Mathematical Methods and Applications

HANDOUT 6

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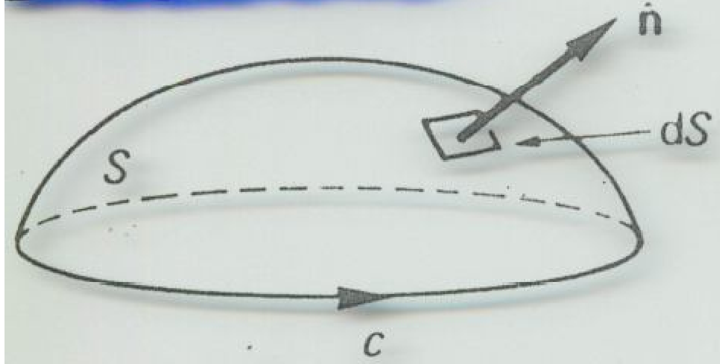
- VECTOR CALCULUS (concluded)

- Stoke's Theorem
  - proof
  - applications
- Conservative Fields – Revisited
  - the five equivalent conditions
  - examples of conservative fields

then …

- Examples of solenoidal fields (zero divergence everywhere)
- Alternative space coordinate systems (reference material)

Stokes Theorem



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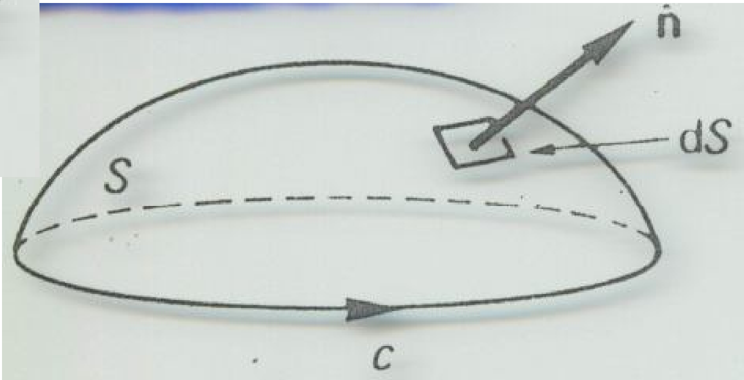
If **F** is a vector field existing over an open surface S and around its boundary closed curve c, then

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$$

where $\oint \underset{\sim}{F} \cdot d\underset{\sim}{r}$ is called THE CIRCULATION OF $\underset{\sim}{F}$ AROUND THE CURVE C.

$$\int_S \mathbf{curl\ F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



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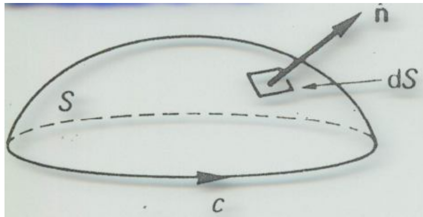
Stoke's theorem expresses the relationship between

$\nabla \times F$ and the 'circulation'.

For example, recall the paddle wheel on page 116.

One can quantify the rotational character of the field by working out the circulation around the paddle wheel.

An important convention ...



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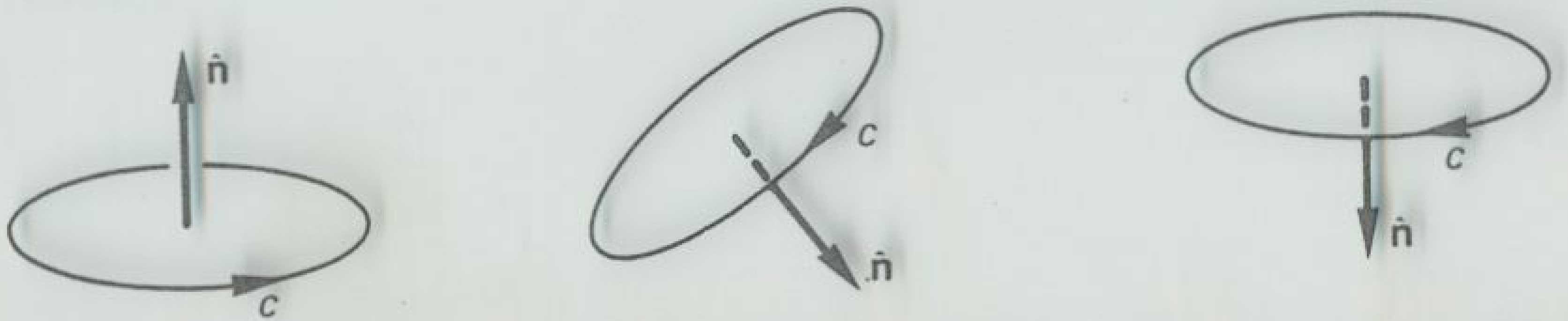
Direction of unit normal vectors to a surface S

When we were dealing with the divergence theorem, the normal vectors were drawn in a direction outward from the enclosed region.

With an open surface, as we now have, there is, in fact, no inward or outward direction. With any general surface, a normal vector can be drawn in either of two opposite directions. To avoid confusion, a convention must therefore be agreed upon and the established rule is as follows.



A unit normal $\hat{\mathbf{n}}$ is drawn perpendicular to the surface S at any point in the direction indicated by applying a right-handed screw sense to the direction of integration round the boundary c.

