Mathematical Methods and Applications

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HANDOUT 3

VECTOR CALCULUS (continued)

- Evaluating triple products
- Applications of the cross-product
- Differentiation of vectors
  - time variation
  - space variation
  - partial differentiation

Mathematical Methods and Applications

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- Grad
  - definition
  - direction derivatives
  - unit normal vectors
- Flux and solid angle

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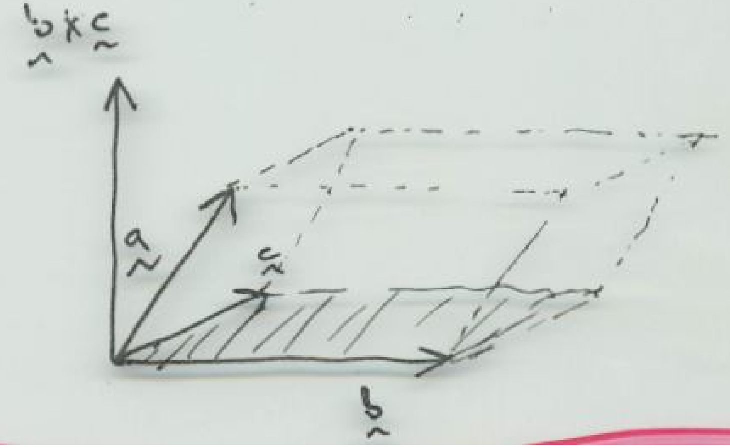
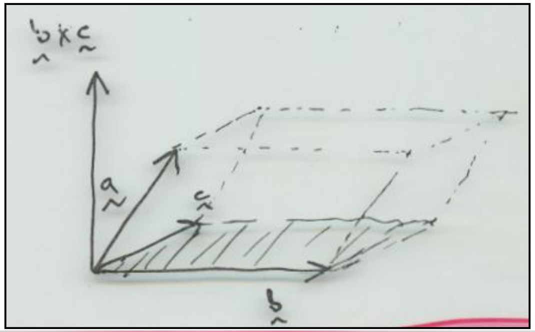
Examples involving the scalar and vector triple products

SCALAR TRIPLE PRODUCT

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) \equiv \pm \text{ volume of parallelipiped with edges } \underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$$

When does the volume of the parallelipiped equal zero?





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→ when $\underset{\sim}{a}$ lies in the plane of $\underset{\sim}{b}$ and $\underset{\sim}{c}$

i.e. $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are "COPLANAR".

In other words,

$\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b} \times \underset{\sim}{c}$

→ dot product is zero

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Ex Show that $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ are COPLANAR.

Ans See if $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = 0$

$$\underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad \text{where} \quad \begin{aligned} \underset{\sim}{b} &= (b_1, b_2, b_3) \\ \underset{\sim}{c} &= (c_1, c_2, c_3) \end{aligned}$$

See if $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = 0$

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$$\underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad \text{where} \quad \underset{\sim}{b} = (b_1, b_2, b_3), \quad \underset{\sim}{c} = (c_1, c_2, c_3)$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= \underset{\sim}{i} \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= \underset{\sim}{i}(1-2) - \underset{\sim}{j}(-2-6) + \underset{\sim}{k}(2+3)$$

$$= -\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k}$$

See if $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = 0$

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$\underset{\sim}{b} \times \underset{\sim}{c} = -\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k}$

(65)

Then, $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}) \cdot (-\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k})$

$$= -1 + 2.8 + (-3)5$$

$$= -1 + 16 - 15 = 0.$$

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VECTOR TRIPLE PRODUCT

Ex If

$$\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$$

$$\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$$

$$\underset{\sim}{c} = 3\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k} ,$$

what vector is given by $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$?

Ans Start by working out $\underset{\sim}{b} \times \underset{\sim}{c}$

i.e. $$\underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 3 & 1 & 3 \end{vmatrix} = \boxed{\underset{\sim}{i}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}$$

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i.e. $\underset{\sim}{b} \times \underset{\sim}{c} = \underset{\sim}{i}(6+1) - \underset{\sim}{j}(3+3) + \underset{\sim}{k}(1-6)$

$= 7\underset{\sim}{i} - 6\underset{\sim}{j} - 5\underset{\sim}{k}$.

Then, $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ 7 & -6 & -5 \end{vmatrix}$,

where $\underset{\sim}{a} = (a_1, a_2, a_3) = (2, -3, 1)$.

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i.e. $$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -3 & 1 \\ 7 & -6 & -5 \end{vmatrix}$$

$$= \underset{\sim}{i} \begin{vmatrix} -3 & 1 \\ -6 & -5 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 2 & 1 \\ 7 & -5 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 2 & -3 \\ 7 & -6 \end{vmatrix}$$

$$= \underset{\sim}{i}(15+6) - \underset{\sim}{j}(-10-7) + \underset{\sim}{k}(-12+21)$$

$$= 21\underset{\sim}{i} + 17\underset{\sim}{j} + 9\underset{\sim}{k} .$$

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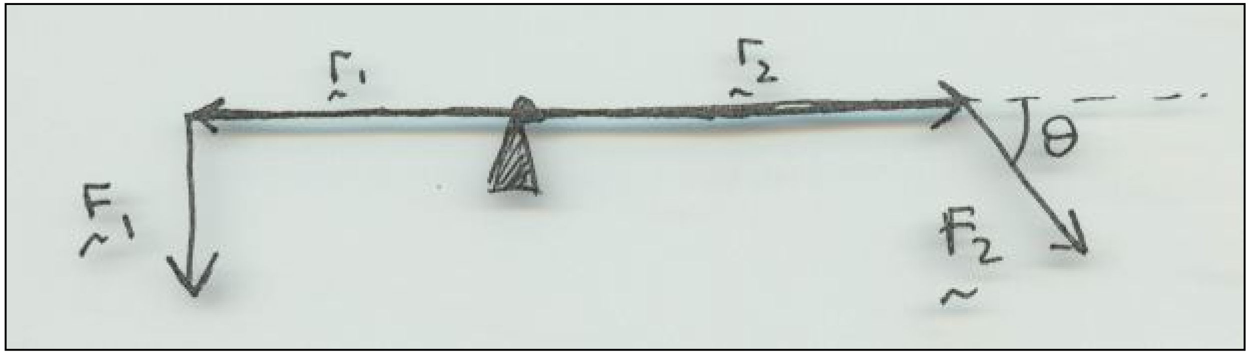
Applications of the cross product

Think about a set of scales:

At one end of the scales $\underset{\sim}{F}_1$ is applied and at the other end $\underset{\sim}{F}_2$ is applied.

$\underset{\sim}{F}_2$ is most effective at lowering the right hand side of the scales when it is perpendicular to $\underset{\sim}{r}_2$, i.e. when θ is 90°.



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P66b!
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i.e. when θ is 90°.

We get maximum MOMENT of $\underset{\sim}{F_2}$ (a scalar)
when $\theta = 90^\circ$.

We get maximum TORQUE of $\underset{\sim}{F_2}$ (a vector)
when $\theta = 90^\circ$.

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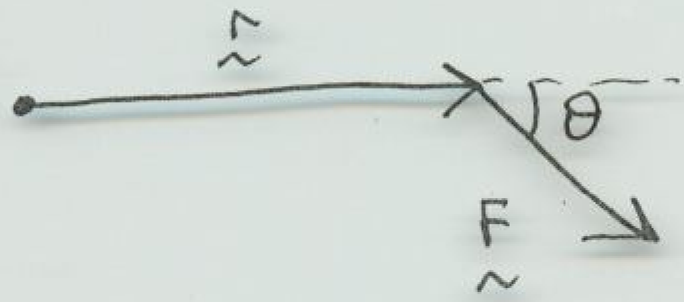
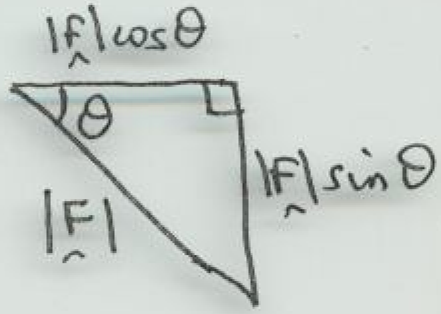
Looking at the right hand side of the scales...



resolve the components of $\underset{\sim}{F}$:



