Mathematical Methods and Applications

HANDOUT 4

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- VECTOR CALCULUS (continued)
  - Flux calculations (surface integrals like $\int_S \underset{\sim}{F} \cdot d\underset{\sim}{s}$)
  - Divergence
  - Curl
    - definition
    - physical examples

Mathematical Methods and Applications

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Contents continued …

- Multiple operations
  - grad div , div grad , curl curl
  - curl grad , div curl , div grad
    (revisited, Laplacian, physical examples)
- Revision summary (so far)

VECTOR PRODUCT → VECTOR AREA → SURFACE INTEGRALS

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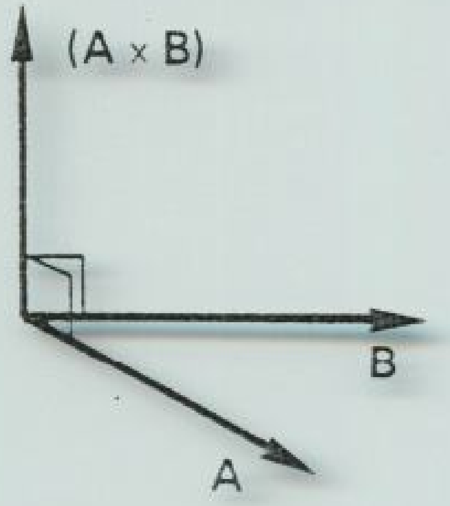
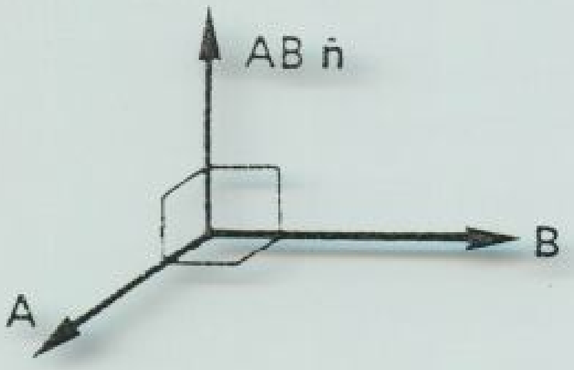
The vector product of two vectors **A** and **B** is defined as



$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$, at right angles to the plane of **A** and **B** to form a right-handed set.



If $\theta = \dfrac{\pi}{2}$, then $|\mathbf{A} \times \mathbf{B}| = AB$, in the direction of the normal. Therefore, if $\hat{\mathbf{n}}$ is a unit normal then

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\hat{\mathbf{n}} = A\,B\,\hat{\mathbf{n}}$$

VECTOR PRODUCT → VECTOR AREA → SURFACE INTEGRALS

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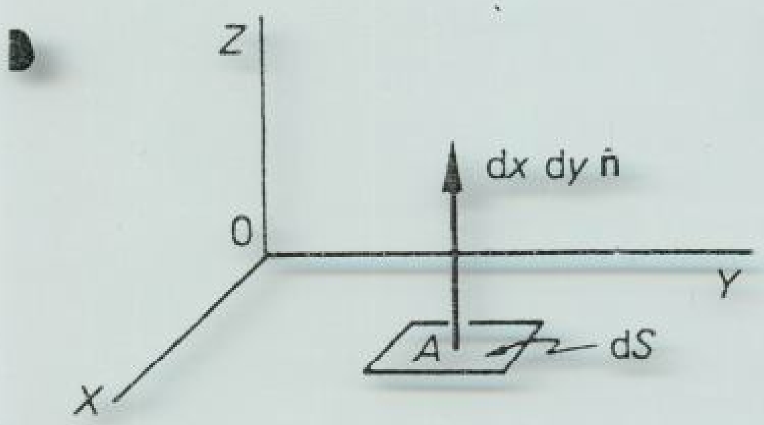
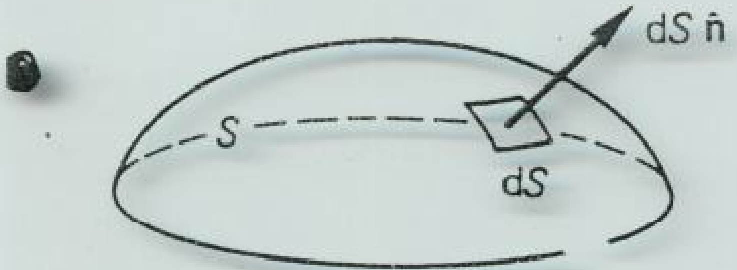


If P(x, y) is a point in the xy-plane, the element of area $d\mathbf{S}$ can be written

$$d\mathbf{S} = (\mathbf{i}\,dx) \times (\mathbf{j}\,dy)$$
$$= dx\,dy\,\hat{\mathbf{n}}$$

i.e. a vector of magnitude $dx\,dy$ acting in the direction of $\hat{\mathbf{n}}$ and referred to as the *vector area*.



For a general surface S in space, each element of surface dS has a *vector area* $d\mathbf{S}$ such that $d\mathbf{S} = dS\,\hat{\mathbf{n}}$.

And…

$$\hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

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Let's work out some surface integrals of the form

$$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

to illustrate the technique.

This is a long example, but we will use the result again when we get to the Divergence Theorem.

Ex Consider a vector field $\underset{\sim}{F}(x,y,z) = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$ and a surface S with flat sides that is bounded by the planes $x=0$, $y=0$, $z=0$, $x=1$, $y=3$, $z=2$.

$$\underset{\sim}{F}(x,y,z) = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$$

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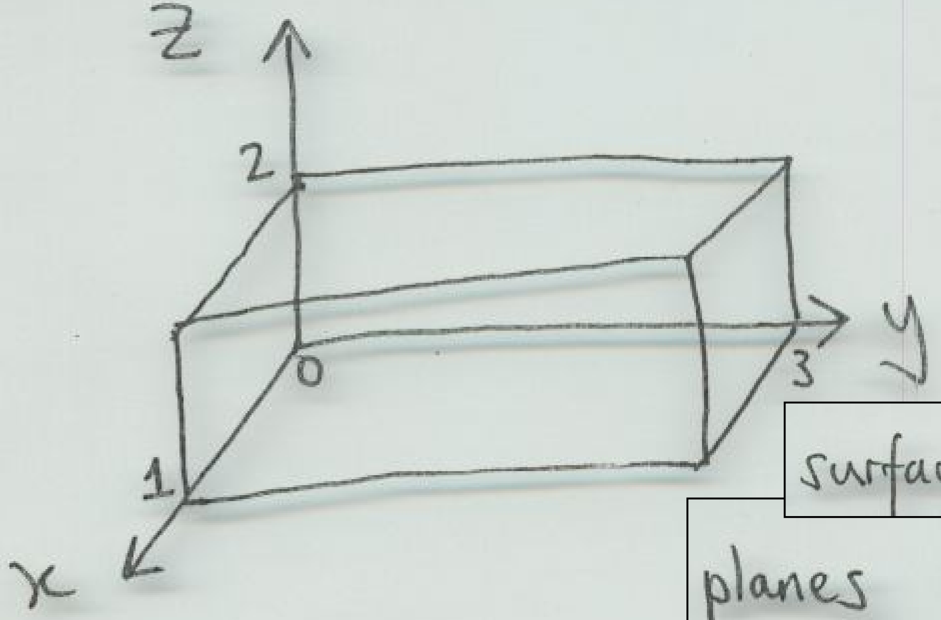
What is the total flux of $\underset{\sim}{F}$ over S ?

In other words, what is $\oint_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$?

Ans The surface S is a "box" in x, y, z ...



surface S with flat sides

planes $x=0$, $y=0$, $z=0$
$x=1$, $y=3$, $z=2$

To work out the flux of $\mathbf{F}$ over the whole surface, consider each side of the box in turn.

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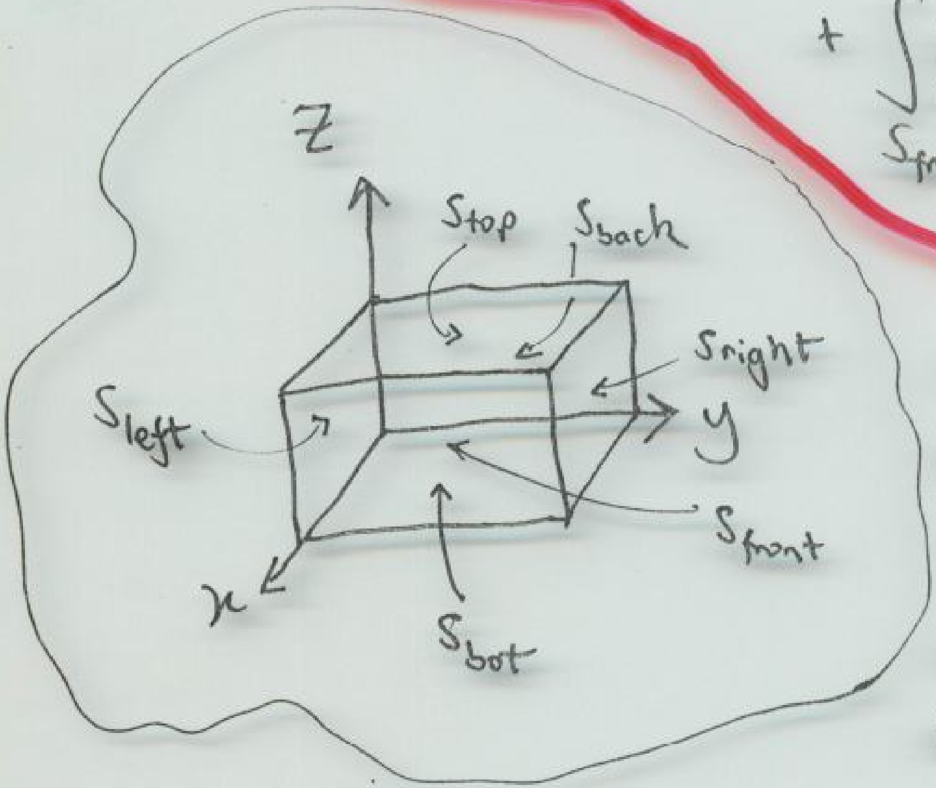
$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \int_{S_{bot}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{top}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{right}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{left}} \mathbf{F}\cdot d\mathbf{S}$$

$$+ \int_{S_{front}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{back}} \mathbf{F}\cdot d\mathbf{S}$$



And recall that, for this closed surface, each $d\mathbf{S}$ will point OUTWARDS from the enclosed volume.