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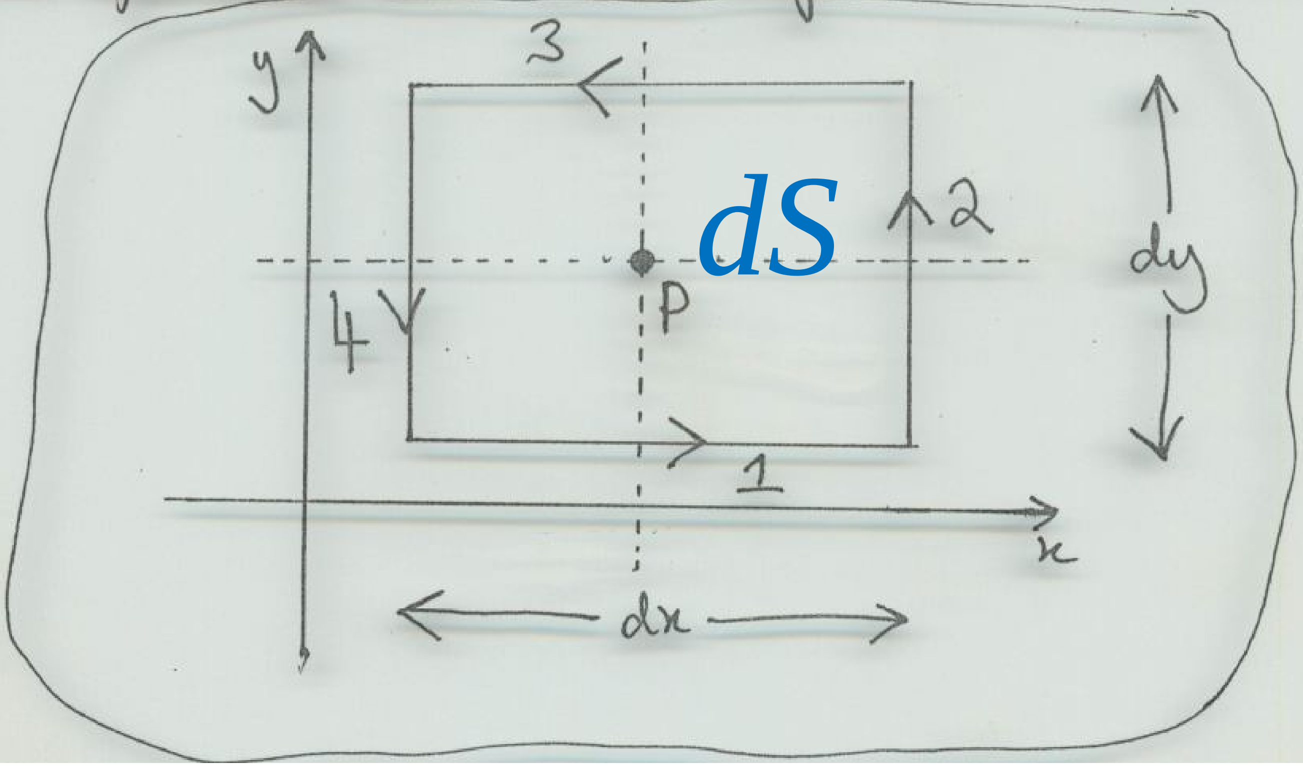
Proof of Stoke's Theorem

1

Consider a vector field $\underset{\sim}{V}$ and an area element $dS = dx dy$ which, for simplicity, lies in the xy plane.

Consider an infinitesimal area (drawn large to aid visualisation!)



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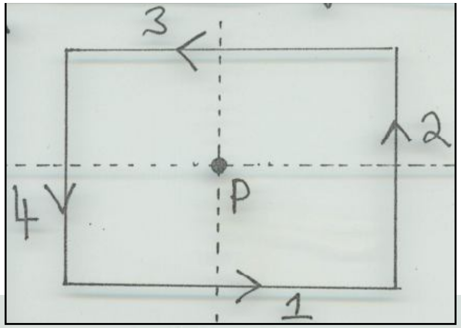

<u>Convention</u> The circulation is defined around the curve such that the area enclosed is kept to the LEFT (as above). This corresponds to a CLOCKWISE navigation around a normal to this area coming OUT of the page.

Let $\underset{\sim}{V} = (V_x, V_y)$ at P in the centre of area element.

The circulation

$$\oint \underset{\sim}{V} \cdot d\underset{\sim}{r} = \int_{\text{along } 1} \underset{\sim}{V} \cdot d\underset{\sim}{r} + \int_{\text{along } 2} \underset{\sim}{V} \cdot d\underset{\sim}{r} + \int_{\text{along } 3} \underset{\sim}{V} \cdot d\underset{\sim}{r} + \int_{\text{along } 4} \underset{\sim}{V} \cdot d\underset{\sim}{r}$$

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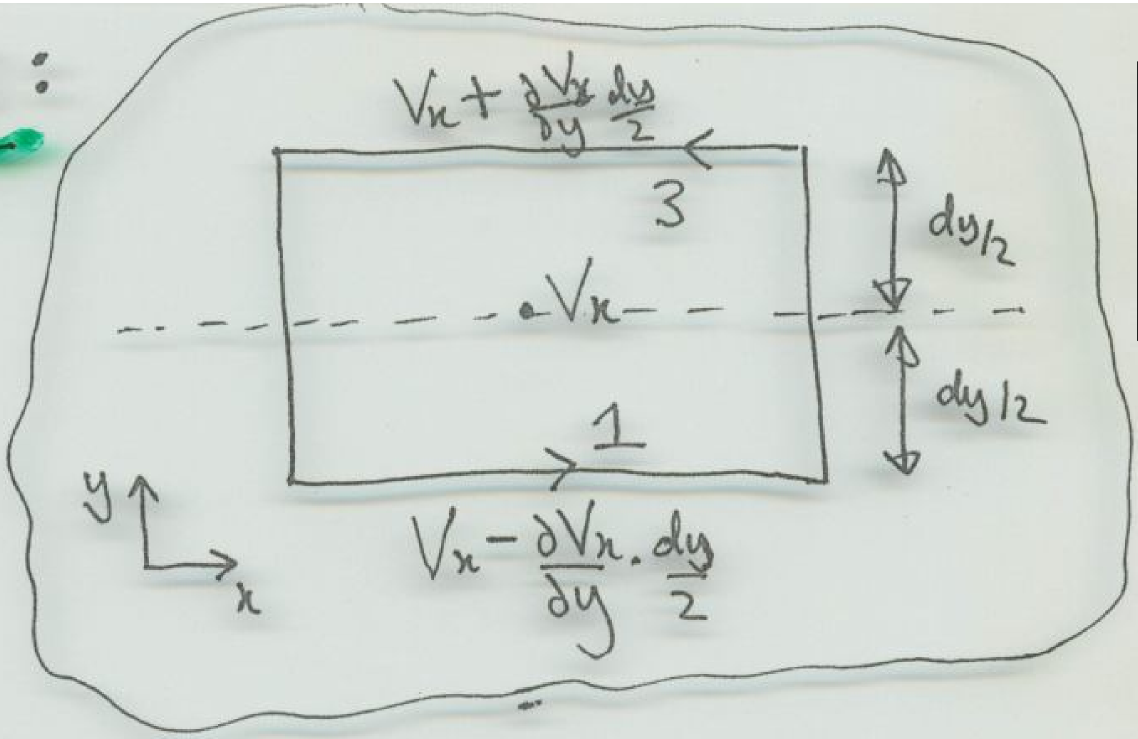
Along sides 1 and 3 :

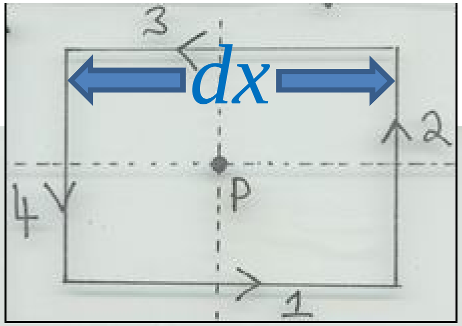


We have $V_x \mp \frac{\partial V_x}{\partial y}\frac{dy}{2}$ and $d\underset{\sim}{r} \to dx$ only.