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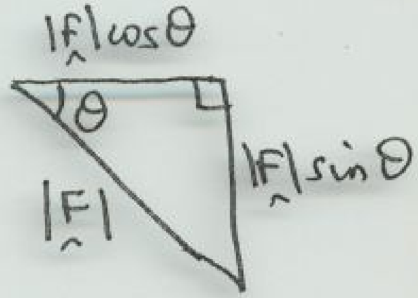
resolve the components of $\underset{\sim}{F}$:



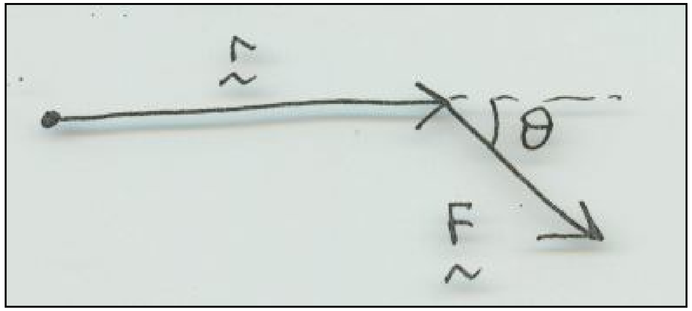
The **MOMENT** of $\underset{\sim}{F}$ is $|\underset{\sim}{F}|\sin\theta\,|\underset{\sim}{r}|$

$$= |\underset{\sim}{r}|\,|\underset{\sim}{F}|\sin\theta$$

$$= |\underset{\sim}{r} \times \underset{\sim}{F}| .$$

The **TORQUE** of $\underset{\sim}{F}$ is $\underset{\sim}{r} \times \underset{\sim}{F}$

i.e. $\underset{\sim}{\tau} = \underset{\sim}{r} \times \underset{\sim}{F}$.



$$\underset{\sim}{\tau} = \underset{\sim}{r} \times \underset{\sim}{F}$$

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bot

Looking along $\underset{\sim}{\tau}$, $\underset{\sim}{r}$ rotates clockwise to $\underset{\sim}{F}$
(definition of direction of vector product)

$\Rightarrow$ in the above $\underset{\sim}{\tau}$ is defined to be

INTO the page.

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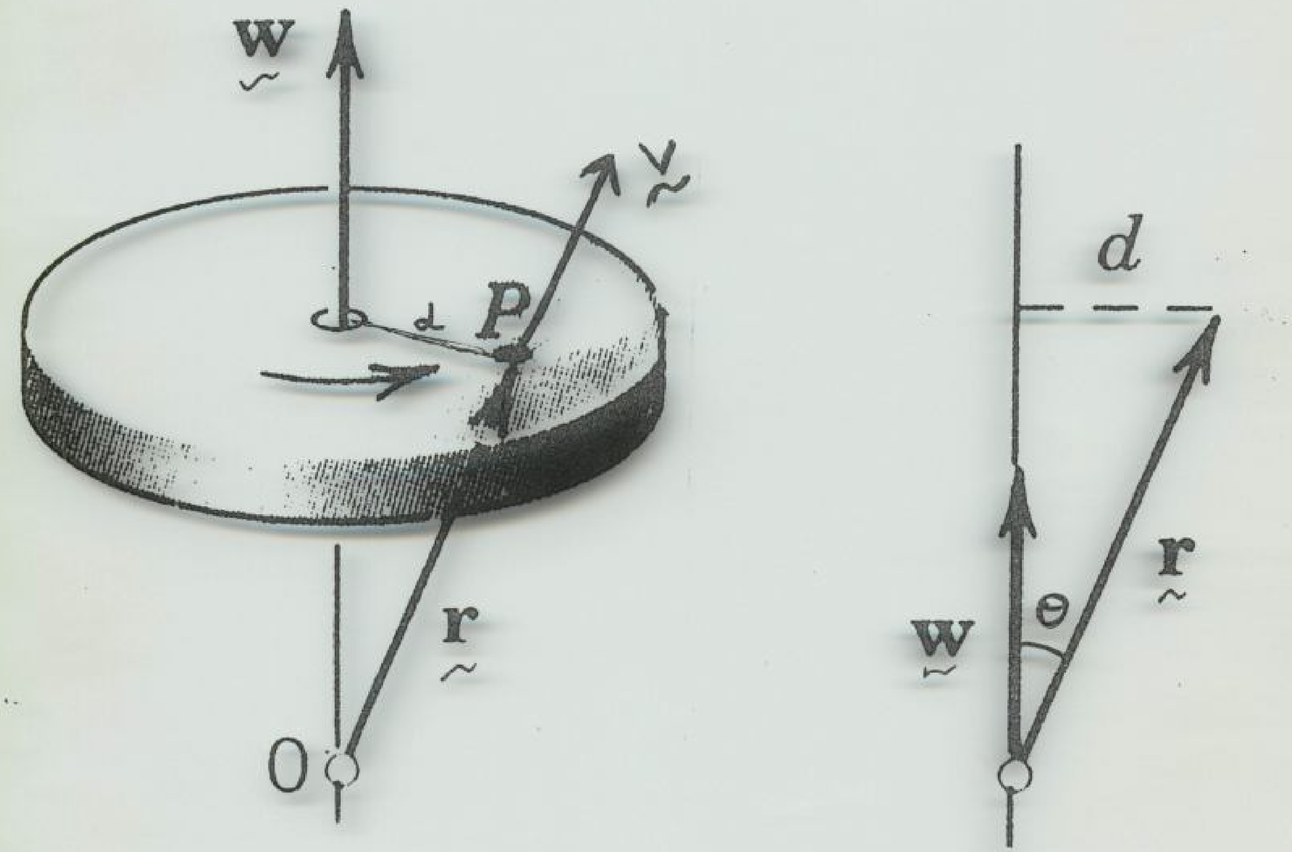
Rotation of a rigid body

In general, angular velocity is a vector $\underset{\sim}{w}$.

In what direction is it defined to point?



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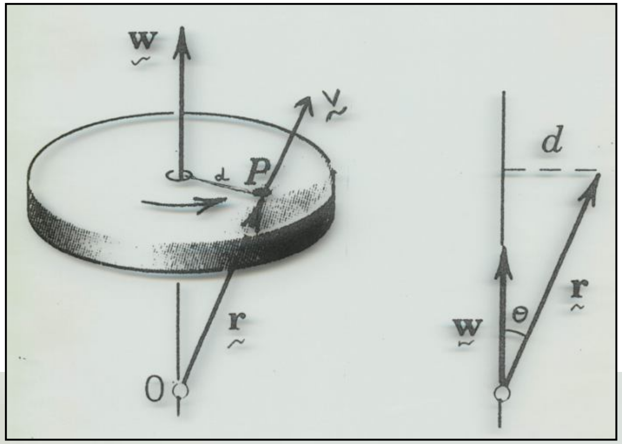

Linear velocity of point P, $|\underset{\sim}{v}| = \omega d$ (scalar)

$= |\underset{\sim}{w}||\underset{\sim}{r}|\sin\theta$

$= |\underset{\sim}{w} \times \underset{\sim}{r}|$

(Vector) velocity of point P, $\underset{\sim}{v} = \underset{\sim}{w} \times \underset{\sim}{r}$.

Looking along $\underset{\sim}{v}$, $\underset{\sim}{w}$ rotates clockwise towards $\underset{\sim}{r}$

i.e. $\underset{\sim}{v}$ is into the page in the right hand diagram.

Differentiation of vectors

Time-varying vectors:

a vector $\underset{\sim}{v}$ will be time-varying if its components vary with time, t

i.e. $\underset{\sim}{v}(t) = \left(v_1(t),\ v_2(t),\ v_3(t)\right)$

If during a time interval Δt

$$\underset{\sim}{v}(t) \longrightarrow \underset{\sim}{v}(t+\Delta t)$$

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then

$$\frac{d}{dt}\underset{\sim}{v} = \lim_{\Delta t \to 0}\left[\frac{\underset{\sim}{v}(t+\Delta t)-\underset{\sim}{v}(t)}{\Delta t}\right]$$

$$= \lim_{\Delta t \to 0}\left(\frac{v_1(t+\Delta t)-v_1(t)}{\Delta t},\ \frac{v_2(t+\Delta t)-v_2(t)}{\Delta t},\ \frac{v_3(t+\Delta t)-v_3(t)}{\Delta t}\right)$$

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$$\lim_{\Delta t \to 0} \left(\frac{v_1(t+\Delta t) - v_1(t)}{\Delta t}, \frac{v_2(t+\Delta t) - v_2(t)}{\Delta t}, \frac{v_3(t+\Delta t) - v_3(t)}{\Delta t} \right)$$

i.e. $\frac{d}{dt} \underset{\sim}{v} = \left(\frac{dv_1}{dt}, \frac{dv_2}{dt}, \frac{dv_3}{dt} \right)$

i.e. $\boxed{\frac{d}{dt} \underset{\sim}{v} = \frac{dv_1}{dt} \underset{\sim}{i} + \frac{dv_2}{dt} \underset{\sim}{j} + \frac{dv_3}{dt} \underset{\sim}{k}}$

i.e. we just differentiate each component separately.