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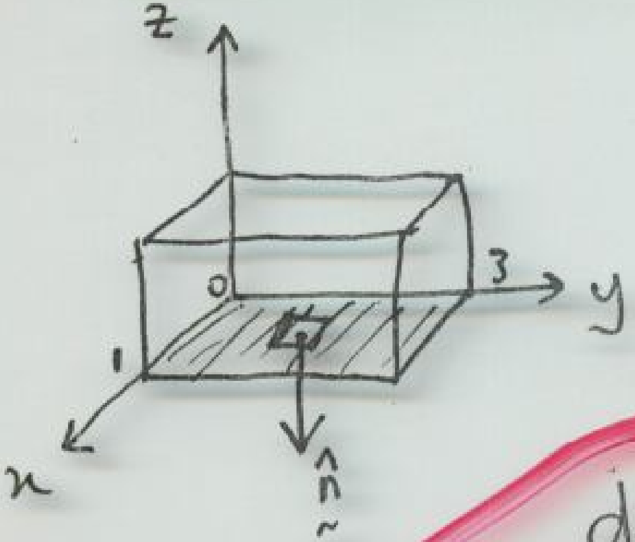
(i) The base of the box, S_{bot}



$\underset{\sim}{F} = x^2 \underset{\sim}{i} + z \underset{\sim}{j} + y \underset{\sim}{k}$

but here $z = 0$, so

$\underset{\sim}{F} = x^2 \underset{\sim}{i} + y \underset{\sim}{k}$.

dS is in the xy plane, i.e. $dS = dx\,dy$

while $\underset{\sim}{\hat{n}} = -\underset{\sim}{k}$ (pointing outwards and therefore downwards).

So we have $\quad d\underset{\sim}{S} = \underset{\sim}{\hat{n}}\, dS = -\underset{\wedge}{k}\, dS$

and $\quad \displaystyle\int_{S_{bot}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{S_{bot}} \underset{\wedge}{F}\cdot \underset{\wedge}{\hat{n}}\, dS$

$$= \int_{S_{bot}} \left(x^2 \underset{\wedge}{i} + y\underset{\wedge}{k}\right)\cdot\left(-\underset{\wedge}{k}\right) dS$$

$$= \int_{S_{bot}} \left[x^2\left(-\underset{\wedge}{k}\cdot\underset{\wedge}{i}\right) + y\left(-\underset{\wedge}{k}\right)\cdot\underset{\wedge}{k}\right] dS$$

$$= \int_{S_{bot}} \left(0 - y\right) dS$$

$$= \int_{S_{bot}} (-y)\, dS$$

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bot

On this surface, $dS = dx\,dy$ and x varies from 0 to 1 and y varies from 0 to 3

$$\therefore \int_{S_{bot}} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{x=0}^{x=1} \int_{y=0}^{y=3} (-y)\, dx\, dy$$

$$= \int_{x=0}^{x=1} \left[-\frac{y^2}{2} \right]_0^3 dx$$

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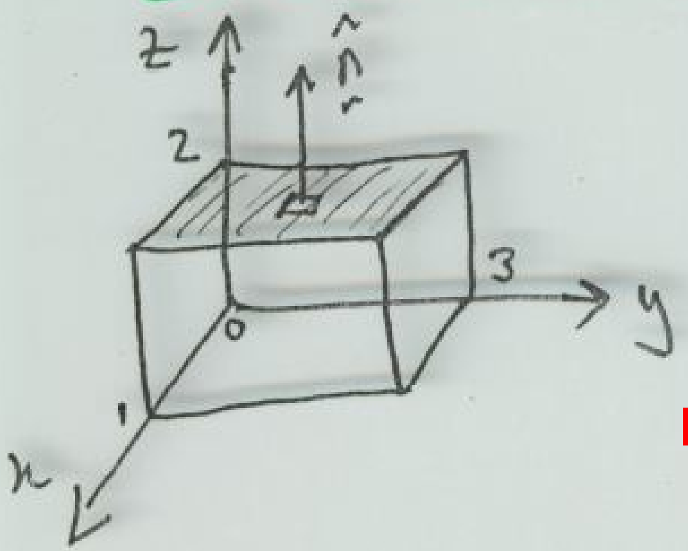
i.e. $\int_{S_{bot}} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_0^1 \left(-\frac{9}{2}\right) dx = \left[-\frac{9}{2}x\right]_0^1$

$= -\frac{9}{2}.$

(ii) The top of the box, S_{top}



$\underset{\sim}{F} = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$

but here $z = 2$, so

$$\boxed{\underset{\sim}{F} = x^2\underset{\sim}{i} + 2\underset{\sim}{j} + y\underset{\sim}{k}.}$$

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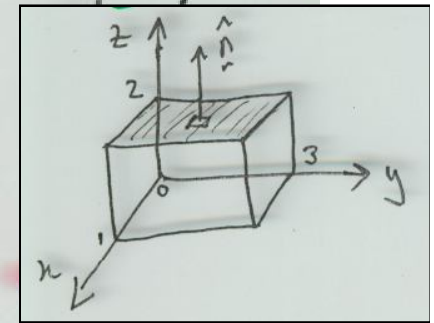
S_{top} is composed of elements dS that, again, can be written in terms of dx and dy i.e. $\boxed{dS = dx\,dy}$

while $\underset{\sim}{\hat{n}} = +\underset{\sim}{k}$ (pointing outwards and therefore upwards).



So we have $\boxed{d\underset{\sim}{S} = \underset{\sim}{\hat{n}}\,dS = +\underset{\sim}{k}\,dx\,dy}$

and $$\int_{S_{top}} \underset{\sim}{F} . d\underset{\sim}{S} = \int_{S_{top}} \underset{\sim}{F} . \underset{\sim}{\hat{n}}\,dS = \boxed{\int_{S_{top}} (x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}) . \underset{\sim}{k}\,dx\,dy}$$

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Since $(x^2\hat{i} + 2\underset{\sim}{j} + y\hat{k}) \cdot \underset{\sim}{k} = y$,

$$\int_{S_{top}} \hat{F} \cdot d\underset{\sim}{S} = \int_{S_{top}} y \, dS .$$

On this surface, $dS = dx\,dy$ and x varies from 0 to 1
and y varies from 0 to 3

$$\therefore \int_{S_{top}} \hat{F} \cdot d\hat{S} = \int_{x=0}^{x=1} \int_{y=0}^{y=3} y \, dx\, dy = \int_{x=0}^{x=1} \left[\frac{y^2}{2} \right]_0^3 dx$$

$$= \left[\frac{9}{2} x \right]_0^1 = +\frac{9}{2} .$$

H4
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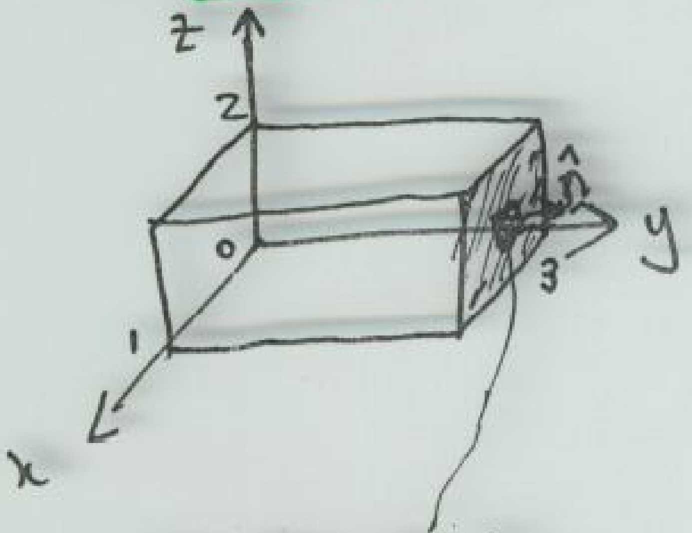
bot

(iii) Right hand side of the box, S_{right}



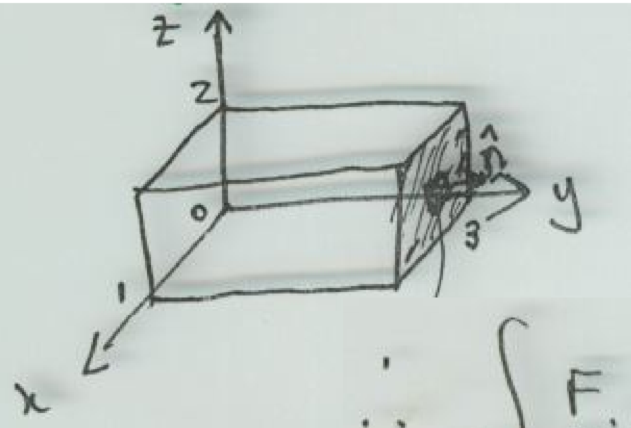
Here, $y=3$ and

$\underset{\sim}{F} = x^2 \underset{\sim}{i} + z \underset{\sim}{j} + 3 \underset{\sim}{k}$

$\underset{\sim}{\hat{n}} = \underset{\sim}{j}$ (outwards along positive y)

$\underset{\sim}{NB}$ $dS = dx\,dz$.

$$\boxed{\begin{aligned} d\underset{\sim}{S}_{right} &= \underset{\sim}{\hat{n}}\, dS \\ &= \underset{\sim}{\hat{n}}\, dx\,dz \\ &= \underset{\sim}{j}\, dx\,dz \end{aligned}}$$



