim}{a} \times \underset{\sim}{b}$ and what is the direction of $\underset{\sim}{b} \times \underset{\sim}{a}$?

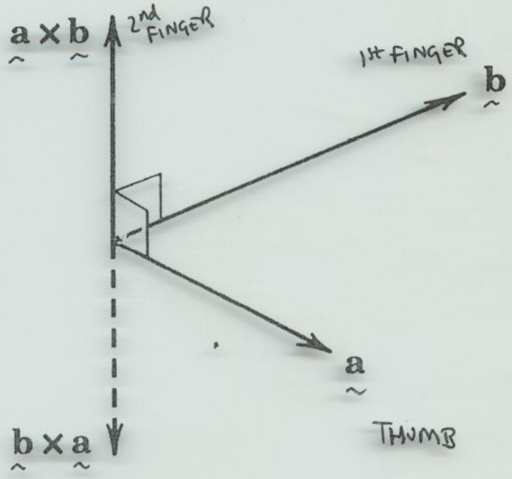
Recall that we defined $\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$, so...

So, $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

In other words,

$\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to the plane that contains $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

RIGHT HAND RULE :



Whereas, $\underset{\sim}{b} \times \underset{\sim}{a} = -(\underset{\sim}{a} \times \underset{\sim}{b})$

i.e. $\underset{\sim}{b} \times \underset{\sim}{a}$ points in precisely the OPPOSITE DIRECTION to $\underset{\sim}{a} \times \underset{\sim}{b}$.

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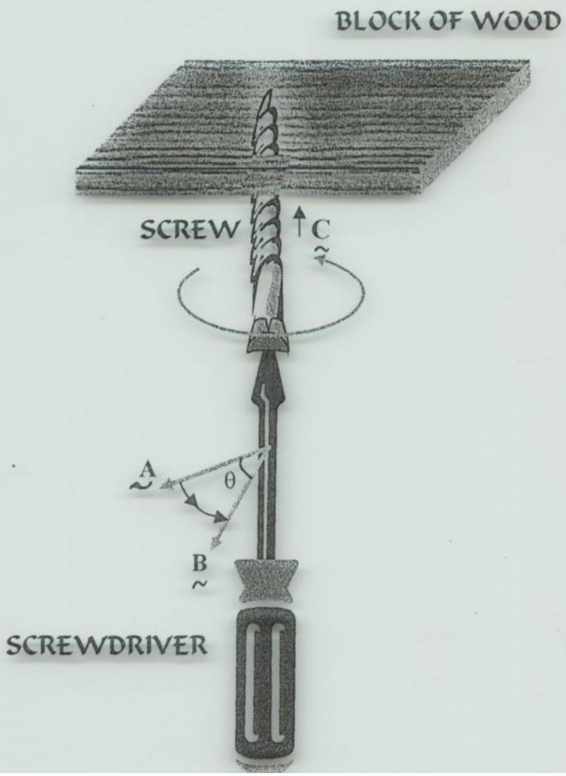
An alternative way of visualising the direction of $\underset{\sim}{A} \times \underset{\sim}{B}$ ...

THE VECTOR PRODUCT



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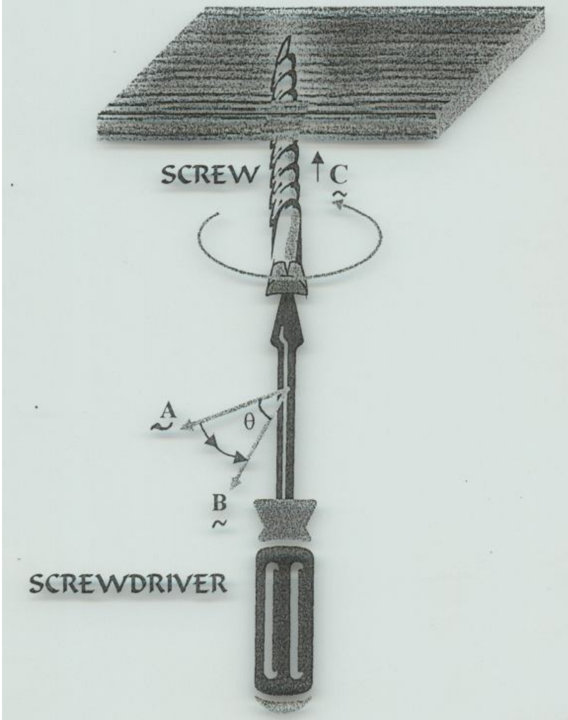

$\underset{\sim}{A}$ ROTATES TOWARDS $\underset{\sim}{B}$ IN A <u>CLOCKWISE</u> SENSE WHEN WE LOOK ALONG $\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B}$.