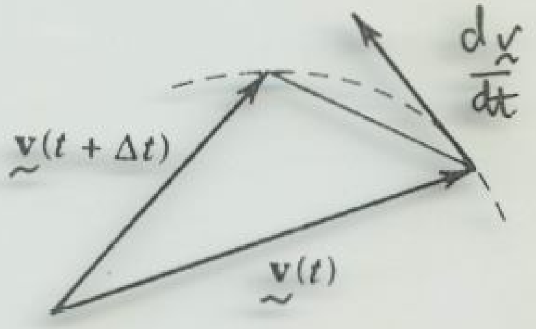
Direction of $\frac{d}{dt}\underset{\sim}{v}$?

when $\underset{\sim}{v}$ is a position vector

$$\frac{d}{dt}\underset{\sim}{v} = \lim_{\Delta t \to 0}\left[\frac{\underset{\sim}{v}(t+\Delta t) - \underset{\sim}{v}(t)}{\Delta t}\right]$$

Imagine a curve that is swept out by the tip of $\underset{\sim}{v}$ as it changes in time - - -

$\frac{d\underset{\sim}{v}}{dt}$ is TANGENTIAL TO THIS CURVE.



Its direction is given by the difference of vectors $\underset{\sim}{v}(t+\Delta t)$ and $\underset{\sim}{v}(t)$ as $\Delta t \to 0$.

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Ex If force $\underset{\sim}{F}(t) = \sin 2t\,\hat{i} + e^{3t}\,\hat{j} + (t^3 - 4t)\underset{\sim}{k}$

what is $\frac{d}{dt}\hat{F}$ when $t = 1$?

Ans $\underset{\sim}{F}(t) = \left(F_1(t), F_2(t), F_3(t) \right)$

where $F_1(t) = \sin 2t$, $F_2(t) = e^{3t}$,

$F_3(t) = t^3 - 4t$

$$\frac{d}{dt}\hat{F} = \left(\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt} \right)$$

$$= \left(2\cos 2t,\ 3e^{3t},\ 3t^2 - 4 \right).$$

$\left(2\cos 2t, \quad 3e^{3t}, \quad 3t^2-4 \right)$

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when $t=1$

$$\frac{d}{dt}\underset{\sim}{F} = \left(2\cos 2, \; 3e^3, \; -1 \right)$$

$$= 2\cos 2\,\underset{\sim}{i} + 3e^3\,\underset{\sim}{j} - \underset{\sim}{k}.$$

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Spatial derivative of $\underset{\sim}{v}(x)$

Similarly, $$\frac{d}{dx}\underset{\sim}{v} = \lim_{\Delta x \to 0}\left[\frac{\underset{\sim}{v}(x+\Delta x) - \underset{\sim}{v}(x)}{\Delta x}\right]$$

and $$\frac{d\underset{\sim}{v}}{dx} = \left(\frac{dv_1}{dx}, \frac{dv_2}{dx}, \frac{dv_3}{dx}\right)$$

i.e. $$\boxed{\frac{d\underset{\sim}{v}}{dx} = \frac{dv_1}{dx}\underset{\sim}{i} + \frac{dv_2}{dx}\underset{\sim}{j} + \frac{dv_3}{dx}\underset{\sim}{k}}$$

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In fact, if we let $u = t$ (time) or $u = x$ (space), then it straightforward to show that:

$$\boxed{\begin{aligned} \frac{d}{du}(\underset{\sim}{a} \cdot \underset{\sim}{b}) &= \underset{\sim}{a} \cdot \frac{d\underset{\sim}{b}}{du} + \frac{d\underset{\sim}{a}}{du} \cdot \underset{\sim}{b} \\ \frac{d}{du}(\underset{\sim}{a} \times \underset{\sim}{b}) &= \underset{\sim}{a} \times \frac{d\underset{\sim}{b}}{du} + \frac{d\underset{\sim}{a}}{du} \times \underset{\sim}{b} \end{aligned}}$$

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Ex If a vector has constant magnitude C then what is the time derivative of this vector?

Ans We have $|\underset{\sim}{v}(t)| = C$, a constant.

Then, $|\underset{\sim}{v}|^2 = C^2$ (another constant)

and $\frac{d}{dt}|\underset{\sim}{v}|^2 = 0$.

Now, $\frac{d}{dt}|\underset{\sim}{v}|^2 = \frac{d}{dt}(\underset{\sim}{v}\cdot\underset{\sim}{v})$

$$= \underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt} + \frac{d\underset{\sim}{v}}{dt}\cdot\underset{\sim}{v}$$

$$= 2\underset{\sim}{v}\cdot\frac{d\underset{\sim}{v}}{dt} = 0.$$

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So, if $|\underset{\sim}{v}(t)| = c$ then $\underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} = 0$.

Either $\frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{0}$ or $\underset{\sim}{v}$ is perpendicular to $\frac{d\underset{\sim}{v}}{dt}$

(dot product is then zero).

Note that when $\underset{\sim}{v} = \underset{\sim}{0}$ then $\frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{0}$ also.

$$|\boldsymbol{p}(t)| = C \quad \Rightarrow \quad \frac{d\boldsymbol{p}}{dt} = \boldsymbol{0} \quad \text{OR} \quad \boldsymbol{p} \text{ perpendicular to } \frac{d\boldsymbol{p}}{dt}$$

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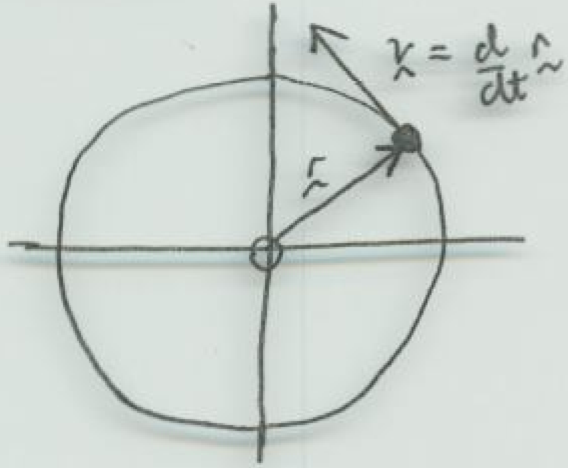
Ex An object moving in a circle



Since the displacement $\underset{\sim}{r}$ has constant magnitude,

the velocity $\underset{\sim}{v} = \frac{d}{dt}\underset{\sim}{r}$ is either zero $(\underset{\sim}{v} = \underset{\sim}{0})$

or it is perpendicular to $\underset{\sim}{r}$ $\left(\underset{\sim}{r} \cdot \frac{d\underset{\sim}{r}}{dt} = 0\right)$.

$$|\boldsymbol{p}(t)| = C \quad \Rightarrow \quad \frac{d\boldsymbol{p}}{dt} = \boldsymbol{0} \quad \text{OR} \quad \boldsymbol{p} \text{ perpendicular to } \frac{d\boldsymbol{p}}{dt}$$

H3
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bot

Ex An object moving in a circle with constant speed $|\underset{\sim}{v}|$

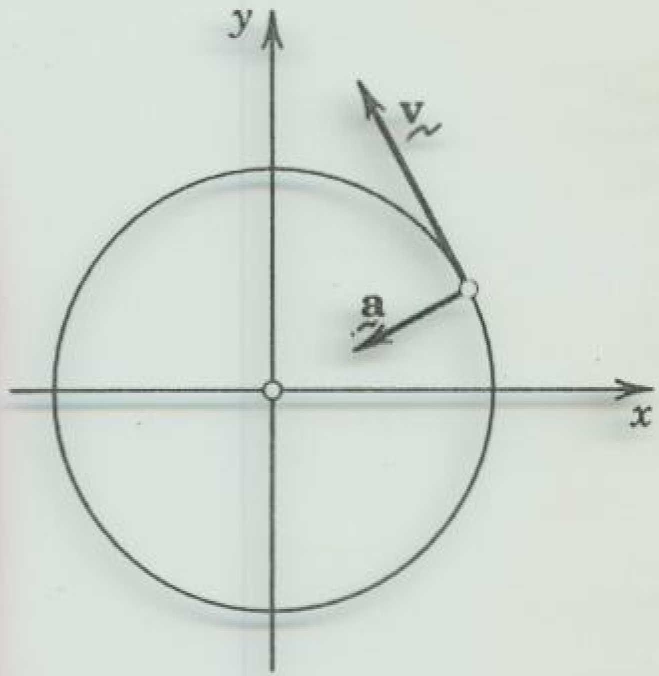
If $|\underset{\sim}{v}| = \text{constant}$ then

the acceleration $\underset{\sim}{a} = \frac{d}{dt}\underset{\sim}{v}$ is

either zero $(\underset{\sim}{a} = \underset{\sim}{0})$

or it is perpendicular

to $\underset{\sim}{v}$ $\left(\underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} = 0\right)$.