$d\underset{\sim}{S}_{right} = \underset{\sim}{j}\, dx\, dz$

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$$\therefore \int_{S_{right}} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{S_{right}} (x^2 \underset{\sim}{i} + z \underset{\sim}{j} + 3\underset{\sim}{k}) \cdot \underset{\sim}{j}\, dx\, dz$$

$$= \iint_{S_{right}} z\, dx\, dz = \int_{x=0}^{x=1} \int_{z=0}^{z=2} z\, dx\, dz$$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^2 dx = \int_0^1 2\, dx$$

$$= \Big[2x\Big]_0^1 = 2.$$

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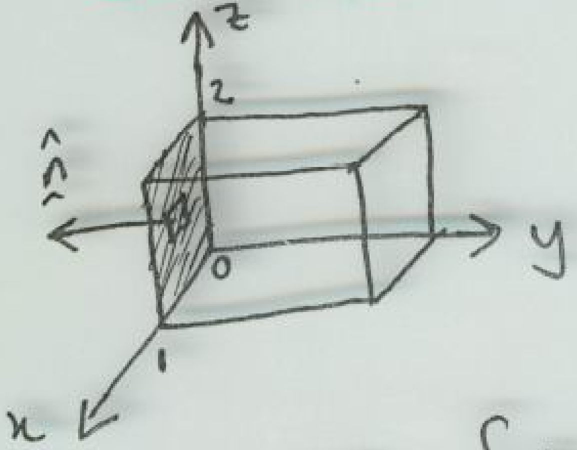
bot

(iv) Left hand side of the box, S_{left}



Here, $y=0$ and $\underset{\sim}{F} = x^2\,\underset{\sim}{i} + z\,\underset{\sim}{j}$.

$$d\underset{\sim}{S}_{left} = -\underset{\sim}{j}\,dx\,dz$$

and

$$\int_{S_{left}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{x=0}^{x=1}\int_{z=0}^{z=2} (-z)\,dz\,dx$$

$$= \int_{x=0}^{x=1} \left[-\frac{z^2}{2}\right]_0^2 dx$$

$$= \int_0^1 (-2)\,dx = -2.$$

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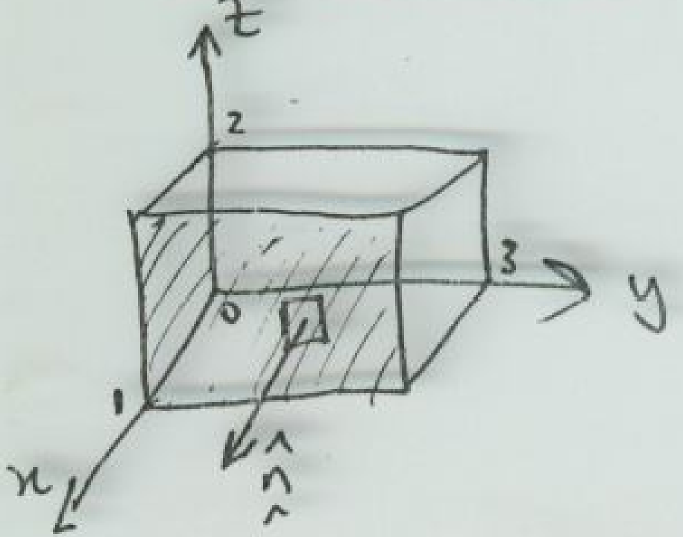
(v) Front side of the box, S_{front}



Here, $x = 1$ and $\underset{\sim}{F} = \underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$.

$$d\underset{\sim}{S}_{front} = \underset{\sim}{i}\, dy\, dz$$

and

$$\int_{S_{front}} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{y=0}^{y=3} \int_{z=0}^{z=2} (1)\, dy\, dz = 6.$$

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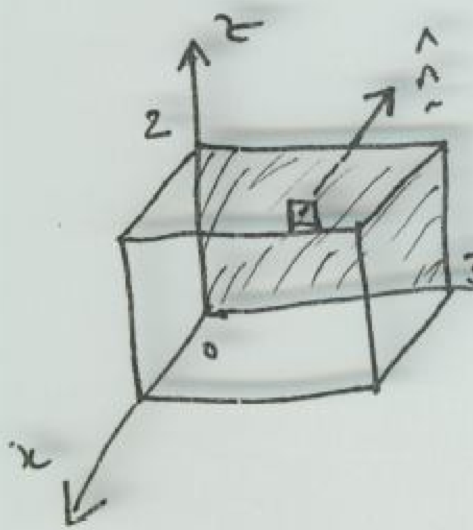
(vi) Back side of the box, S_{back}



(in negative x direction)

Here, $x = 0$ and

$$\underset{\sim}{F} = z\underset{\sim}{j} + y\underset{\sim}{k}$$

$$d\underset{\sim}{S}_{back} = -\underset{\sim}{i}\,dy\,dz$$

and

$$\int_{S_{back}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{S_{back}} \left(z\underset{\sim}{j} + y\underset{\sim}{k}\right)\cdot\left(-\underset{\sim}{i}\right) dS$$

$$= \int_{S_{back}} (0)\, dS = 0.$$

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Finally, the flux of $\underset{\sim}{F} = x^2 \underset{\sim}{i} + z \underset{\sim}{j} + y \underset{\sim}{k}$ over the whole (closed) surface S is

$$\oint_S \underset{\sim}{F} \cdot d\underset{\sim}{S} = \text{" sum of the integrals over the sides "}$$

$$= -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6 .$$

DIV (THE DIVERGENCE OF A VECTOR FUNCTION)

The div operator is the second differential operator that we will define, but first lets clarify what is meant by an "operator" and an "operation".

The scalar product of two vectors is an operation between two vectors that yields a scalar result.

This can also be thought of in terms of an operator that acts on a vector ....

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....

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \text{`a scalar'}$$

But we may consider

$$\underset{\sim}{a} \cdot \quad = \text{`an operator'}$$

that acts on $\underset{\sim}{b}$.

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bot

An operator is like a function

e.g. $f(x) = x^2$ is function f acting upon x,

where f is the "square it" operator.

Similarly, if $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$

then $\underset{\sim}{a}\,.$ is an operator acting upon $\underset{\sim}{b}$

that gives the result $a_1 b_1 + a_2 b_2 + a_3 b_3$.

To define the differential operator div,

we replace $\underset{\sim}{a}$ with $\underset{\sim}{\nabla}$

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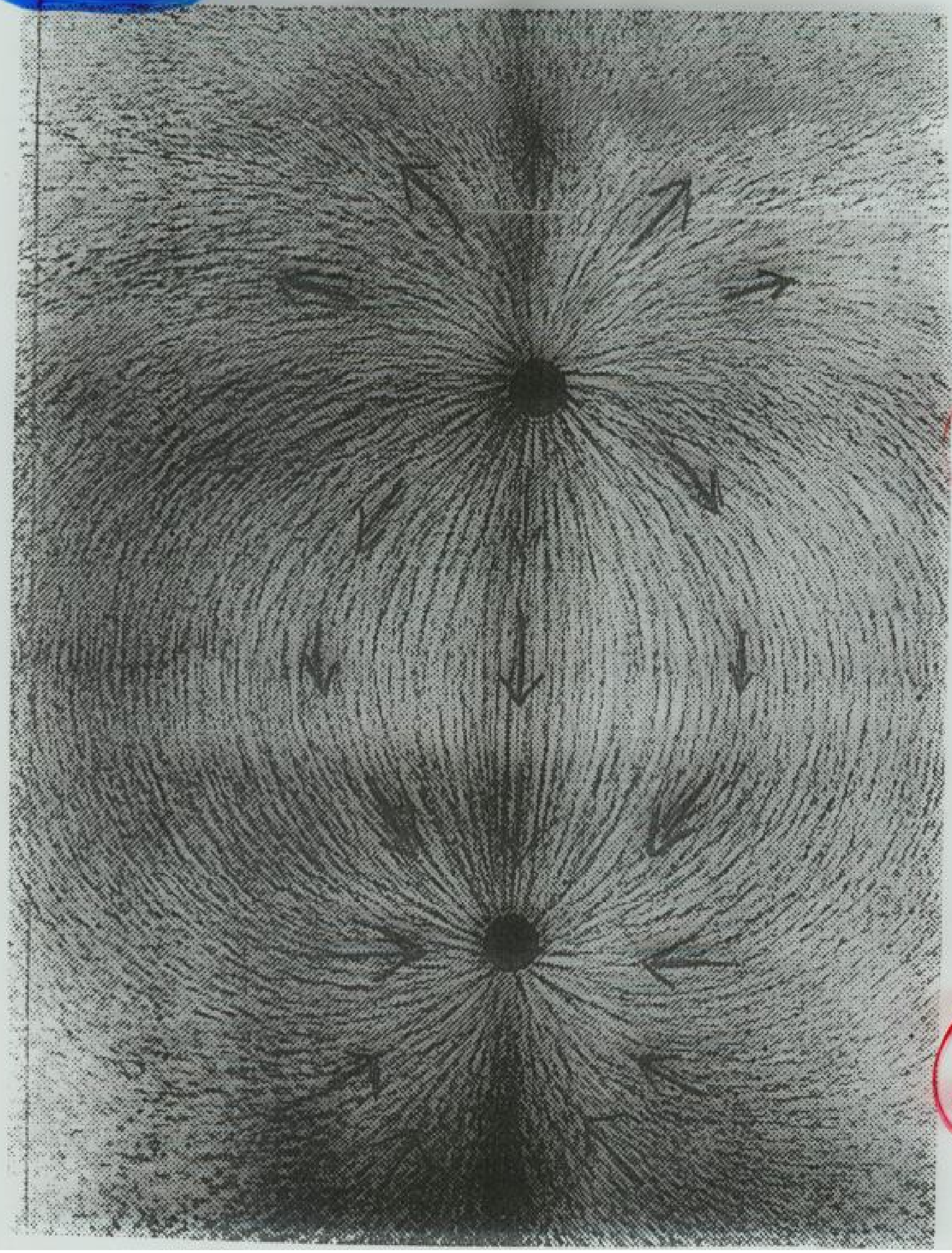
div



SOURCE
(OUTFLOW OF FLUX)

SINK
(INFLOW OF FLUX)

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If $\underset{\sim}{V} = V_x \underset{\sim}{i} + V_y \underset{\sim}{j} + V_z \underset{\sim}{k}$ then

$$\text{div}\,\underset{\sim}{V} \equiv \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x \underset{\sim}{i} + V_y \underset{\sim}{j} + V_z \underset{\sim}{k}\right)$$

$$\Rightarrow \boxed{\underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

→ net outflow per unit volume (at a point)
OFFLUX

$\boxed{\text{div (vector)} = \text{scalar}}$

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In other words,

- the div operator $\underset{\sim}{\nabla}\cdot$ acts on a vector and gives a scalar

- $\underset{\sim}{\nabla}\cdot\underset{\sim}{V}$ has the physical interpretation of the net outflow OF FLUX per unit volume (at a point) of the vector field $\underset{\sim}{V}$. This can be deduced from the "Divergence Theorem" that is covered later.