$$\therefore \int_{\text{along } 1} \underset{\sim}{V} \cdot d\underset{\sim}{r} = \left(V_x - \frac{\partial V_x}{\partial y}\frac{dy}{2}\right)dx \,, \quad \int_{\text{along } 3} \underset{\sim}{V} \cdot d\underset{\sim}{r} = -\left(V_x + \frac{\partial V_x}{\partial y}\frac{dy}{2}\right)dx$$

(we are going in the negative x-direction this time)

$dr \to dy$ only



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Along sides 2 and 4 :

$$\int_{\text{along } 2} \underset{\sim}{V} \cdot d\underset{\sim}{r} = \left(V_y + \frac{\partial V_y}{\partial x} \frac{dx}{2} \right) dy \,, \quad \int_{\text{along } 4} \underset{\sim}{V} \cdot d\underset{\sim}{r} = - \left(V_y - \frac{\partial V_y}{\partial x} \frac{dx}{2} \right) dy$$

$$\therefore \boxed{\oint \underset{\sim}{V} \cdot d\underset{\sim}{r} = \underbrace{\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)}_{z\text{-component of } \nabla \times V} dx\,dy}$$

when contributions from the four sides are added

z-component of $\underset{\sim}{\nabla} \times \underset{\sim}{V}$ since

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\oint \underset{\sim}{V}\cdot d\underset{\sim}{r} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy$$

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Denote the element area as $dS = dx\,dy$

and the unit normal to this area as $\hat{\underset{\sim}{n}}$.

Then the z-component of curl $\underset{\sim}{V}$ is $(\text{curl}\, \underset{\sim}{V})\cdot\hat{\underset{\sim}{n}}$

i.e.
$$\boxed{\oint \underset{\sim}{V}\cdot d\underset{\sim}{r} = (\underset{\sim}{\nabla}\times\underset{\sim}{V})\cdot\hat{\underset{\sim}{n}}\,dS}$$

or,
$$\boxed{\oint \underset{\sim}{V}\cdot d\underset{\sim}{r} = (\underset{\sim}{\nabla}\times\underset{\sim}{V})\cdot d\underset{\sim}{S}}$$

FOR AN ELEMENT OF AREA $d\underset{\sim}{S}$

- The above is true also for an element in 3D space by allowing the normal to this element to point in any appropriate direction

- Let's build up a 3D surface by tiling it with lots of *no gaps* such elements...

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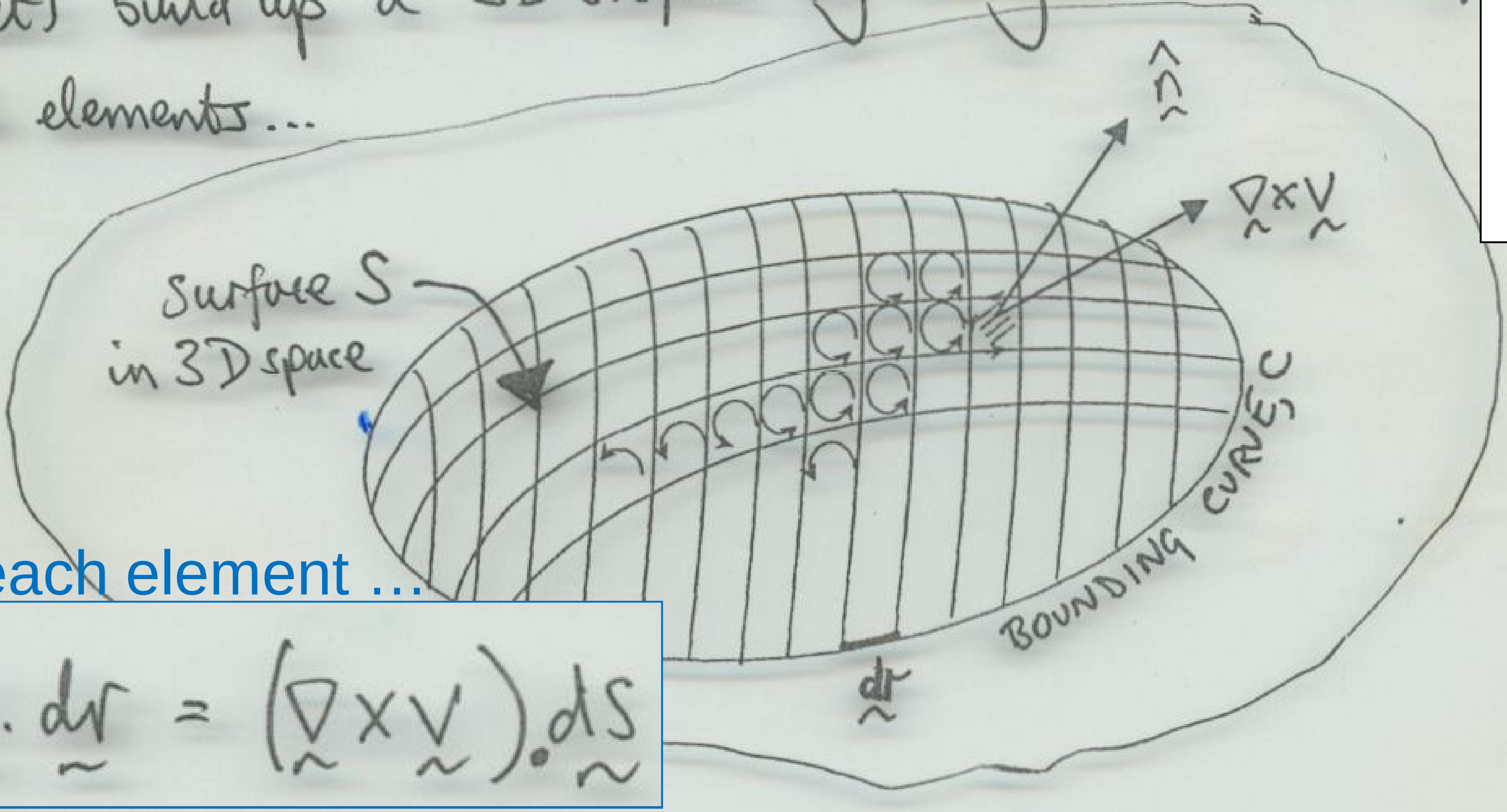


For each element ...

$$\oint \mathbf{V} \cdot d\mathbf{r} = (\nabla \times \mathbf{V}) \cdot d\mathbf{S}$$