Centripetal acceleration

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Partial derivatives of vectors

Consider a 2D vector space $\underset{\sim}{V}(x,y)$

e.g. a vector $\underset{\sim}{V}$ that can assume different values across the x-y plane.

We now have two independent variables x and y.

If we seek the derivative of $\underset{\sim}{V}$ with respect to x only then we need to consider holding y constant, ie we need to form the partial derivative of $\underset{\sim}{V}$ with respect to x.

to x.

So if $\underset{\sim}{V} = (V_1, V_2)$

then $\underset{\sim}{V}(x,y) = \left(V_1(x,y) , V_2(x,y) \right)$

and $\dfrac{\partial}{\partial x} \underset{\sim}{V} = \left(\dfrac{\partial V_1}{\partial x} , \dfrac{\partial V_2}{\partial x} \right)$

i.e. $\boxed{\dfrac{\partial \underset{\sim}{V}}{\partial x} = \dfrac{\partial V_1}{\partial x} \underset{\sim}{i} + \dfrac{\partial V_2}{\partial x} \underset{\sim}{j}}$

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bot

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Ex If vector field $\underset{\sim}{V}(x,y,z)$ is given by

$$\underset{\sim}{V}(x,y,z) = a\cos x\, \underset{\sim}{i} + a\sin x\, \underset{\sim}{j} + y\, \underset{\sim}{k} \quad (a = \text{constant})$$

then what are $\dfrac{\partial \underset{\sim}{V}}{\partial x}$ and $\dfrac{\partial \underset{\sim}{V}}{\partial y}$?

Ans $\underset{\sim}{V} = V_1 \underset{\sim}{i} + V_2 \underset{\sim}{j} + V_3 \underset{\sim}{k}$

where $V_1 = a\cos x$, $V_2 = a\sin x$, $V_3 = y$

$V_1 = a\cos x \quad , V_2 = a\sin x \quad , V_3 = y$

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$$\frac{\partial \underset{\sim}{V}}{\partial x} = \frac{\partial V_1}{\partial x}\underset{\sim}{i} + \frac{\partial V_2}{\partial x}\underset{\sim}{j} + \frac{\partial V_3}{\partial x}\underset{\sim}{k}$$

$$= -a\sin x\,\underset{\sim}{i} + a\cos x\,\underset{\sim}{j} + 0\,\underset{\sim}{k}$$

$$\frac{\partial \underset{\sim}{V}}{\partial y} = \frac{\partial V_1}{\partial y}\underset{\sim}{i} + \frac{\partial V_2}{\partial y}\underset{\sim}{j} + \frac{\partial V_3}{\partial y}\underset{\sim}{k}$$

$$= 0\,\underset{\sim}{i} + 0\,\underset{\sim}{j} + 1.\underset{\sim}{k}$$

$$= \underset{\sim}{k}$$

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The properties of partial derivatives of vectors are similar to those of ordinary derivatives and it is also quite straightforward to show that

$$\boxed{\begin{aligned} \frac{\partial}{\partial x}(\underset{\sim}{a} \cdot \underset{\sim}{b}) &= \underset{\sim}{a} \cdot \frac{\partial \underset{\sim}{b}}{\partial x} + \frac{\partial \underset{\sim}{a}}{\partial x} \cdot \underset{\sim}{b} \\ \frac{\partial}{\partial x}(\underset{\sim}{a} \times \underset{\sim}{b}) &= \underset{\sim}{a} \times \frac{\partial \underset{\sim}{b}}{\partial x} + \frac{\partial \underset{\sim}{a}}{\partial x} \times \underset{\sim}{b} \end{aligned}}$$

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bot

GRAD (THE VECTOR GRADIENT OF A SCALAR FUNCTION)

If $\phi(x,y,z)$ defines a scalar field, i.e. ϕ is a scalar function of x, y, z, then the GRADIENT of ϕ is defined as the vector

$$\mathrm{grad}\,\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$$

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grad itself is an example of a

VECTOR DIFFERENTIAL OPERATOR,

$$\text{grad}\,\phi = \left\{ \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} \right\} \phi$$

and is denote by the symbol $\underset{\sim}{\nabla}$

(pronounced 'del' or 'nabla'),

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i.e. $\text{grad}\,\phi = \underset{\sim}{\nabla}\phi$

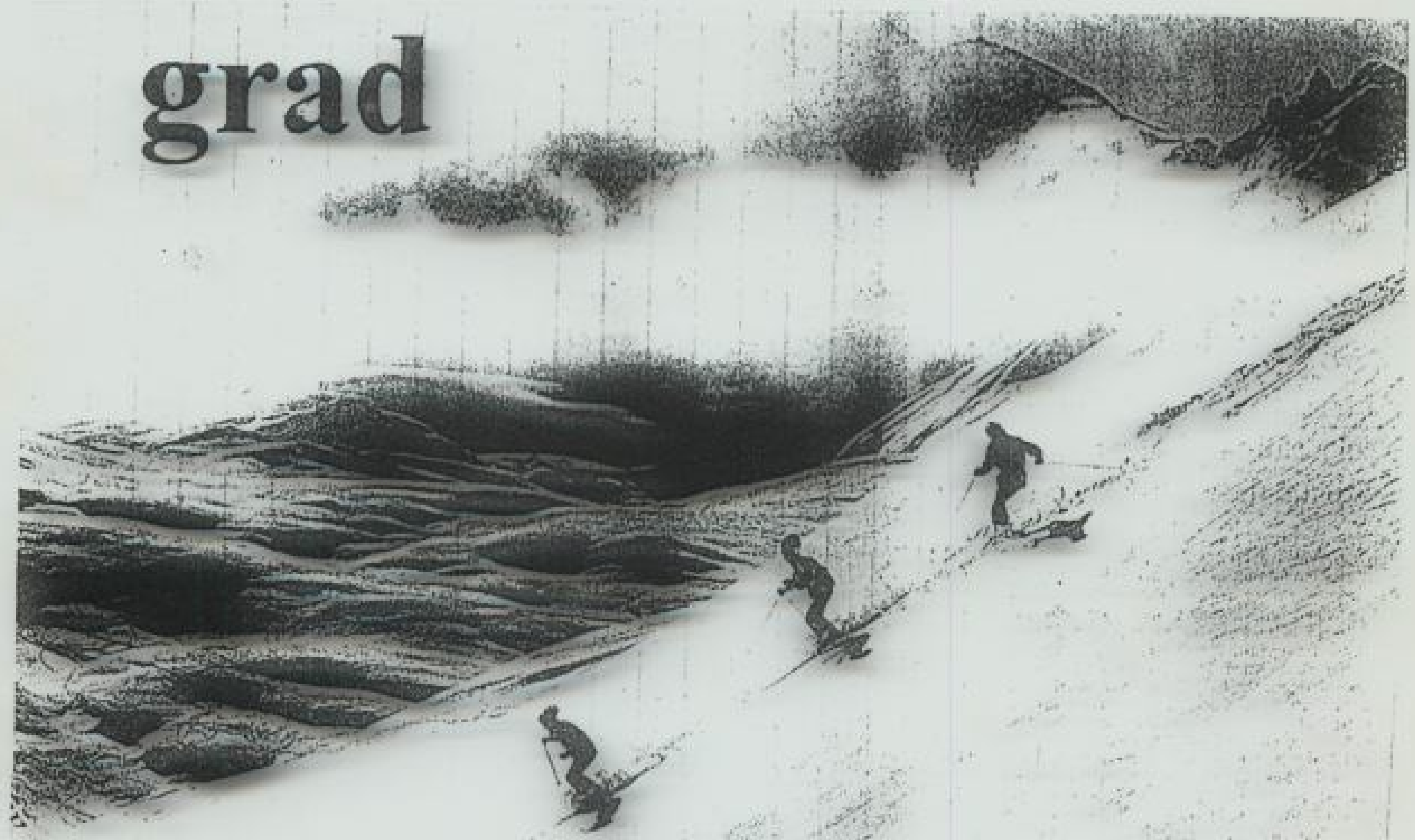
where $\underset{\sim}{\nabla} = \underset{\wedge}{i}\dfrac{\partial}{\partial x} + \underset{\wedge}{j}\dfrac{\partial}{\partial y} + \underset{\wedge}{k}\dfrac{\partial}{\partial z}$