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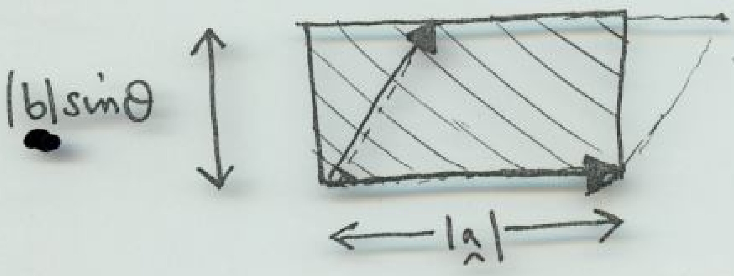
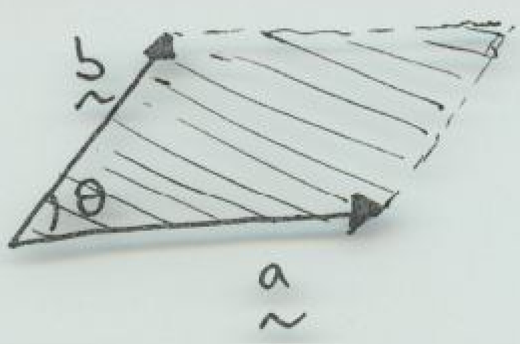
Is there a way to visualise the magnitude of $\underset{\sim}{a} \times \underset{\sim}{b}$?

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$$

i.e. in the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ we have



The area of the equivalent rectangle is $|\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$



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The area of the equivalent rectangle is $|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$

i.e. $|\underset{\sim}{a}\times\underset{\sim}{b}|$ is the area defined in the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(as in the top shaded area).

In fact, it is often convenient to think of

AN AREA <u>AS</u> A VECTOR

WITH DIRECTION PERPENDICULAR TO THE PLANE

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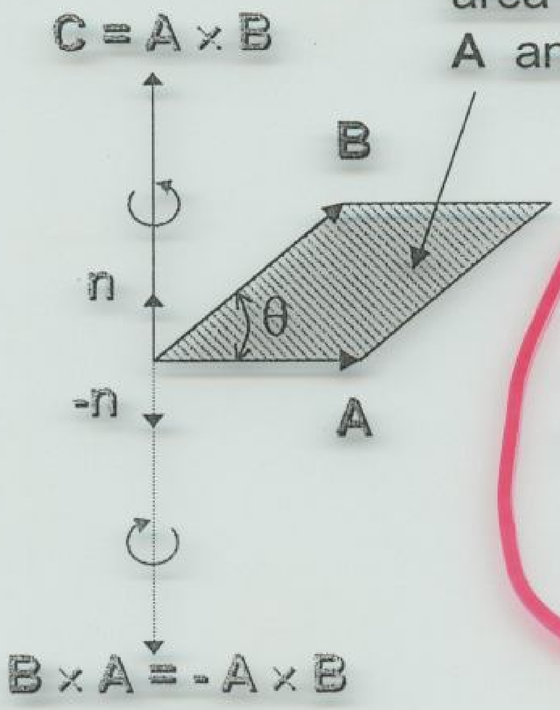
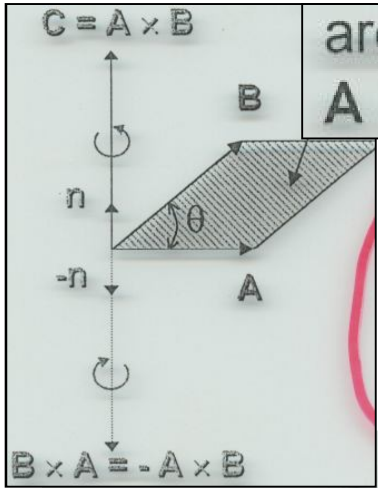
THE VECTOR PRODUCT

area of plane made by
A and **B** is AB sinθ



"VECTOR AREA"
WITH DIRECTION GIVEN
BY THE NORMAL VECTOR
$\underset{\sim}{n}$.