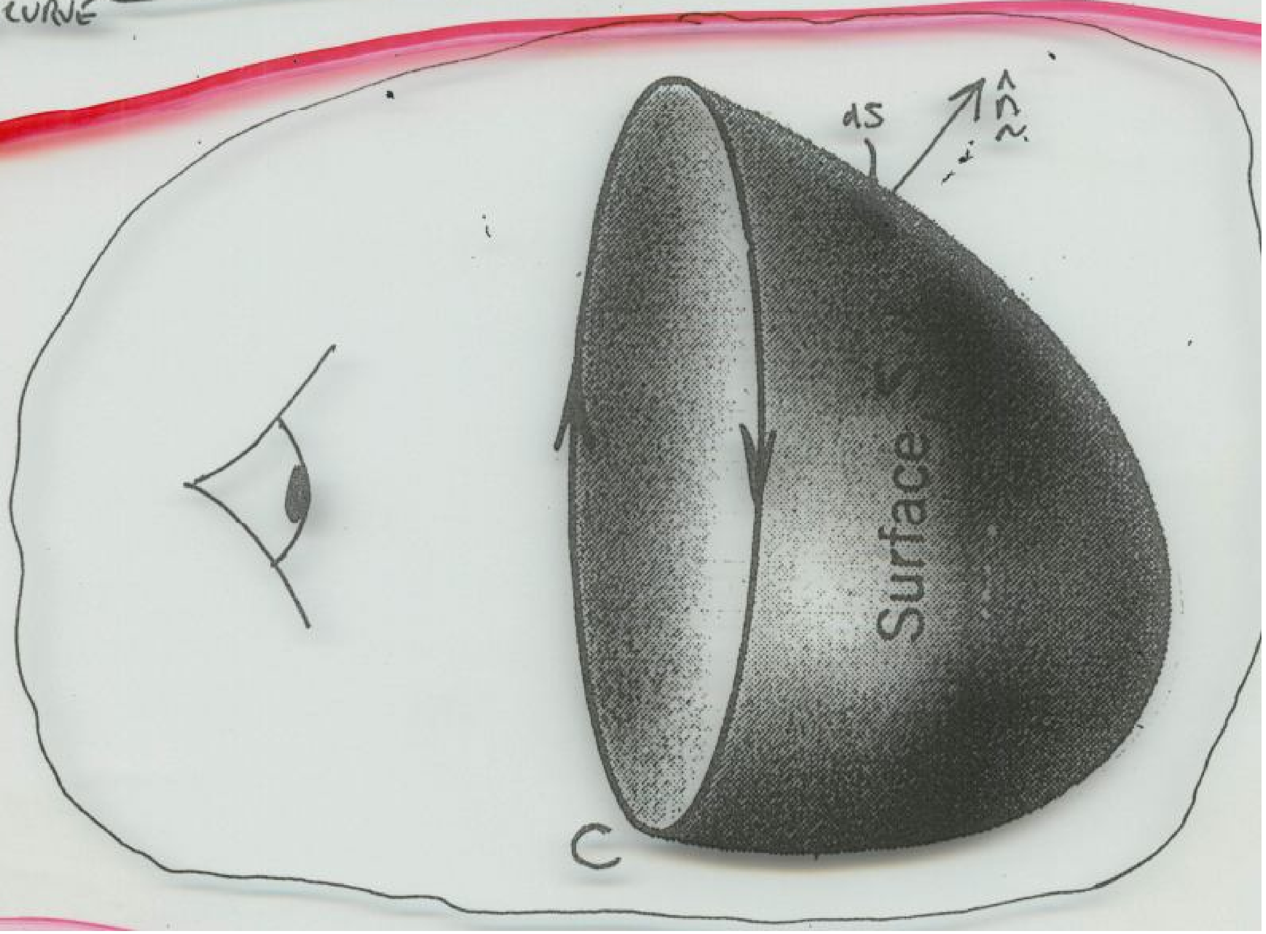
- Adjacent line integrals cancel out, leaving just the line integral around the bounding curve.
- Adding up all the surface elements turns the right-hand side of $\oint \mathbf{V} \cdot d\mathbf{r}$ into a surface integral.

This gives **STOKE'S THEOREM**

$$\oint_C \mathbf{V}\cdot d\mathbf{r} = \int_S (\nabla\times\mathbf{V})\cdot d\mathbf{S}$$

LINE INTEGRAL AROUND THE BOUNDING CURVE

SURFACE INTEGRAL OF $\nabla\times\mathbf{V}$



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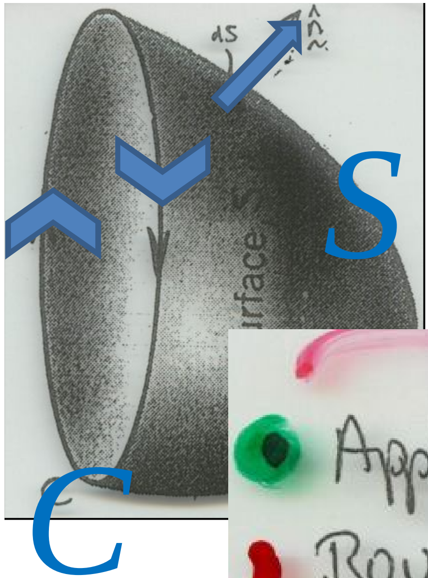

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- Applies to an OPEN SURFACE S having a BOUNDING CURVE C.
- All the $\hat{n}$'s point outwards and this gives the CLOCKWISE directional sense around C.
- It's like a fishing net where the surface is composed of all the elemental loops (the net itself) and the bounding curve is the rim of the net.

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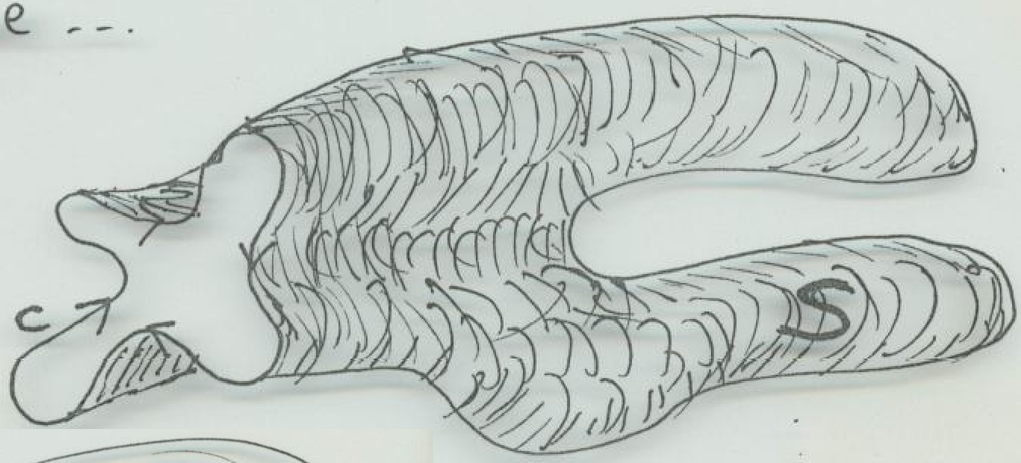
AND the bounding curve doesn't need to lie in a plane