Note that this is an OPERATOR

that operates on a SCALAR FUNCTION/FIELD

and gives a result that IS A VECTOR.

grad



$$\boxed{\text{grad}\,\phi \equiv \underset{\sim}{\nabla}\phi = \underset{\wedge}{i}\frac{\partial \phi}{\partial x} + \underset{\sim}{j}\frac{\partial \phi}{\partial y} + \underset{\wedge}{k}\frac{\partial \phi}{\partial z}}$$

∇ magnitude <u>and</u> direction of maximum rate of change of ϕ with respect to space

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H3
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bot

i.e. $\boxed{\text{grad (scalar)} = \text{vector}}$

For example,

$\phi(x,y)$ only representing height.

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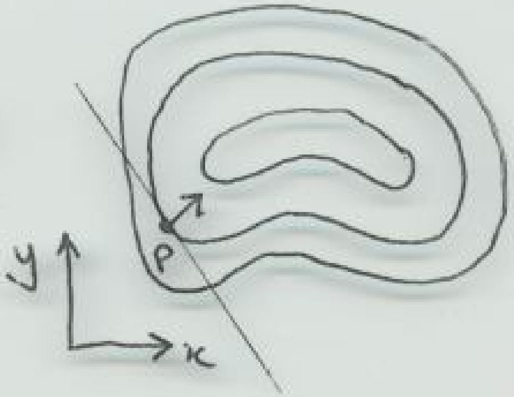
$\boxed{\text{grad} = \underset{\sim}{\nabla}}$

Scalar field ϕ. e.g. height $h(x,y)$, temperature $T(x,y,z)$, gravitational/electric potential $V(x,y,z)$

$\boxed{2D}$ $\phi(x,y)$



"HILL"

- contours = lines of constant ϕ
- gradient = max. rate of change of ϕ, $\perp^{ar}$ contour (points "uphill")

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bot

$\boxed{3D}$ $\phi(x,y,z)$

gradient, $\underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$

ie vector sum of components along x, y, z

$$(\underset{\sim}{\nabla}\phi)\cdot(\underset{\sim}{\nabla}\phi) = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$$

$\Rightarrow$ Magnitude of grad ϕ, $\boxed{|\underset{\sim}{\nabla}\phi| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}}$

H3
p81
top

Ex If scalar field $\phi(x,y,z) = x^2yz^3 + xy^2z^2$

then determine the (vector) gradient, i.e. grad ϕ,

at the point $P(1,3,2)$.

Ans $\underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$

where $\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x^2yz^3 + xy^2z^2) = 2xyz^3 + y^2z^2$

$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x^2yz^3 + xy^2z^2) = x^2z^3 + 2xyz^2$

$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(x^2yz^3 + xy^2z^2) = 3x^2yz^2 + 2xy^2z$

H3
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bot

$$\therefore \nabla\phi = (2xyz^3 + y^2z^2)\,\underset{\sim}{i} + (x^2z^3 + 2xyz^2)\,\underset{\sim}{j} + (3x^2yz^2 + 2xy^2z)\,\underset{\sim}{k}$$

At point $P(1,3,2)$, $x=1$, $y=3$, $z=2$,

giving $$\nabla\phi = 84\,\underset{\sim}{i} + 32\,\underset{\sim}{j} + 72\,\underset{\sim}{k}.$$

i.e. magnitude and direction of greatest rate of change of ϕ at P

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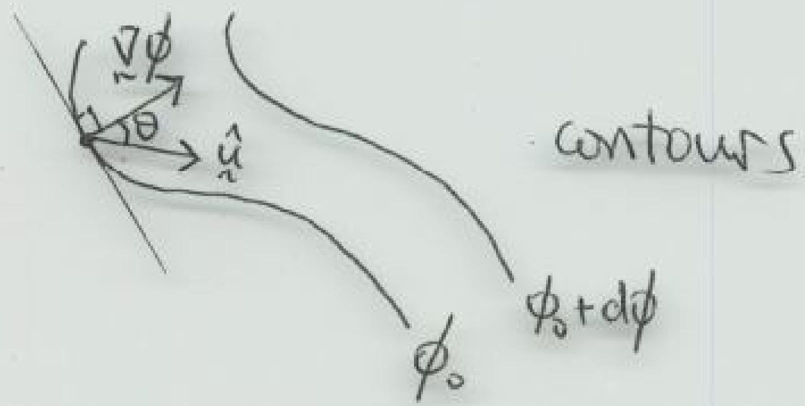
gradϕ points in the direction of greatest (positive) change ϕ at any particular point.

How does ϕ change in other directions?

Consider a 2D scalar field $\phi(x,y)$ and two contours along which ϕ has constant values ϕ_0 and $\phi_0 + d\phi$ ...



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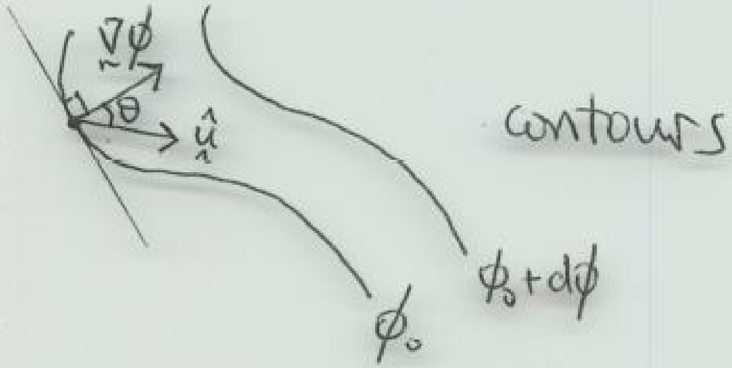


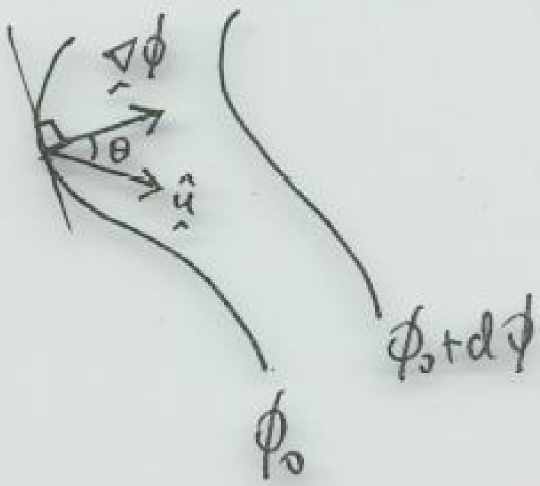
$\hat{\underset{\sim}{u}}$ = unit vector in <u>any</u> direction

ds = small distance along $\hat{\underset{\sim}{u}}$

$\frac{d\phi}{ds}$ = rate of change of ϕ along $\hat{\underset{\sim}{u}}$ = DIRECTION DERIVATIVE

$= \underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{u}} = |\underset{\sim}{\nabla}\phi||\hat{\underset{\sim}{u}}|\cos\theta$ (component of $\underset{\sim}{\nabla}\phi$ along $\hat{\underset{\sim}{u}}$)

$\frac{d\phi}{ds} = |\underset{\sim}{\nabla}\phi|\cos\theta$, THE PROJECTION OF $\underset{\sim}{\nabla}\phi$ ALONG $\hat{\underset{\sim}{u}}$.