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bot

- The "outflow" of a vector field can be related to the presence of "sources" and "sinks" of flux within the vector field.

Ex. If $\underset{\sim}{V} = x^2 y\, \underset{\sim}{i} - xyz\, \underset{\sim}{j} + yz^2\, \underset{\sim}{k}$

then work out $\operatorname{div} \underset{\sim}{V} = \underset{\sim}{\nabla} \cdot \underset{\sim}{V}$.

Ans

$$\underset{\sim}{V} = (V_x, V_y, V_z)$$

$$= (x^2 y, -xyz, yz^2)$$

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So,

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \left(\underset{\sim}{i} \frac{\partial}{\partial x} + \underset{\sim}{j} \frac{\partial}{\partial y} + \underset{\sim}{k} \frac{\partial}{\partial z} \right) \cdot \left(V_x \underset{\sim}{i} + V_y \underset{\sim}{j} + V_z \underset{\sim}{k} \right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \, ,$$

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mid

where $$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}(x^2 y) = 2xy$$

$$\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}(-xyz) = -xz$$

$$\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}(yz^2) = 2yz.$$

$$\therefore \operatorname{div} \underset{\sim}{V} = \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 2xy + (-xz) + 2yz.$$

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bot

$$\operatorname{div} \underset{\sim}{V} = \hat{\nabla} \cdot \underset{\sim}{V} = 2xy + (-xz) + 2yz$$

This is the divergence of vector field $\underset{\sim}{V}(x, y, z)$.

At any point (x, y, z), we can work out the divergence of $\underset{\sim}{V}$ but substituting the values of x, y and z in the right hand side of the above.

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Ex If vector field $\underset{\sim}{A} = 2x^2y\,\underset{\sim}{i} - 2(xy^2 + y^3z)\,\underset{\sim}{j} + 3y^2z^2\,\underset{\sim}{k}$

then determine $\underset{\sim}{\nabla} \cdot \underset{\sim}{A}$.

Ans $\underset{\sim}{A} = (A_x, A_y, A_z)$ where

$A_x = 2x^2y$

$A_y = -2(xy^2 + y^3z)$

$A_z = 3y^2z^2$

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$$\therefore \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

$$\boxed{= 4xy - 2(2xy + 3y^2 z) + 6y^2 z}$$

$$= 4xy - 4xy - 6y^2 z + 6y^2 z = 0 .$$

- Any such vector field $\underset{\sim}{A}$ for which $\underset{\sim}{\nabla} \cdot \underset{\sim}{A} = 0$ at all points (x, y, z), as in the above example, is termed SOLENOIDAL.

CURL (THE CURL OF A VECTOR FUNCTION)

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The cross product of two vectors is an operation between two vectors that yields a vector result.

This can also be thought of in terms of an operator that acts on a vector ...

$$\underset{\sim}{a} \times \underset{\sim}{b} = \text{'a vector'}$$

and we may consider

$$\underset{\sim}{a} \times \; = \text{'an operator' that acts on } \underset{\sim}{b} \text{ and gives a vector result.}$$

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$\underset{\sim}{a} \times$ = "an operator that acts on $\underset{\wedge}{b}$ and gives a vector result".

To define the differential operator curl,

we replace $\underset{\wedge}{a}$ with $\underset{\sim}{\nabla}$.

curl is most concisely defined in terms of a 3x3 determinant but can also be written in terms of the expansion of this determinant ---.

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Recall that
$$\underset{\sim}{a} \times \underset{\wedge}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\wedge}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and
$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}.$$

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

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mid

curl $\underset{\sim}{V}$ (also called rot $\underset{\sim}{V}$) of a vector field

$\underset{\sim}{V}(x,y,z) = (V_x, V_y, V_z)$ is given by

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

i.e. $$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

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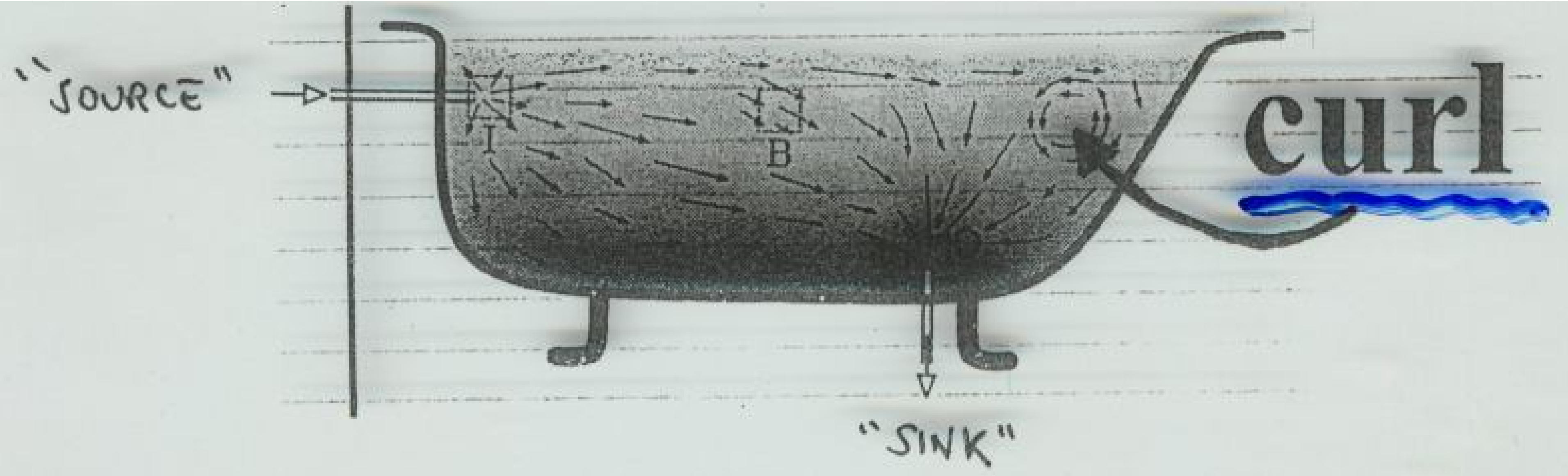

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Since grad is a vector operator,
div and curl yield properties of a vector field.

div ⇒ the flux of the field originating from a point
curl ⇒ the "circulation" there is about a point

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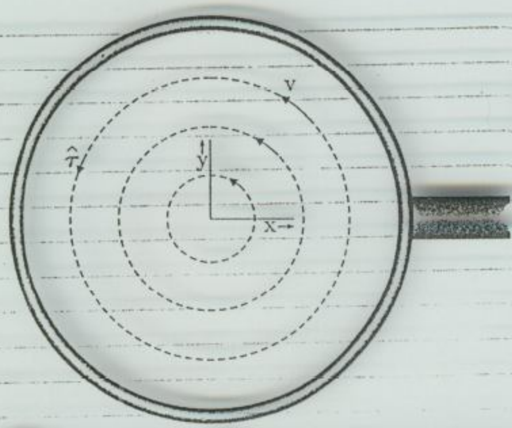
From the small ...



Storm in a teacup

twist

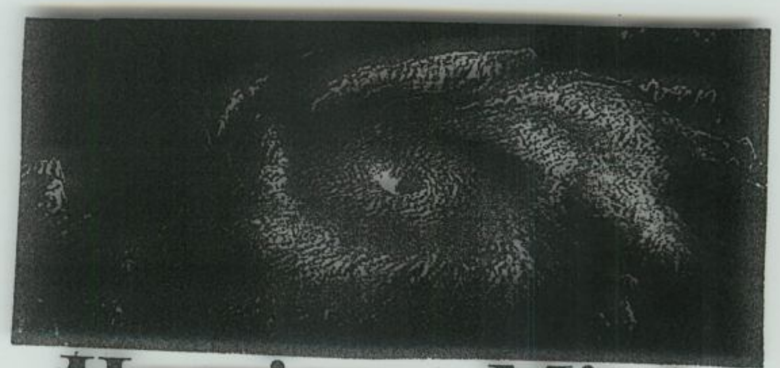
swirl

rotation *(rot)*

circulation

curl

... to the large



Hurricane Mitch

A couple of things that are older than nearly everyone in this room ☺ !

H4 digression ...

Optics Communications 94 (1992) 469–476
North-Holland

OPTICS COMMUNICATIONS

Full length article

Optical vortices in beam propagation through a self-defocussing medium

G.S. McDonald, K.S. Syed [1] and W.J. Firth

Department of Physics and Applied Physics, University of Strathclyde, 107 Rottenrow, Glasgow G4 0NG, UK