- vector product has *magnitude* equal to product of factors and sine of angle between them
- direction of product is perpendicular to plane containing **A** and **B**



area of plane made by **A** and **B** is AB sinθ

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A × **B** is directed in such a sense that an observer sees, in the direction of an arrow, a CLOCKWISE (RIGHT-HANDED) rotation about axis **A** × **B**. It is through the smaller angle and brings **A** into position occupied by **B**

- $\mathbf{C} = \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} = AB\sin\theta\,\mathbf{n}$

unit vector perpendicular to **A** and **B** and gives sense of direction

(in books vectors are often denoted with bold symbols)

TRIPLE PRODUCTS

There are 3 meaningful (i.e. consistent) ways that one can form the product of three vectors...

$$ - \; (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c} \quad , \quad \underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) \quad , \quad \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) $$

where

I. $$\underbrace{\underbrace{(\underset{\sim}{a} \cdot \underset{\sim}{b})}_{\text{scalar}}\underset{\sim}{c}}_{\text{parallel to } \underset{\sim}{c}} \neq \underbrace{\underset{\sim}{a}\underbrace{(\underset{\sim}{b} \cdot \underset{\sim}{c})}_{\text{scalar}}}_{\text{parallel to } \underset{\sim}{a}}$$

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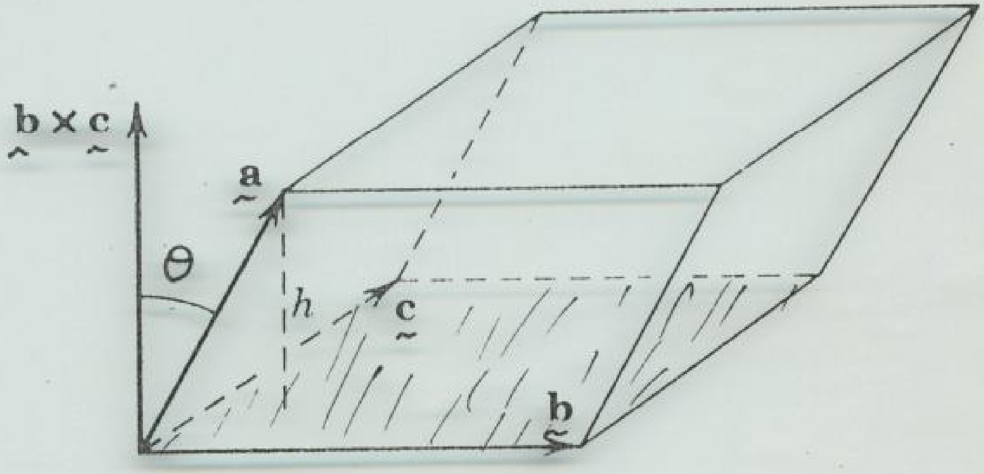
II.

$$\underbrace{\hat{a} \cdot \underbrace{(\hat{b} \times \hat{c})}_{\text{vector}}}_{\text{scalar}}$$

THE SCALAR TRIPLE PRODUCT

(i.e. it's a scalar!)

The scalar triple product has a geometrical interpretation as a volume



$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = |\underset{\sim}{a}||\underset{\sim}{b} \times \underset{\sim}{c}| \cos\theta$$

where $|\underset{\sim}{b} \times \underset{\sim}{c}|$ = shaded area

and $h = |\underset{\sim}{a}| \cos\theta$

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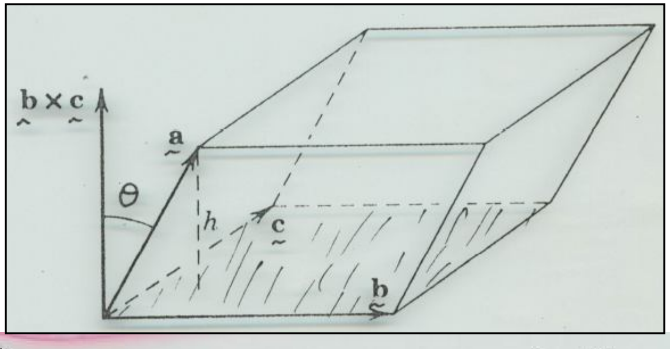

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i.e. $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \pm$ volume of the parallelipiped with edges $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$

[+ sign when θ acute, − sign when θ obtuse].

It can also be shown that

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\underset{\sim}{a} = (a_1, a_2, a_3)$
$\underset{\sim}{b} = (b_1, b_2, b_3)$
$\underset{\sim}{c} = (c_1, c_2, c_3)$

(see examples later)

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III. THE VECTOR TRIPLE PRODUCT

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) \neq (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}$$

(i.e. it's a vector!)

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In fact, there are two useful identities for the vector triple product ---

$$\text{(i)} \quad \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$$

$$\text{(ii)} \quad (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c} = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a}$$

--- which demonstrate that

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) \neq (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}$$

[think of the direction of RHS's of (i) and (ii)]

--- more details in examples sheet.