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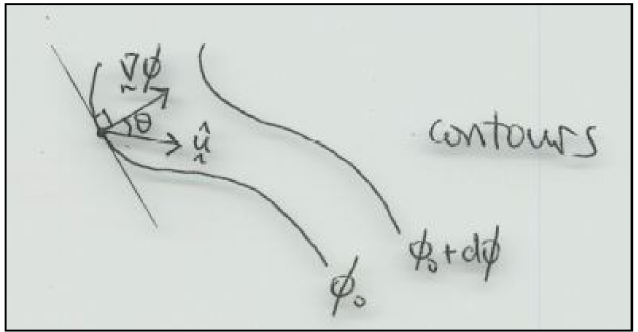
Direction derivative,

$$\frac{d\phi}{ds} = \nabla \phi \cdot \hat{u}$$

i.e. $$\frac{d\phi}{ds} = |\nabla \phi| \cos \theta$$

gives the rate of change of ϕ (with respect to distance) along the direction of $\hat{u}$.



$$\frac{d\phi}{ds} = \text{rate of change of } \phi \text{ along } \underset{\sim}{\hat{u}}$$

$$= \underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\hat{u}} = |\underset{\sim}{\nabla}\phi||\underset{\sim}{\hat{u}}|\cos\theta$$

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$\frac{d\phi}{ds}$ is

- maximum along $\underset{\sim}{\nabla}\phi$ (i.e. $\theta = 0$)
- zero along a contour of constant ϕ (i.e. $\theta = 90°$)

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top

Ex Find direction derivative of scalar field

$$\phi(x,y,z) = x^2 z + 2xy^2 + yz^2$$

at point $(1,2,-1)$ in direction of

vector $\underset{\sim}{A} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\hat{k}$.

Ans $\dfrac{d\phi}{ds} = \underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{u}}$. We need $\underset{\sim}{\nabla}\phi$ and $\hat{\underset{\sim}{u}}$ (unit vector along $\underset{\sim}{A}$)

$$\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial\phi}{\partial x} + \underset{\sim}{j}\frac{\partial\phi}{\partial y} + \underset{\sim}{k}\frac{\partial\phi}{\partial z},$$

where $\dfrac{\partial\phi}{\partial x} = 2xz + 2y^2$, $\dfrac{\partial\phi}{\partial y} = 4xy + z^2$, $\dfrac{\partial\phi}{\partial z} = x^2 + y2z$

i.e. $\underset{\sim}{\nabla}\phi = (2xz + 2y^2)\underset{\sim}{i} + (4xy + z^2)\underset{\sim}{j} + (x^2 + 2yz)\underset{\sim}{k}$

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and $\underset{\sim}{\nabla}\phi = (-2+8)\underset{\sim}{i} + (8+1)\underset{\sim}{j} + (1-4)\underset{\sim}{k}$

i.e $\underset{\sim}{\nabla\phi} = 6\underset{\sim}{i} + 9\underset{\sim}{j} - 3\underset{\sim}{k}$, at point $(1,2,-1)$ ..

Unit vector along $\underset{\sim}{A}$.

$$\boxed{\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{A}}{|\underset{\sim}{A}|}}$$

$\underset{\sim}{A} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

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where $|\underset{\sim}{A}| = \left(2^2 + 3^2 + (-4)^2\right)^{\frac{1}{2}}$

$= (4 + 9 + 16)^{\frac{1}{2}}$

$= \sqrt{29}$

i.e. $\hat{\underset{\sim}{u}} = \dfrac{\underset{\sim}{A}}{|\underset{\sim}{A}|} = \dfrac{1}{\sqrt{29}}(2, 3, -4)$

$$\therefore \frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{u}} = \boxed{\begin{aligned} &= \left(6\underset{\sim}{i} + 9\underset{\sim}{j} - 3\underset{\sim}{k}\right) \cdot \frac{1}{\sqrt{29}}(2, 3, -4) \\ &= \frac{1}{\sqrt{29}}(12 + 27 + 12) = \frac{51}{\sqrt{29}}. \end{aligned}}$$

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Physical examples ....

Ex A skier goes fastest downhill in the $-\underset{\wedge}{\nabla} h$ direction, where $h(x,y)$ = height,

i.e. in direction towards maximum lower gravitational potential

("lower" gives minus sign).

Ex Electrostatics ...

V = scalar potential field

$\underset{\sim}{E}$ = vector electric field

In one dimension x,

$$E = -\frac{dV}{dx}$$

In three dimensions x, y, z,

$$\underset{\sim}{E} = -\underset{\sim}{\nabla} V$$

i.e. force on a positive charge in scalar potential field V points to path that will give maximum decrease in the electrostatic potential.

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Specifically, $V = \dfrac{Q}{4\pi\varepsilon_0 r}$

and it can be shown that $\underset{\sim}{\nabla}\left(\dfrac{1}{r}\right) = -\dfrac{\hat{\underset{\sim}{r}}}{r^2}$

giving $\underset{\sim}{E} = -\underset{\sim}{\nabla} V = \dfrac{Q\,\hat{\underset{\sim}{r}}}{4\pi\varepsilon_0 r^2}$.

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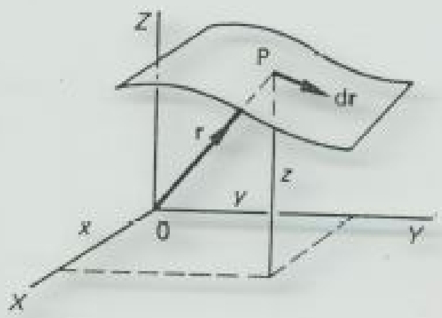
Unit normal vectors

If $\phi(x,y,z) = \text{constant}$ then (rather than a 2D contour) we define a surface in 3D.



For a given constant, we get a particular surface.

If $\underset{\sim}{dr}$ is a displacement on this surface $d\phi$ along $\underset{\sim}{dr}$ is zero, since ϕ is constant over this surface.

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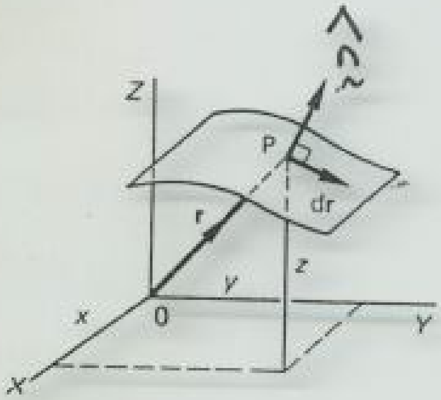
bot

Generalising the notation of direction derivative to 3D, the maximum change in ϕ (with space) is given by $|\text{grad}\,\phi|$ and the direction of grad ϕ is perpendicular to the surface.

A unit normal vector is thus given by



$$\boxed{\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{\nabla}\phi}{|\underset{\sim}{\nabla}\phi|}}$$

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top

Ex Find the unit normal vector to the surface

$$x^3y + 4xz^2 + xy^2z + z = 0$$

at the point $(1, 3, -1)$

Ans A normal vector is given by

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (3x^2y + 4z^2 + y^2z)\hat{i} + (x^3 + 2xyz)\hat{j}$$

$$+ (8xz + xy^2)\hat{k}$$

H3
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bot

At $(1,3,-1)$, $x=1$, $y=3$, $z=-1$ giving - -

$$\underset{\sim}{\nabla}\phi = 4\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k} \quad \text{at point } (1,3,-1)$$

Unit normal at this point, $\hat{\underset{\sim}{n}} = \dfrac{\underset{\sim}{\nabla}\phi}{|\underset{\sim}{\nabla}\phi|} = \dfrac{4\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{4^2+5^2+1^2}}$

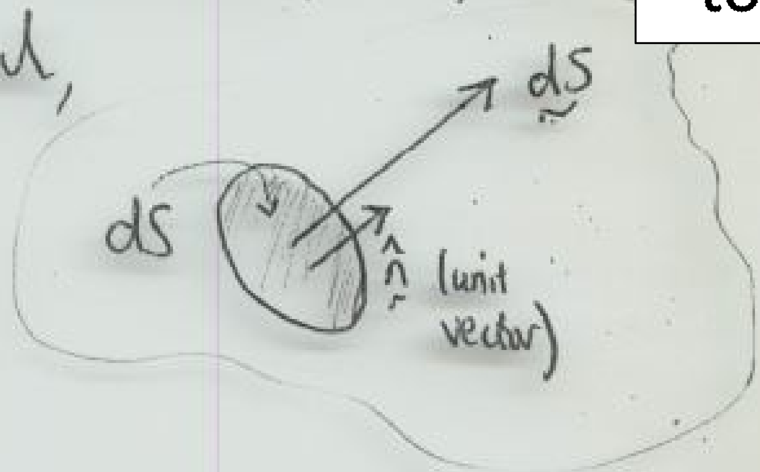
i.e. $\hat{\underset{\sim}{n}} = \dfrac{1}{\sqrt{42}}\left(4\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}\right)$

H3 p89 top

Flux of a vector over a surface

A small surface area dS can be treated as a vector with direction given by its normal,

$$\boxed{\underset{\sim}{dS} = \underset{\sim}{\hat{n}}\, dS}$$



Let $\underset{\sim}{V}$ be <u>any</u> vector defined over dS.