Received 21 April 1992

Optics Communications 95 (1993) 281–288
North-Holland

OPTICS COMMUNICATIONS

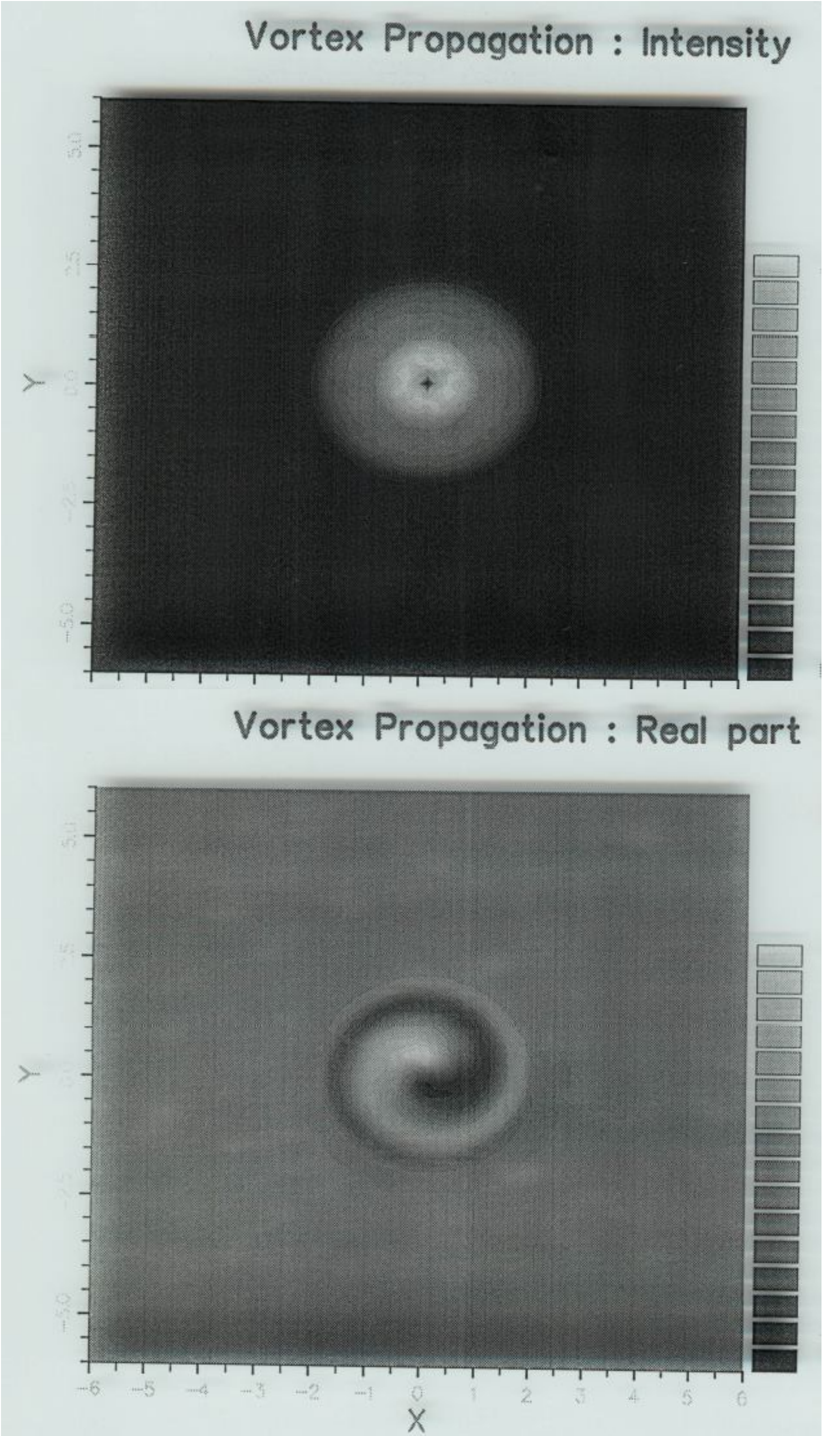
Dark spatial soliton break-up in the transverse plane

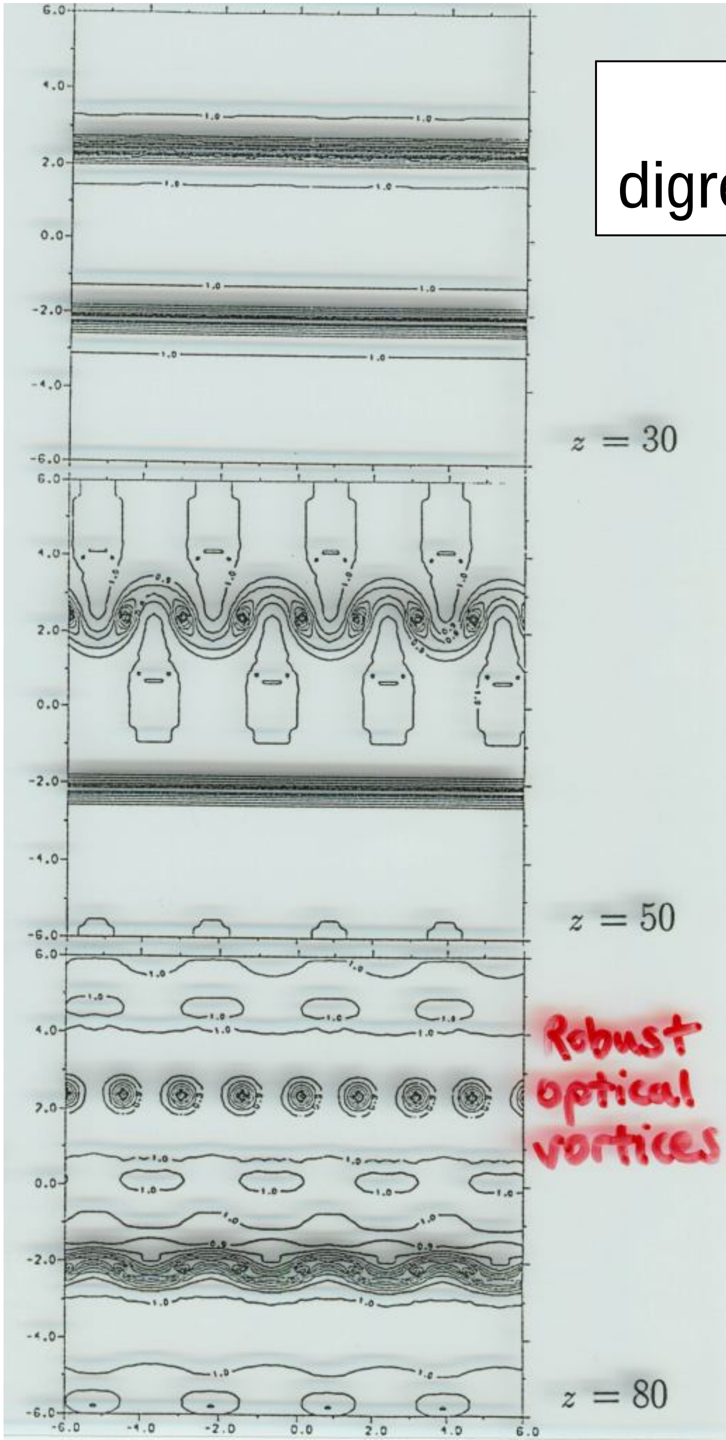
G.S. McDonald [1], K.S. Syed [2] and W.J. Firth

Department of Physics and Applied Physics, University of Strathclyde, 107 Rottenrow, Glasgow G4 0NG, UK

Received 3 April 1992

H4
digression ...





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p110
top

And be clear that while

$$\boxed{\text{div (vector) = scalar}}\ \text{(scalar product)},$$

$$\boxed{\text{curl (vector) = vector}}\ \text{(vector product)}.$$

H4
p110
bot

Ex If vector field $\underset{\sim}{V} = (y^4 - x^2z^2)\underset{\sim}{i} + (x^2+y^2)\underset{\sim}{j} - x^2yz\underset{\sim}{k}$

then determine $\text{curl}\,\underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V}$.

Ans $\underset{\sim}{V} = (V_x, V_y, V_z)$ where

$$V_x = y^4 - x^2z^2$$
$$V_y = x^2 + y^2$$
$$V_z = -x^2yz$$

and $\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$

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top

$$\operatorname{curl} \underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \underset{\sim}{i}\left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right) - \underset{\sim}{j}\left(\frac{\partial}{\partial x}V_z - \frac{\partial}{\partial z}V_x\right) + \underset{\sim}{k}\left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$$

$$\underset{\sim}{\text{curl}\,V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \underset{\sim}{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$V_x = y^4 - x^2 z^2$$

$$V_y = x^2 + y^2$$

$$V_z = -x^2 y z$$

H4
p111
mid

$$= \underset{\sim}{i}\left[\frac{\partial}{\partial y}(-x^2 y z) - \frac{\partial}{\partial z}(x^2 + y^2)\right]$$

$$- \underset{\sim}{j}\left[\frac{\partial}{\partial x}(-x^2 y z) - \frac{\partial}{\partial z}(y^4 - x^2 z^2)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2 z^2)\right]$$

H4
p111
bot

$$= \underset{\sim}{i}\left[\frac{\partial}{\partial y}\left(-x^2yz\right) - \frac{\partial}{\partial z}\left(x^2+y^2\right)\right]$$

$$- \underset{\sim}{j}\left[\frac{\partial}{\partial x}\left(-x^2yz\right) - \frac{\partial}{\partial z}\left(y^4-x^2z^2\right)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}\left(x^2+y^2\right) - \frac{\partial}{\partial y}\left(y^4-x^2z^2\right)\right]$$

$$= \underset{\sim}{i}\left[-x^2z + 0\right] - \underset{\sim}{j}\left[-2xyz + 2x^2z\right]$$

$$+ \underset{\sim}{k}\left[2x - 4y^3\right]$$

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top

Ex Determine curl $\underset{\sim}{F}$ at the point $(2,0,3)$

where $\underset{\sim}{F} = ze^{2xy}\,\underset{\sim}{i} + 2xz\cos y\,\underset{\sim}{j} + (x+2y)\,\underset{\sim}{k}$.

Ans

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ze^{2xy} & 2xz\cos y & x+2y \end{vmatrix}$$

$$= \underset{\sim}{i}\left[2 - 2x\cos y\right] - \underset{\sim}{j}\left[1 - e^{2xy}\right] + \underset{\sim}{k}\left[2z\cos y - 2xze^{2xy}\right]$$

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bot

$$= \underset{\sim}{i}\left[2 - 2x\cos y\right] - \underset{\sim}{j}\left[1 - e^{2xy}\right] + \underset{\sim}{k}\left[2z\cos y - 2xze^{2xy}\right].$$