AND we haven't specified what the surface looks like

we could have ...



or ...

and we still have

$$\oint_C \underset{\sim}{v} \cdot d\underset{\sim}{r} = \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{v}) \cdot d\underset{\sim}{S} \ !$$

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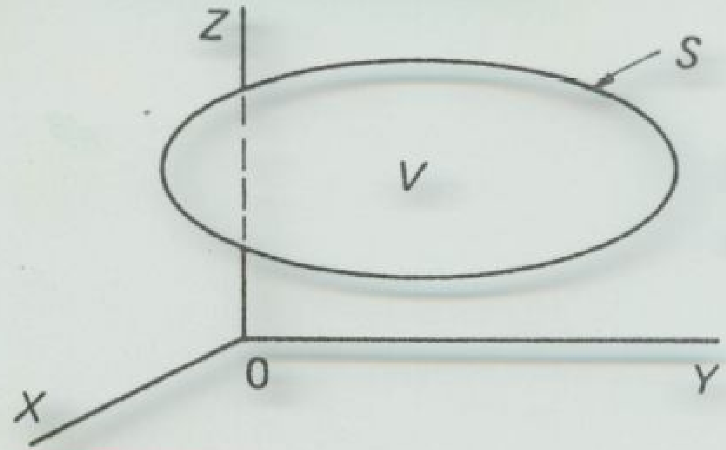
A comparison ...

Divergence theorem (Gauss' theorem)



Closed surface S enclosing a region V in a vector field $\mathbf{F}$.

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

Quantifies the net flux through S and relates this to $\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$ (the volume density of sources and sinks of flux).

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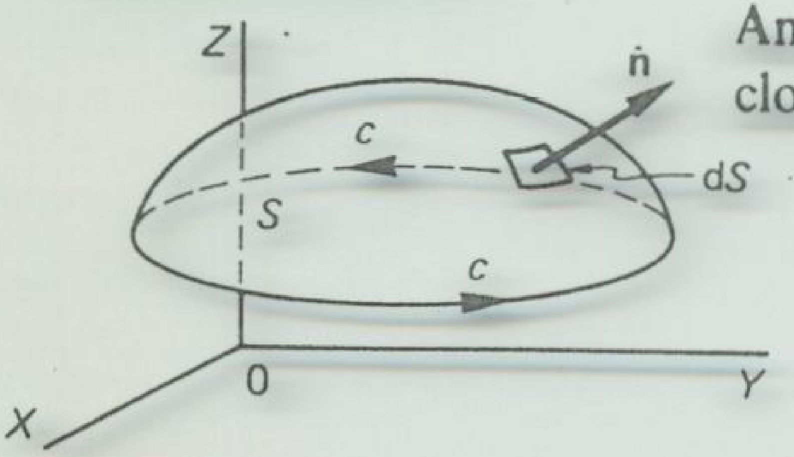
Stokes theorem



An open surface S bounded by a simple closed curve c, then

$$\int_S \text{curl } \mathbf{F} \cdot dS = \oint_c \mathbf{F} \cdot d\mathbf{r}$$

Quantifies the circulation (twist/swirl/rotation/vorticity) of the field around curve C and relates this to $(\underset{\sim}{\nabla} \times \underset{\sim}{\mathbf{F}}) \cdot \underset{\sim}{\hat{n}}$ (the surface density of the circulation).

George Gabriel Stokes

Born: 13 Aug 1819 in Skreen, County Sligo, Ireland
Died: 1 Feb 1903 in Cambridge, Cambridgeshire, England

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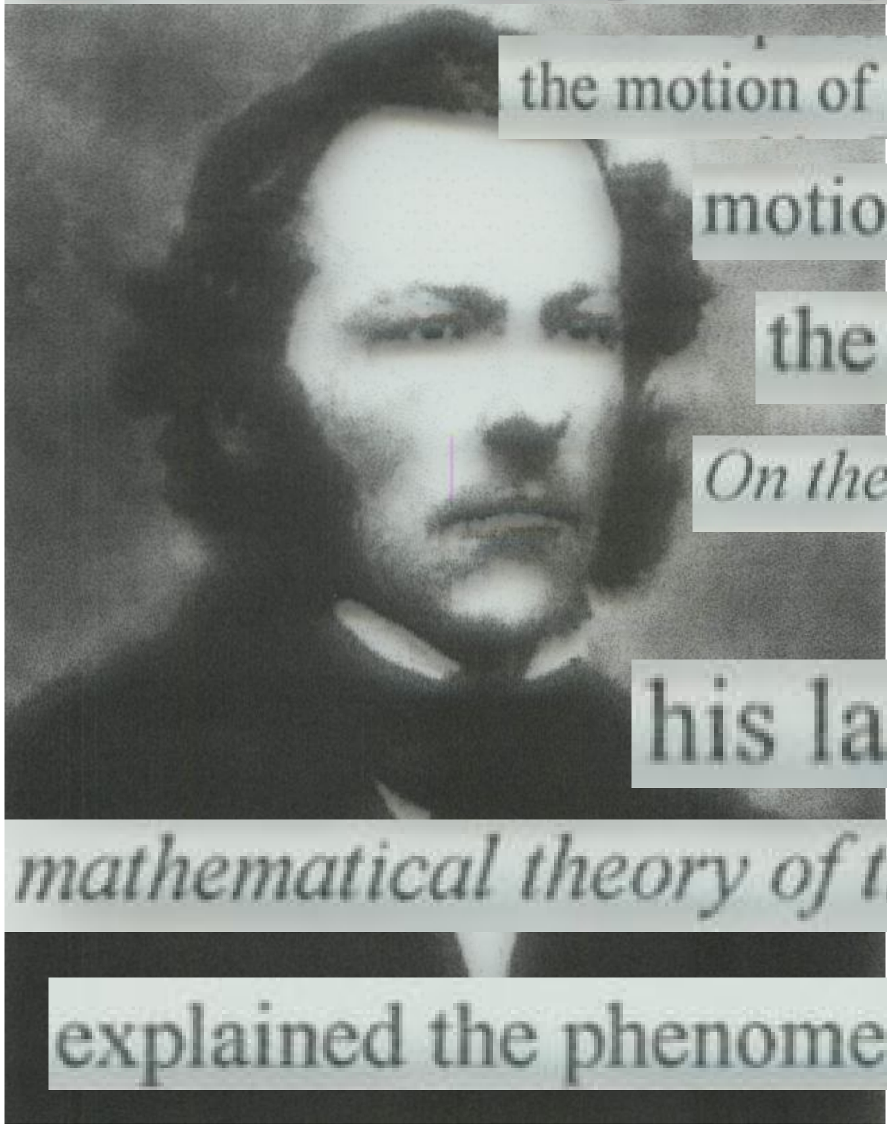