If $\underset{\sim}{V}$ constant over dS then

$$\boxed{\text{THE FLUX OF } \underset{\sim}{V} \text{ OVER } dS = V\, dS \cos\theta = \underset{\sim}{V} \cdot \underset{\sim}{dS}}$$

THE FLUX OF $\underset{\sim}{V}$ OVER dS = $V\,dS\cos\theta$ = $\underset{\sim}{V}\cdot\underset{\sim}{dS}$

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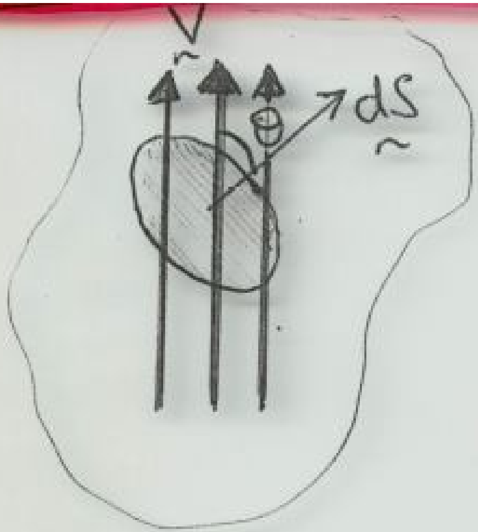

ie. $\underset{\sim}{V}$ $\parallel^{al}$ $\underset{\sim}{dS}$ $\rightarrow$ max. flux

$\underset{\sim}{V}$ $\perp^{ar}$ $\underset{\sim}{dS}$ $\rightarrow$ zero flux

Flux of $\underset{\sim}{V}$ over larger surface S is the sum of the fluxes over all the constituent dS elements of S.

ie. $\boxed{\text{FLUX OF } \underset{\sim}{V} \text{ OVER } S = \int_S \underset{\sim}{V}\cdot\underset{\sim}{dS}}$

Surface integral = double integral

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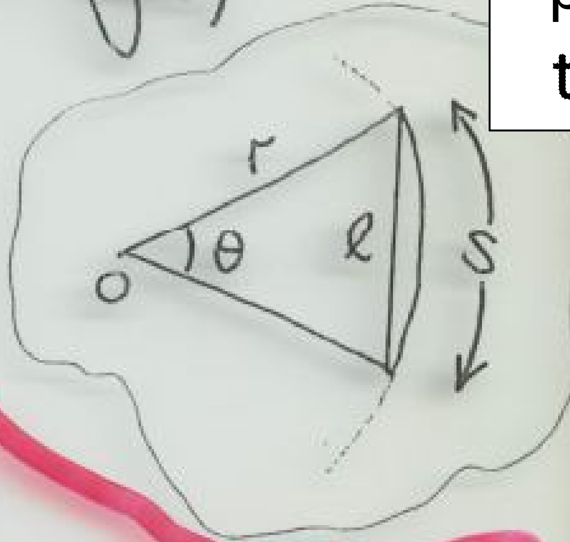
Useful related quantity ... (solid angle)

2D plane angle, $\theta = \frac{s}{r}$ (radians)



where s = arc length, r = radius

small angle $d\theta = \frac{ds}{r} = \frac{d\ell}{r}$, where $d\ell$ = chord length

3D solid angle, $\boxed{\Omega = \frac{S}{r^2}}$

where s = arc length, r = radius

small angle $d\theta = \frac{ds}{r} = \frac{dl}{r}$

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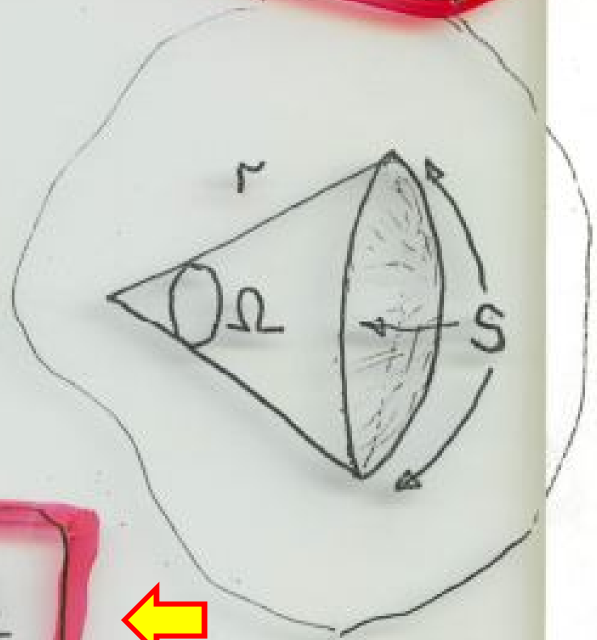
mid

3D

solid angle, $\boxed{\Omega = \frac{S}{r^2}}$

(steradians)

where S = surface area subtended at sphere



small solid angle, $\boxed{d\Omega = \frac{dS}{r^2} = \frac{dA}{r^2}}$

similar to l, dA is a (flat) plane across the "mouth" of the cone

H3
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bot

2D Full plane angle,
circle : $s \to 2\pi r$

$$\Rightarrow \theta = \frac{2\pi r}{r} = 2\pi$$

3D Full solid angle,
area of sphere $= 4\pi r^2$

$$\Rightarrow \Omega = \frac{4\pi r^2}{r^2} = 4\pi$$

See page opposite …

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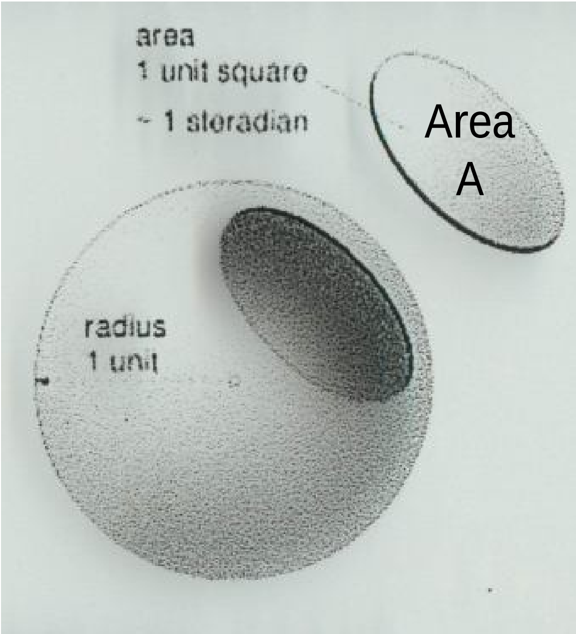