At point $(2,0,3)$, $\quad \underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{i}\left[2 - 4\cos 0\right] - \underset{\sim}{j}\left[1 - e^{0}\right] + \underset{\sim}{k}\left[6\cos 0 - 12e^{0}\right]$

$$= -2\underset{\sim}{i} + 0 - 6\underset{\sim}{k}$$

$$= -2\left(\underset{\sim}{i} + 3\underset{\sim}{k}\right).$$

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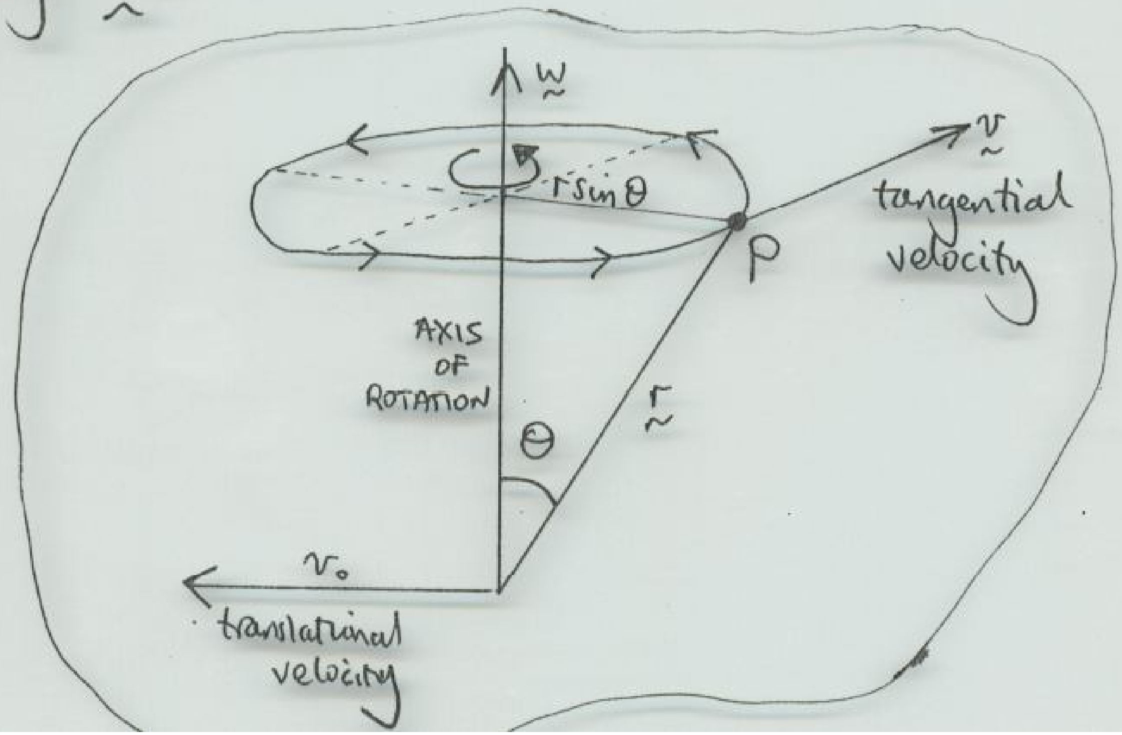
top

Let's look at two physical examples to see if some justification can be found that 'curl expresses rotation'.

An example from mechanics ...

Ex Consider a rotating body with constant angular velocity $\underset{\sim}{\omega}$ and that is also moving with a translational velocity $\underset{\sim}{v}_0$



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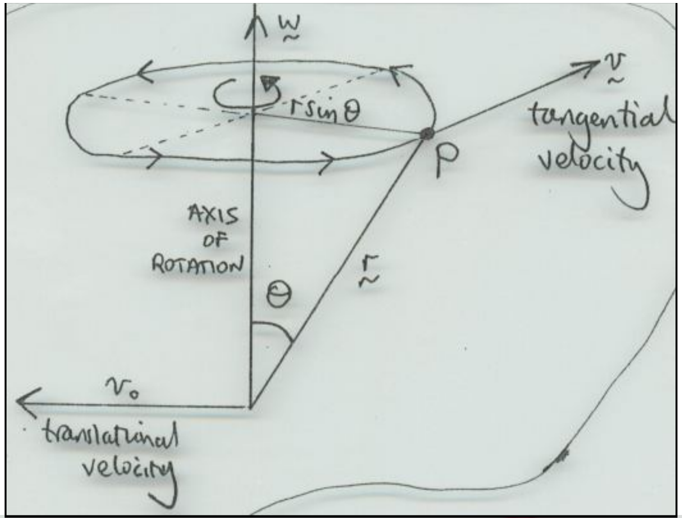
bot



TOTAL VELOCITY AT ANY POINT P ON THE TRANSLATING AND ROTATING BODY,

$$\underset{\sim}{V} = \underset{\sim}{v_0} + \underset{\sim}{v}$$

$$= \underset{\sim}{v_0} + \underset{\sim}{w} \times \underset{\sim}{r}$$

$$\text{curl } \underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \left(\underset{\sim}{\nabla} \times \underset{\sim}{v_0}\right) + \left(\underset{\sim}{\nabla} \times \underset{\sim}{w} \times \underset{\sim}{r}\right)$$

$$= \underset{\sim}{\nabla} \times \underset{\sim}{w} \times \underset{\sim}{r}$$

, for a <u>constant</u> translational velocity i.e. not space dependent.

$$\operatorname{curl} \underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \left(\underset{\sim}{\nabla} \times \underset{\sim}{v_0}\right) + \left(\underset{\sim}{\nabla} \times \underset{\sim}{w} \times \underset{\sim}{r}\right)$$

$$= \underset{\sim}{\nabla} \times \underset{\sim}{w} \times \underset{\sim}{r}$$

for

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- Evaluate $\underset{\sim}{w} \times \underset{\sim}{r}$.

$$\underset{\sim}{w} = w_x \underset{\sim}{i} + w_y \underset{\sim}{j} + w_z \underset{\sim}{k}$$

$$\underset{\sim}{r} = x \underset{\sim}{i} + y \underset{\sim}{j} + z \underset{\sim}{k}$$

$$\therefore \underset{\sim}{w} \times \underset{\sim}{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix}$$

$$= \underset{\sim}{i}\left(w_y z - w_z y\right) - \underset{\sim}{j}\left(w_x z - w_z x\right) + \underset{\sim}{k}\left(w_x y - w_y x\right)$$

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bot

Evaluate $\underset{\sim}{\nabla} \times (\underset{\sim}{w} \times \underset{\sim}{r})$, noting $\underset{\sim}{w}$ not a function of x, y or z.

$$\underset{\sim}{\nabla} \times (\underset{\sim}{w} \times \underset{\sim}{r}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_y z - w_z y & -(w_x z - w_z x) & w_x y - w_y x \end{vmatrix}$$

$$= \underset{\sim}{i}(w_x + w_x) + \underset{\sim}{j}(w_y + w_y) + \underset{\sim}{k}(w_z + w_z)$$

$$= 2\underset{\sim}{w}$$

$$\therefore \boxed{\text{curl } \underset{\sim}{V} = 2\underset{\sim}{w}}$$

i.e. curl expresses both the direction and magnitude of the rotational property of the total velocity vector.

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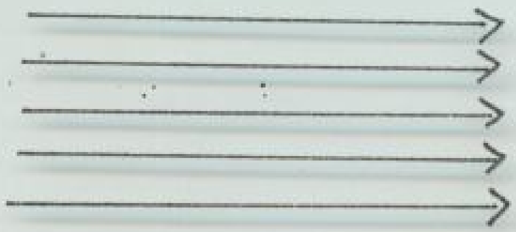
Examples from fluid dynamics ...

Ex (a)



$\underset{\sim}{V}$

UNIFORM FLOW OF FLUID
(space independent)

Say, $\underset{\sim}{V} = \rho \underset{\sim}{v}$ where ρ = fluid density
$\underset{\sim}{v}$ = fluid velocity

and $\underset{\sim}{V} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ (i.e. an example of a constant vector)

$$\Rightarrow \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & 2 & -4 \end{vmatrix} = \underset{\sim}{0} \Rightarrow$$

$\underset{\sim}{v}$ field is
IRROTATIONAL.

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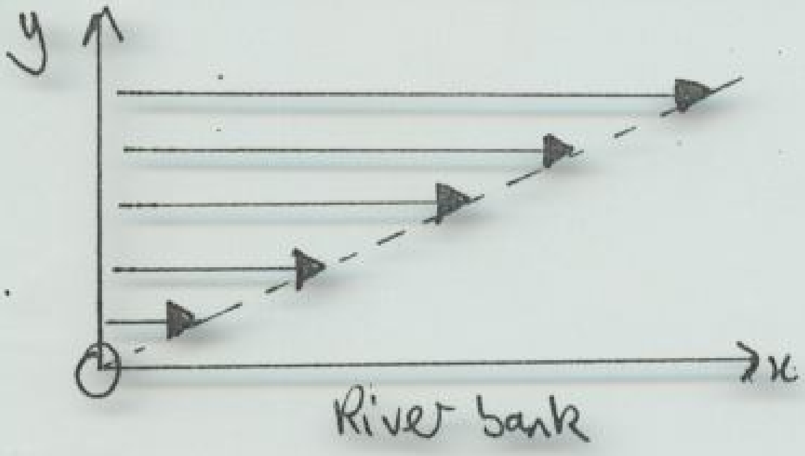
bot

(3) Flow near a river bank (at $y=0$)



$\underset{\sim}{V} = \alpha y \underset{\sim}{i}$,

for example.

i.e. x-component depends on y
and α = constant

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha y & 0 & 0 \end{vmatrix} = \underset{\sim}{k}\left(0 - \frac{\partial}{\partial y}(\alpha y)\right) = -\alpha \underset{\sim}{k}$$

(INTO the paper)

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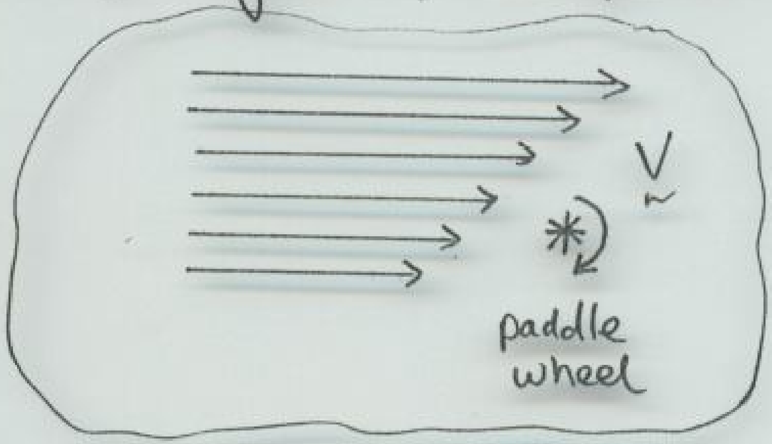
In this case, $\underset{\sim}{\nabla} \times \underset{\sim}{V} \neq \underset{\sim}{0}$: the field is ROTATIONAL and has some "circulation".

To see this, imagine a small paddle wheel in the fluid flow ...



If the fluid velocity is not uniform across the side of the wheel then the wheel will turn i.e. the paddle measures $\underset{\sim}{\nabla} \times \underset{\sim}{V}$ (or the circulation) of the flow at that point.

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bot

(c) But not all non-uniform flows have such circulation.