the motion of incompressible fluids in 1842 and 1843

motion of pendulums in fluids

the aberration of light

On the variation of gravity at the surface of the earth in 1849

his law of viscosity

mathematical theory of the phenomenon of diffraction

explained the phenomenon of fluorescence in 1852

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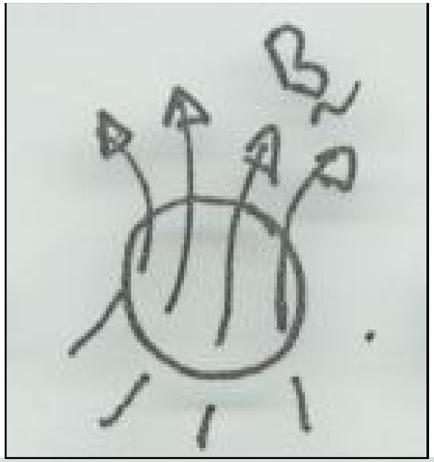
Applications of Stoke's theorem

Ex 1 Induction currents.

Consider a loop of wire 

. If we wave this through a magnetic field $\underset{\sim}{B}$ then currents are induced in the wire.

More precisely, the change in the magnetic flux Φ gives rise to an electric field $\underset{\sim}{E}$ that, in turn, gives rise to the current.



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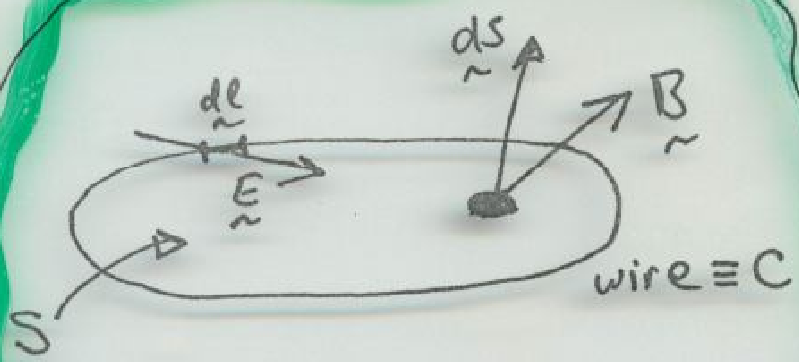
The circuital law,

$$\oint_C \underset{\sim}{E}\cdot d\underset{\sim}{l} = -\frac{\partial \Phi}{\partial t}$$

$$= -\frac{\partial}{\partial t}\int_S \underset{\sim}{B}\cdot d\underset{\sim}{S}$$



dS is area element of open surface bounded by curve C

$$\therefore \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{E})\cdot d\underset{\sim}{S} = -\frac{\partial}{\partial t}\int_S \underset{\sim}{B}\cdot d\underset{\sim}{S}$$

(using Stoke's theorem)

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$$\therefore \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{E}) \cdot d\underset{\sim}{S} = -\frac{\partial}{\partial t} \int_S \underset{\sim}{B} \cdot d\underset{\sim}{S} \qquad \text{(using Stoke's theorem)}$$

i.e.

$$\boxed{\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}}$$

This is the differential form of one of Maxwell's equations.

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Ex 2 There is an interplay between the magnetic field and moving charges such that moving charges give rise to twists (circulation) in the magnetic field.

Ampère's law in differential form:

$$\boxed{\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}}$$

where $\underset{\sim}{J}$ = electric current density (A/m^2)

Stoke's theorem $\Rightarrow$ $\displaystyle\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{B}) . d\underset{\sim}{S} = \oint_C \underset{\sim}{B} . d\underset{\sim}{r}$

while current, $I = \displaystyle\int_S \underset{\sim}{J} . d\underset{\sim}{S}$

$$\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{B} \cdot d\underset{\sim}{r}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$$

$$I = \int_S \underset{\sim}{J} \cdot d\underset{\sim}{S}$$

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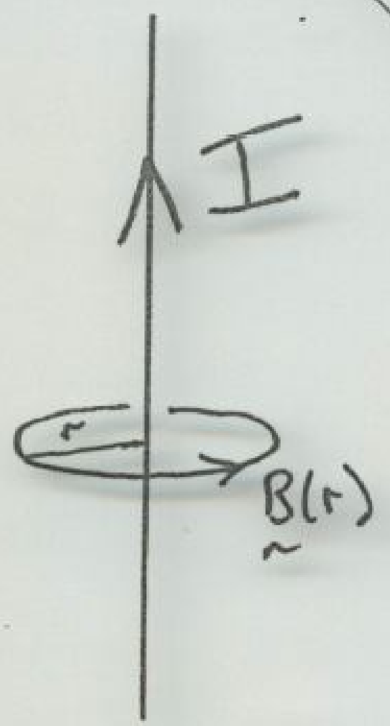
i.e. $$\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{r} = \mu_0 \int_S \underset{\sim}{J} \cdot d\underset{\sim}{S} = \mu_0 I$$

... Ampères law in integral form



$$B \times 2\pi r = \mu_0 I$$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