Solid angle: $\Omega = A / r^2$

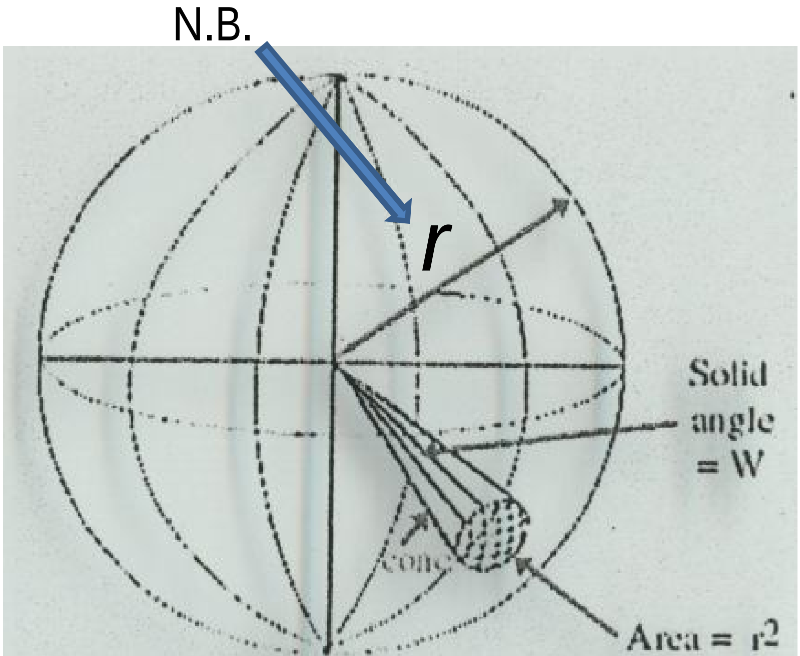

Figure 4
Solid angle of one steradian

$\theta = \frac{s}{r}$

$s = r \rightarrow \theta = 1 \text{ rad}$

$A = r^2 \rightarrow \boldsymbol{\Omega} = 1 \text{ sr}$

Physical example:

cone-shaped area of illumination from a spotlight

Now, back to previous page … p91

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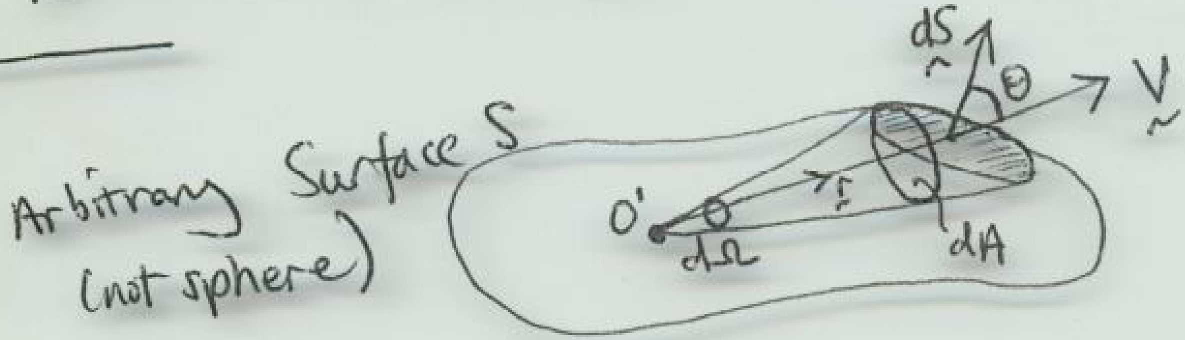
Eg. Now consider a vector field obeying an inverse-square law

i.e. $\underset{\sim}{V}(\underset{\sim}{r}) = \dfrac{\eta\,\hat{\underset{\sim}{r}}}{r^2}$ where η = constant, and summing up the flux of

$\underset{\sim}{V}$ over a closed surface with the origin O' inside.



NB The element $\underset{\sim}{dS}$ is tiny but drawn large for illustration.

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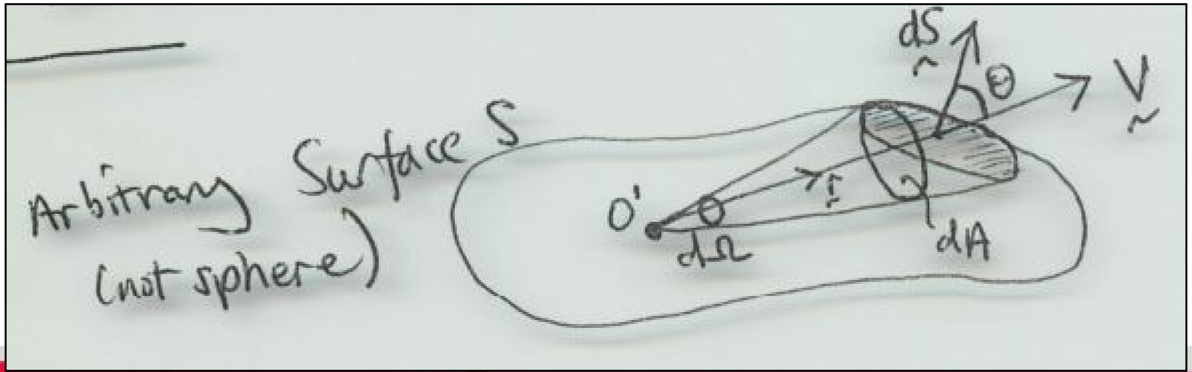
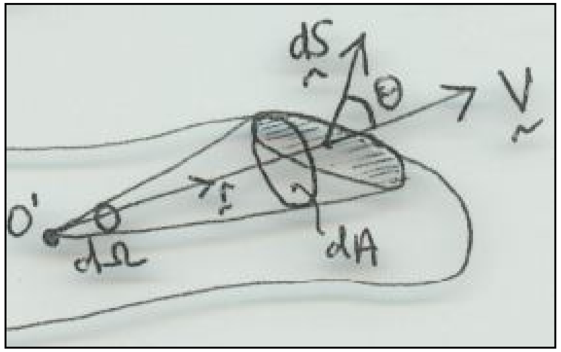


$$\text{Flux of } \underset{\sim}{V} \text{ over } d\underset{\sim}{S} = \underset{\sim}{V} \cdot d\underset{\sim}{S}$$

$$= V\,dS\cos\theta$$

$$= V\,dA = V r^2 d\Omega \qquad \left(\text{solid angle } d\Omega = \frac{dA}{r^2}\right)$$

$d\underset{\sim}{S}$ has been projected onto the plane across the "mouth" of the cone of the solid angle. This plane has area dA and $\underset{\sim}{r}$ is perpendicular. For small solid angles, this is equivalent to projecting onto a sphere centred at O' and having radius r.



Flux of $\underset{\sim}{V}$ over dS $= \underset{\sim}{V} \cdot d\underset{\sim}{S}$

$= V\, dS \cos\theta$

$= V\, dA = V r^2 d\Omega$

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bot

Flux of $\underset{\sim}{V}$ over whole S $= \oint_S \underset{\sim}{V} \cdot d\underset{\sim}{S}$

$= \oint_S V r^2 d\Omega$

Now $|\underset{\sim}{V}| = \dfrac{\eta}{r^2} \Rightarrow$ Flux over S $= \eta \oint_S d\Omega = 4\pi\eta$.

(4π is the full solid angle)

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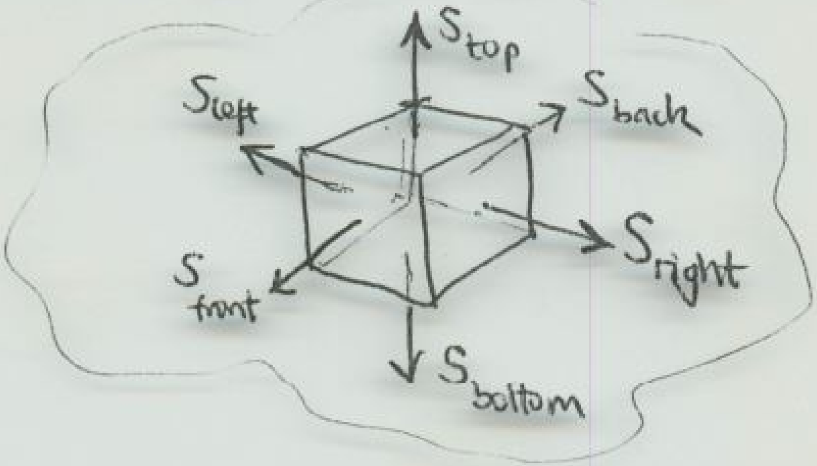
Ex If $\underset{\sim}{J}$ = electric current density then total current I (flux of current density $\underset{\sim}{J}$) over surface S is given by

$$I = \int_S \underset{\sim}{J} \cdot d\underset{\sim}{S}$$

<u>Conventions</u> • For a <u>closed</u> surface,

$\underset{\sim}{\hat{n}}$ and $d\underset{\sim}{S}$ point OUTWARDS

e.g. surface of a cube



Conventions (continued)

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For a closed surface : $\int_S \rightarrow \oint_S$

Similarly,

closed curve : $\int_C \rightarrow \oint_C$