Consider $\underset{\sim}{V} = V_x(x)\,\underset{\sim}{i} + V_y(y)\,\underset{\sim}{j} + V_z(z)\,\underset{\sim}{k}$

i.e. x, y, z components are functions of x, y, z, respectively.

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x(x) & V_y(y) & V_z(z) \end{vmatrix} = \underset{\sim}{i}\left[\frac{\partial V_z(z)}{\partial y} - \frac{\partial}{\partial z} V_y(y)\right] - \underset{\sim}{j}\left[\frac{\partial V_z(z)}{\partial x} - \frac{\partial}{\partial z} V_x(x)\right] + \underset{\sim}{k}\left[\frac{\partial V_y(y)}{\partial x} - \frac{\partial}{\partial y} V_x(x)\right]$$

$= \underset{\sim}{0}$, since each component is <u>only</u> a function of stated coord.

Summary of grad, div and curl

(a) *Grad* operator ∇ acts on a *scalar* field to give a *vector* field
(b) *Div* operator $\nabla.$ acts on a *vector* field to give a *scalar* field
(c) *Curl* operator $\nabla \times$ acts on a *vector* field to give a *vector* field.
(d) With a *scalar function* $\phi(x, y, z)$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a *vector function* $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

(i) $$\text{div}\,\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

(ii) $$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

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Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

Ex
$$\boxed{\text{grad div } \underset{\sim}{A} = \underset{\sim}{\nabla}(\underset{\wedge}{\nabla} . \underset{\wedge}{A})}$$

If $\underset{\sim}{A} = x^2 y \underset{\wedge}{i} + yz^3 \underset{\sim}{j} - zx^3 \underset{\sim}{k} = a_x \underset{\wedge}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k}$

then $\text{div} \underset{\sim}{A} = \underset{\wedge}{\nabla} . \underset{\wedge}{A} = \dfrac{\partial a_x}{\partial x} + \dfrac{\partial a_y}{\partial y} + \dfrac{\partial a_z}{\partial z}$

$= 2xy + z^3 - x^3$

$= \phi(x,y,z)$, a scalar field.

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top

$$\phi(x,y,z) = \underset{\sim}{\nabla} \cdot \underset{\sim}{A} = 2xy + z^3 - x^3$$

Now, div and curl both act on vector fields but grad acts on a scalar field.

e.g.
$$\text{grad}\,(\text{div}\,\underset{\sim}{A}) = \underset{\sim}{\nabla}\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{A}\right)$$

$$= \frac{\partial \phi}{\partial x}\underset{\sim}{i} + \frac{\partial \phi}{\partial y}\underset{\sim}{j} + \frac{\partial \phi}{\partial z}\underset{\sim}{k}$$

$$= (2y - 3x^2)\underset{\sim}{i} + 2x\underset{\sim}{j} + 3z^2\underset{\sim}{k}.$$

Ex $\boxed{\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi)}$

H4
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bot

If scalar field $\phi = xyz - 2y^2 z + x^2 z^2$

then $\text{grad } \phi = \underset{\sim}{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$

$$= (yz + 2xz^2)\underset{\sim}{i} + (xz - 4yz)\underset{\sim}{j} + (xy - 2y^2 + 2x^2 z)\underset{\sim}{k}$$

i.e. a vector field

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top

$$\text{grad}\,\phi = \underset{\sim}{\nabla}\phi = (yz + 2xz^2)\underset{\sim}{i} + (xz - 4yz)\underset{\sim}{j} + (xy - 2y^2 + 2x^2z)\underset{\sim}{k}$$

and

$$\text{div grad}\,\phi = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\phi)$$

$$= \frac{\partial}{\partial x}(yz + 2xz^2) + \frac{\partial}{\partial y}(xz - 4yz) + \frac{\partial}{\partial z}(xy - 2y^2 + 2x^2z)$$

$$= 2z^2 - 4z + 2x^2 .$$

H4
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bot

Ex $\boxed{\text{curl curl } \mathbf{A} = \nabla \times (\nabla \times \mathbf{A})}$

If vector field $\mathbf{A} = x^2yz\,\mathbf{i} + xyz^2\,\mathbf{j} + y^2z\,\mathbf{k}$

then $\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix}$

$$= \mathbf{i}\,(2yz - 2xyz) - \mathbf{j}\,(-x^2y) + \mathbf{k}\,(yz^2 - x^2z).$$

$$\text{curl}\,\underset{\sim}{A} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = \underset{\sim}{i}\left(2yz - 2xyz\right) - \underset{\sim}{j}\left(-x^2 y\right) + \underset{\sim}{k}\left(yz^2 - x^2 z\right)$$

H4
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top

The result of $\underset{\sim}{\nabla} \times \underset{\sim}{A}$ is another vector field so we can take the curl of this new field...

$$\text{curl}\,\text{curl}\,\underset{\sim}{A} = \underset{\sim}{\nabla} \times \left(\underset{\sim}{\nabla} \times \underset{\sim}{A}\right) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz - 2xyz & x^2 y & yz^2 - x^2 z \end{vmatrix}$$

$$= \underset{\sim}{i}\, z^2 - \underset{\sim}{j}\left(-2xz - 2y + 2xy\right) + \underset{\sim}{k}\left(2xy - 2z + 2xz\right).$$

H4
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bot

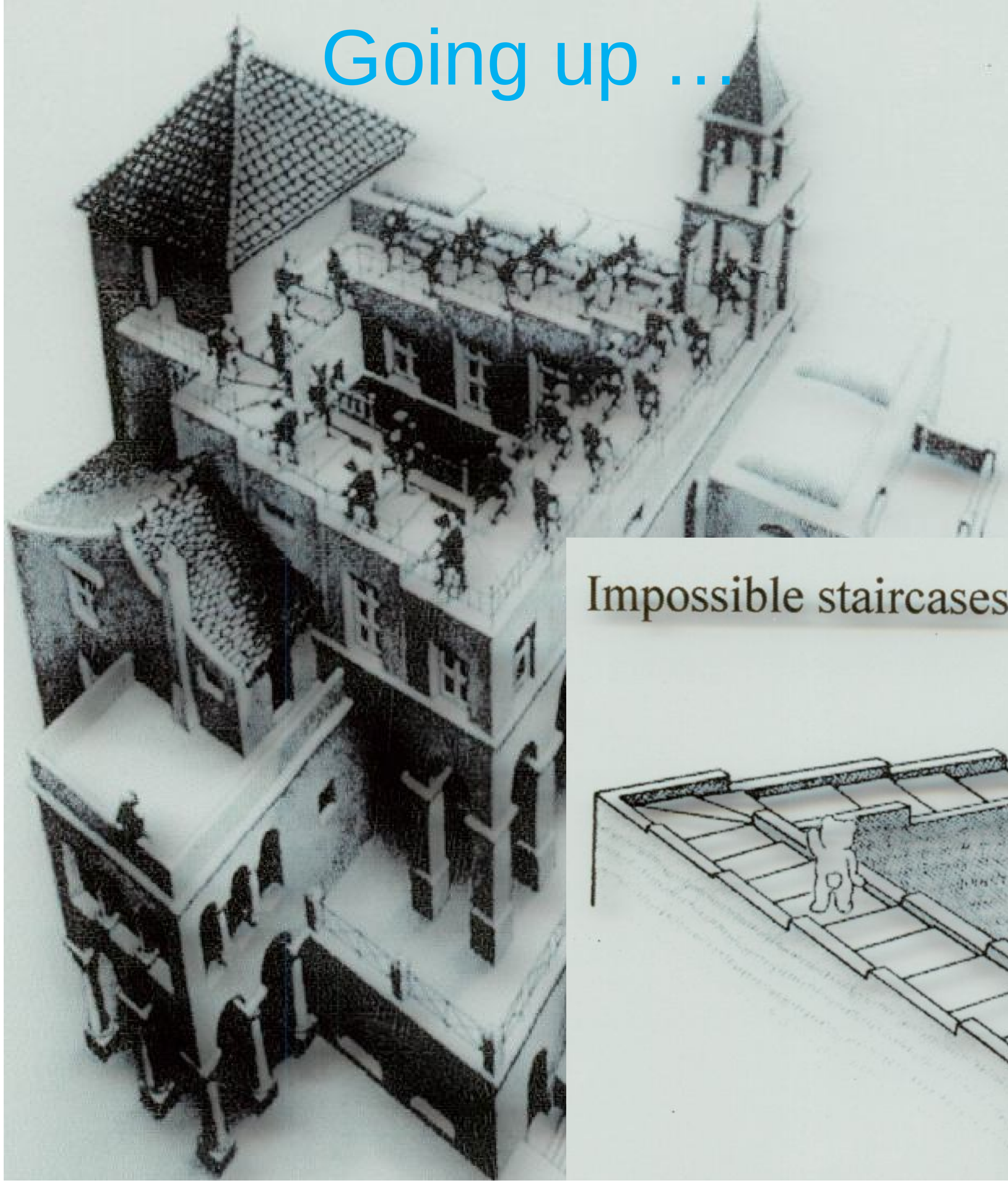
The following three multiple operations lead to general results (often called 'vector identities')