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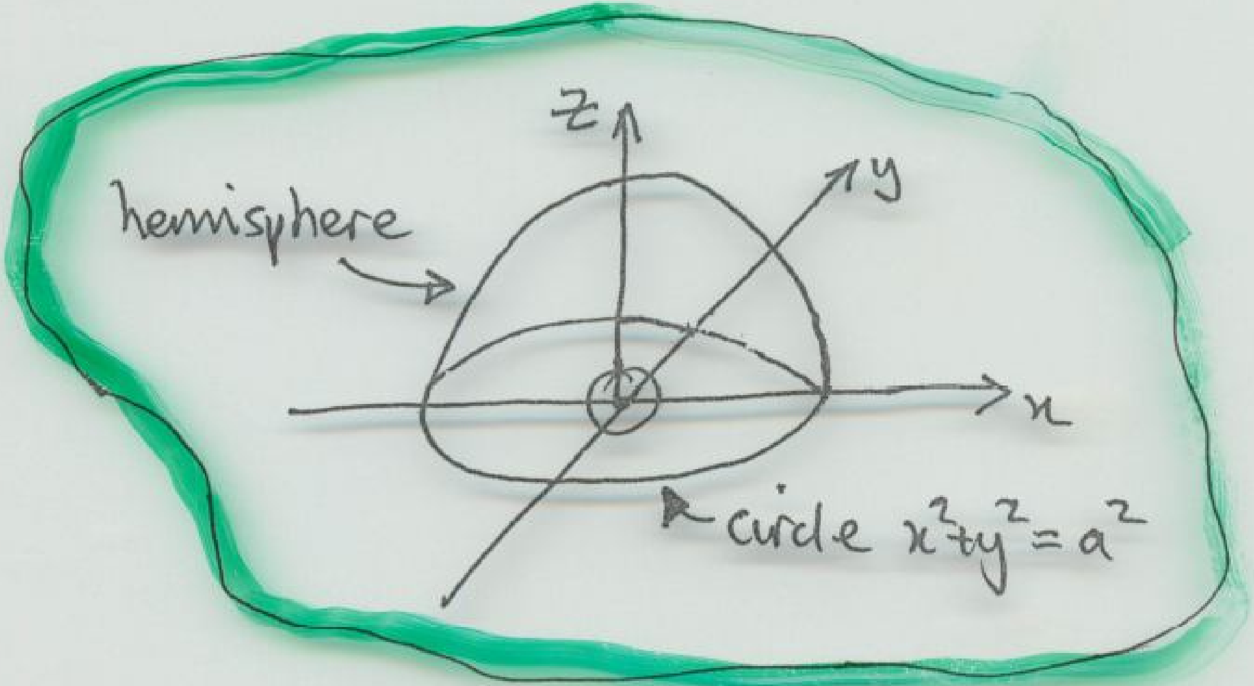
Ex 3 Given a vector field $\underset{\sim}{V} = 4y\underset{\sim}{i} + x\underset{\sim}{j} + 2z\underset{\sim}{k}$

find $\int (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \hat{\underset{\sim}{n}} \, dS$ over the hemisphere

$x^2+y^2+z^2=a^2, \; z \geqslant 0$

i.e.



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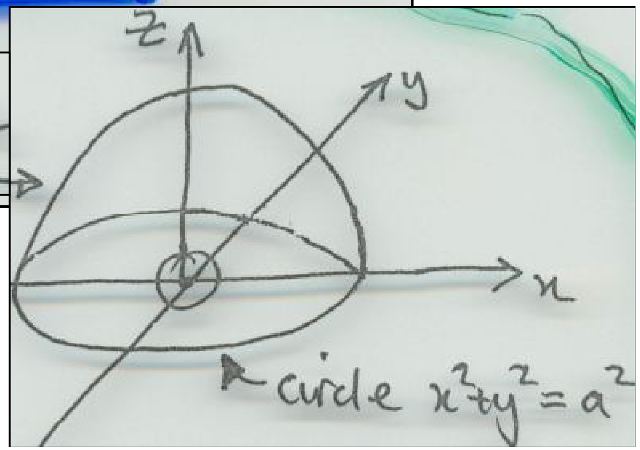
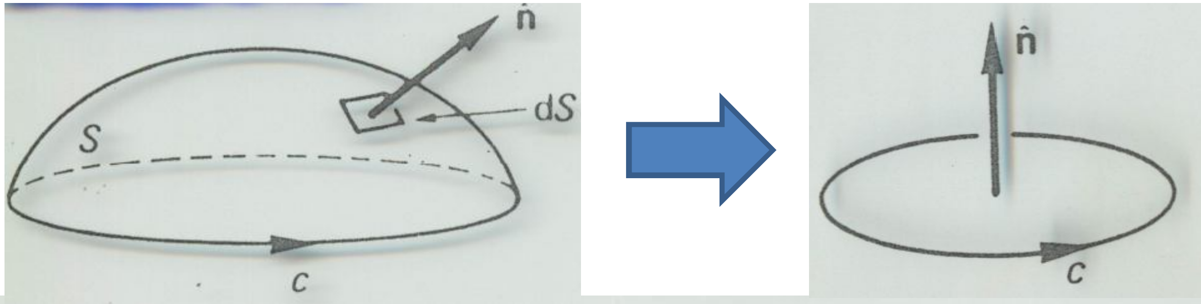
Ans Looks like this integral might be a bit difficult.

BUT, Stoke's theorem implies that the integral is the same over any surface bounded by the circle at $z=0$ i.e. the bounding curve given by $x^2+y^2=a^2$.

So, let's use the plane area inside the circle for this surface integral.

hemisphere





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Recall that all the unit normal vectors $\hat{\underset{\sim}{n}}$ of the hemisphere point outwards. Imagine the hemisphere "deflating" onto the circle with the $\hat{\underset{\sim}{n}}$ vectors still pointing outwards.

Then, for the circle let's choose $\hat{\underset{\sim}{n}} = \underset{\sim}{k}$ (pointing upwards). This would then define which direction one would calculate the circulation around the circle.

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On the xy plane, we have $z=0$ and $\underset{\sim}{V} = 4y\,\underset{\sim}{i} + x\,\underset{\sim}{j}$.

$$\underset{\sim}{V} = 4y\,\underset{\sim}{i} + x\,\underset{\sim}{j} + 2z\,\underset{\sim}{k}$$

$$\int (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \hat{\underset{\sim}{n}}\, dS$$

Then,

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & x & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}\left[\frac{\partial(0)}{\partial y} - \frac{\partial(x)}{\partial z}\right] - \underset{\sim}{j}\left[\frac{\partial(0)}{\partial x} - \frac{\partial(4y)}{\partial z}\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial(x)}{\partial x} - \frac{\partial}{\partial y}(4y)\right]$$

no gaps

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i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + (1-4)\underset{\sim}{k} = -3\underset{\sim}{k}$

We want to calculate

$$\int (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot d\underset{\sim}{S}$$

$$= \int (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \underset{\sim}{\hat{n}}\, dS$$

where $\underset{\sim}{\hat{n}} = \underset{\sim}{k}$ across the whole circle.

$$\int (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \underset{\sim}{\hat{n}}\, dS \qquad \underset{\sim}{\nabla} \times \underset{\sim}{V} = -3\underset{\sim}{k} \qquad \underset{\sim}{\hat{n}} = \underset{\sim}{k}$$

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$$(\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \underset{\sim}{\hat{n}} = -3\underset{\sim}{k} \cdot \underset{\sim}{\hat{n}} = -3\underset{\sim}{k} \cdot \underset{\sim}{k} = -3.$$

$$\therefore \int_{\substack{\text{circular}\\ \text{disk}}} (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot \underset{\sim}{\hat{n}}\, dS = \int_{\substack{\text{circular}\\ \text{disk}}} (-3)\, dS = -3 \int_{\substack{\text{circular}\\ \text{disk}}} dS.$$

$$= -3\pi a^2.$$

Conservative Fields – Revisited

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Earlier, we obtained three equivalent conditions for a vector field $\underset{\sim}{V}$ to be conservative.