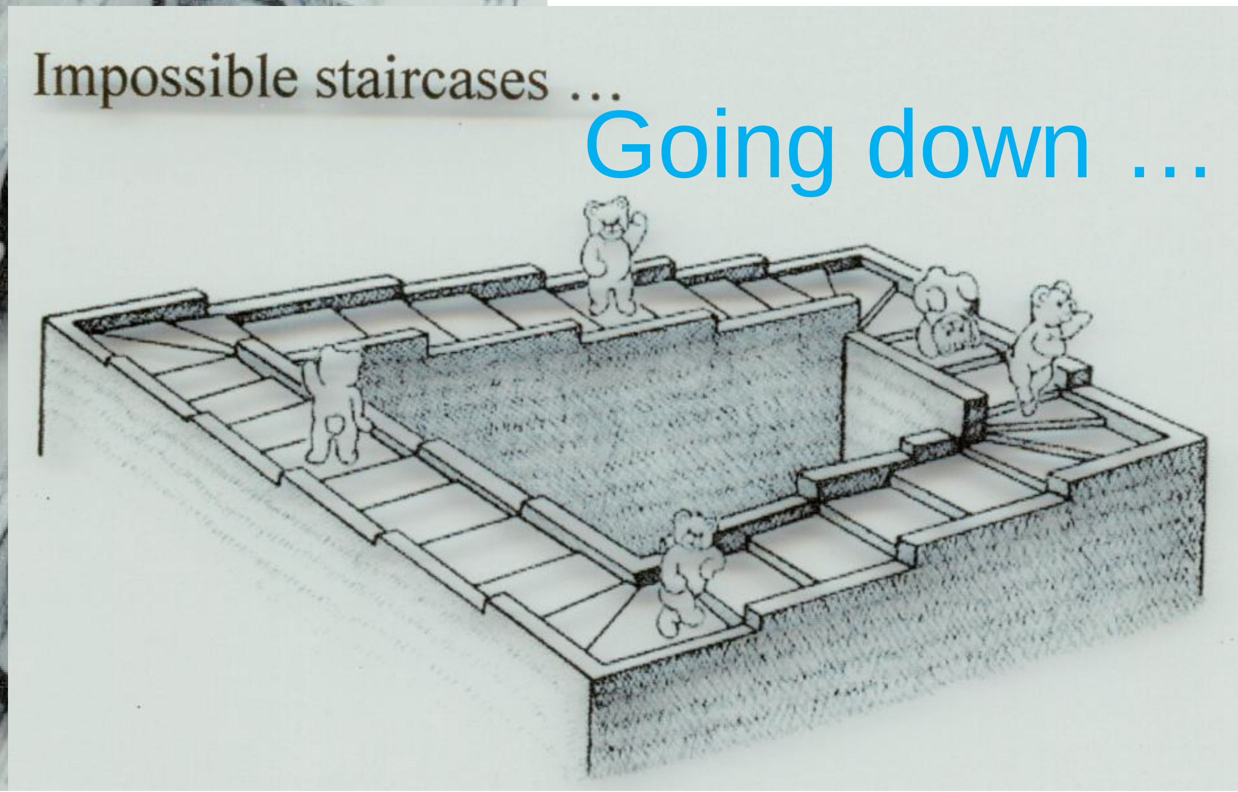
(a) $\boxed{\text{curl grad } \phi \quad \text{, where } \phi \text{ is scalar field}}$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \underset{\sim}{i} + \frac{\partial \phi}{\partial y} \underset{\sim}{j} + \frac{\partial \phi}{\partial z} \underset{\sim}{k}$$

H4
extra

curl grad ϕ ?





$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x}\underset{\sim}{i} + \frac{\partial \phi}{\partial y}\underset{\sim}{j} + \frac{\partial \phi}{\partial z}\underset{\sim}{k}$$

H4
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top

Then, $\text{curl}\,\text{grad}\,\phi = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$

$$= \underset{\sim}{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right] - \underset{\sim}{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right] + \underset{\sim}{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$= \underset{\sim}{0}$$

H4
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bot

i.e. $$\text{curl grad}\,\phi = \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\phi) = \underset{\sim}{0}$$

TRUE FOR ANY SCALAR FIELD ϕ

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top

(b) $\boxed{\operatorname{div} \operatorname{curl} \underset{\sim}{A} = \underset{\sim}{\nabla} \cdot \left(\underset{\sim}{\nabla} \times \underset{\sim}{A} \right)}$

Let $\underset{\sim}{A} = a_x \underset{\sim}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k}$, then

$$\operatorname{curl} \underset{\sim}{A} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \underset{\sim}{i} \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) - \underset{\sim}{j} \left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right) + \underset{\sim}{k} \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right)$$

$$\underset{\sim}{A} = a_x \underset{\sim}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k}$$

H4
p122
mid

$$\text{curl}\,\underset{\sim}{A} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = \underset{\sim}{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

and

$$\text{div curl}\,\underset{\sim}{A} = \underset{\sim}{\nabla}\cdot\left(\underset{\sim}{\nabla}\times\underset{\sim}{A}\right) = \frac{\partial}{\partial x}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y}$$

$$= 0.$$

H4
p122
bot

i.e.

$$\boxed{\begin{array}{l} \text{div curl } \underset{\sim}{A} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{A}) = 0 . \\ \\ \text{TRUE FOR ANY VECTOR FIELD } \underset{\sim}{A} \end{array}}$$

H4
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top

(c) $\boxed{\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi)}$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \underset{\sim}{i} + \frac{\partial \phi}{\partial y} \underset{\sim}{j} + \frac{\partial \phi}{\partial z} \underset{\sim}{k}$$

$$\therefore \text{ div grad} \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

i.e. $\boxed{\text{div grad} \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$

div grad

H4
p123
bot

In physics, we commonly write the divgrad operator as

$$\boxed{\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}.$$

It is an important operator in its own right and it is usually called

THE LAPLACIAN.

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$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Note that the Laplacian is written without an underscore; it is a scalar differential operator.

This means that either $\nabla^2 \phi$ or $\nabla^2 \underset{\sim}{V}$ are possible, where ϕ is a scalar field and $\underset{\sim}{V}$ is a vector field.

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bot

The Laplacian appears in numerous important equations such as...

$$\nabla^2 \phi = 0$$: LAPLACE'S EQUATION

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$: THE WAVE EQUATION

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial \phi}{\partial t}$$: DIFFUSION or HEAT CONDUCTION EQUATION

∇^2 arises in heat, hydrodynamics, electricity, magnetism, aerodynamics, elasticity, optics, quantum mechanics, and more!!

Particular examples of the Laplacian in electrostatics ...

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"Gauss's Law"

$$\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho/\varepsilon_0}$$

where $\underset{\sim}{E}$ = electric field

ρ = charge density

Then, since $\underset{\sim}{E} = -\underset{\sim}{\nabla} V$

where V = (scalar) potential function

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho / \varepsilon_0$$

$$\underset{\sim}{E} = -\underset{\sim}{\nabla} V$$

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$$\underset{\sim}{\nabla} \cdot \left(-\underset{\sim}{\nabla} V \right) = \rho / \varepsilon_0$$

giving

"POISSON'S EQUATION"

$$\boxed{\nabla^2 V = -\rho / \varepsilon_0}$$

If there is no charge

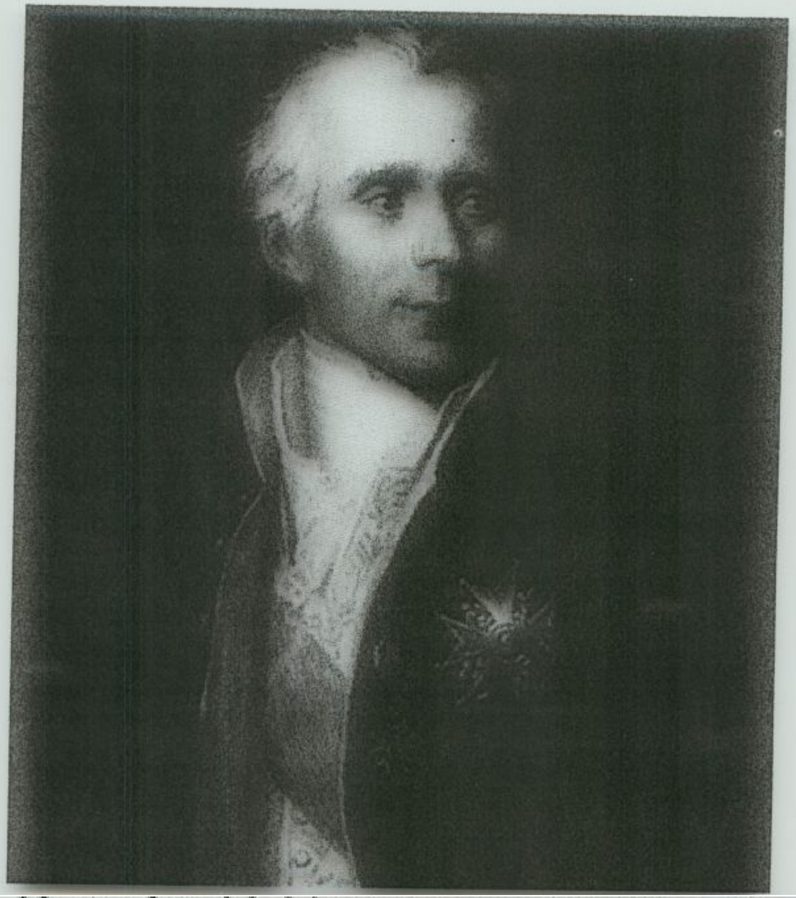
i.e. $\rho = 0$, we get

"LAPLACE'S EQUATION"

$$\boxed{\nabla^2 V = 0}$$

Pierre-Simon Laplace

Born: 23 March 1749 in Beaumont-en-Auge, Normandy, France
Died: 5 March 1827 in Paris, France



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degree, and went to Paris. He took with him a letter of introduction to d'Alembert from Le Canu, his teacher at Caen. Although Laplace was only 19 years old when he arrived in Paris he quickly impressed d'Alembert. Not only did d'Alembert begin to direct Laplace's mathematical studies, he also tried to find him a position to earn enough money to support himself in Paris. Finding a

Revision Summary

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If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; $\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$;
then we have the following relationships.

1. *Scalar product* (dot product) $\mathbf{A}.\mathbf{B} = AB\cos\theta$

$$\mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A} \quad \text{and} \quad \mathbf{A}.(\mathbf{B}+\mathbf{C}) = \mathbf{A}.\mathbf{B} + \mathbf{A}.\mathbf{C}$$

etc...

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4. *Scalar triple product* $\mathbf{A}.(\mathbf{B}\times\mathbf{C})$

From Determinant properties ***OR*** Parallelepiped volumes

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \mathbf{B}.(\mathbf{C}\times\mathbf{A}) = \mathbf{C}.(\mathbf{A}\times\mathbf{B})$$

Unchanged by cyclic change of vectors.
Sign reversed by non-cyclic change of vectors.

7. *Differentiation of vectors*

If $\mathbf{A}$, a_x, a_y, a_z are functions of u,

$$\frac{d\mathbf{A}}{du} = \frac{da_x}{du}\mathbf{i} + \frac{da_y}{du}\mathbf{j} + \frac{da_z}{du}\mathbf{k}$$

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Note similarity,
where *u* is a scalar variable
such as single time or space variable

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9. *Integration of vectors*

$$\int_a^b \mathbf{A}\,du = \mathbf{i}\int_a^b a_x\,du + \mathbf{j}\int_a^b a_y\,du + \mathbf{k}\int_a^b a_z\,du$$