These were ...

(I) • the existence of a scalar potential $\phi(x,y,z)$

such that
$$\int_A^B \underset{\sim}{V} \cdot d\underset{\sim}{r} = \int_A^B d\phi = \phi_B - \phi_A$$

[path independence]

no gaps

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(II) • for $\underset{\sim}{V} \cdot d\underset{\sim}{r} = d\phi = V_x\,dx + V_y\,dy + V_z\,dz$

$\left[\underset{\sim}{V} \cdot d\underset{\sim}{r} = d\phi\right.$, an exact differential$\left.\right]$

(III) • the reciprocity relations : $\dfrac{\partial V_x}{\partial y} = \dfrac{\partial V_y}{\partial x}$; $\dfrac{\partial V_x}{\partial z} = \dfrac{\partial V_z}{\partial x}$

and $\dfrac{\partial V_y}{\partial z} = \dfrac{\partial V_z}{\partial y}$

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With the help of the vector algebra that has been developed, we can re-cast these three conditions in terms of five equivalent conditions.

Firstly, note that if $\underset{\sim}{V} = (V_x, V_y, V_z)$ then

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \underset{\sim}{j}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \underset{\sim}{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$

correction needed!

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reciprocity relations : $\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}$; $\frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}$ and $\frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}$

i.e. $$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \hat{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \hat{j}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \hat{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}(0) - \underset{\sim}{j}(0) + \underset{\sim}{k}(0)$, using the reciprocity relations (III).

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$

$\therefore$ $\boxed{\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0} \quad \text{if } \underset{\sim}{V} \text{ is conservative.}}$

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(II) • for $\underset{\sim}{V}.\underset{\sim}{dr} = d\phi$

Secondly, note condition (II) requiring $d\phi$ to be an exact differential implies that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz.$$

But this is just the dot product of $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial \phi}{\partial x} + \underset{\sim}{j}\frac{\partial \phi}{\partial y} + \underset{\sim}{k}\frac{\partial \phi}{\partial z}$

and $\underset{\sim}{dr} = \underset{\sim}{i}\,dx + \underset{\sim}{j}\,dy + \underset{\sim}{k}\,dz$

•
• •

$$d\phi = \underset{\sim}{\nabla}\phi . \underset{\sim}{dr}.$$

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(I) $\int_A^B \underset{\sim}{V} \cdot d\underset{\sim}{r} = \int_A^B d\phi$

and $d\phi = \underset{\sim}{\nabla}\phi \cdot d\underset{\sim}{r}$

$\Rightarrow$ If we can write a vector field $\underset{\sim}{V}$ as

$\underset{\sim}{V} = \underset{\sim}{\nabla}\phi$, where ϕ is a scalar field,

then $\underset{\sim}{V}$ is a conservative field.

no gaps

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Let's also note that

when $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ (implying $\underset{\sim}{V}$ conservative)

Stoke's theorem gives

$$\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{V}) \cdot d\underset{\sim}{s} = \oint_C \underset{\sim}{V} \cdot d\underset{\sim}{r} = 0$$

So that $\oint_C \underset{\sim}{V} \cdot d\underset{\sim}{r} = 0$ around any closed curve

when $\underset{\sim}{V}$ conservative.

no gaps

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$$\oint_C \underset{\sim}{V} \cdot d\underset{\sim}{r} = 0 \text{ when } \underset{\sim}{V} \text{ conservative.}$$

This is consistent with condition (I)

which gives

$$\int_A^A \underset{\sim}{V} \cdot d\underset{\sim}{r} = \int_A^A d\phi$$

$$= \phi_A - \phi_A$$

$$= 0.$$

Let's re-state the five equivalent conditions for $\underset{\sim}{V}$ to be a conservative field ...

no gaps

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(i) $\nabla \times \mathbf{V} = \mathbf{0}$

(ii) $\oint \mathbf{V} \cdot d\mathbf{r} = 0$ around every simple closed curve

(iii) $\int_A^B \mathbf{V} \cdot d\mathbf{r}$ is path-independent

(iv) $\mathbf{V} \cdot d\mathbf{r} = d\phi$ = an exact differential

(v) $\mathbf{V} = \nabla \phi$, ϕ a (single-valued) scalar field.

→ equivalent conditions for $\mathbf{V}$ conservative ←

Notes (a) A "simple" closed curve does not cross itself and thus a single circulation direction of the curve and directions for the surface normals are possible.

(b) Any scalar or vector field is defined in a region of space. Thus, (i) to (v) apply to a region of space. This region needs to be "simply connected" $\Rightarrow$ any simple closed curve C can be shrunk down to a point within the region.

e.g.

boundary of region R

C around a "hole"
$\Rightarrow$ not simply connected.

large hole!
$\equiv$ tyre tube
$\Rightarrow$ not simply connected.

no gaps

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Ex It was shown earlier that any vector field obeying a radial inverse square law is conservative.

Let's now test a 'central field' of the form

$$\underset{\sim}{V} = r^n \underset{\sim}{\hat{r}}.$$

Note that inverse square $(n=-2)$ is a special case.

no gaps

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Ans. Recall that $\hat{\underset{\sim}{r}} = \dfrac{\underset{\sim}{r}}{|\underset{\sim}{r}|} = \dfrac{\underset{\sim}{r}}{r}$

where $\underset{\sim}{r} = (x, y, z) = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

$\therefore \underset{\sim}{V} = r^n \hat{\underset{\sim}{r}} = \dfrac{r^n \underset{\sim}{r}}{r} = r^{n-1}(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})$

i.e. $\underset{\sim}{V} = (V_x, V_y, V_z)$ where

$V_x = r^{n-1}x$

$V_y = r^{n-1}y$

$V_z = r^{n-1}z$.

no gaps

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$$V_x = r^{n-1}x$$

$$V_y = r^{n-1}y$$

$$V_z = r^{n-1}z$$

$$\therefore \quad \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

i.e. $$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^{n-1}x & r^{n-1}y & r^{n-1}z \end{vmatrix}$$

i.e. $$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}y)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}x)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}(r^{n-1}y) - \frac{\partial}{\partial y}(r^{n-1}x)\right]$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \hat{i}\left[\frac{\partial}{\partial y}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(r^{n-1}y) - \frac{\partial}{\partial y}(r^{n-1}x)\right]$$

no gaps

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Now note that implicit partial differentiation of $r^2 = x^2 + y^2 + z^2$

gives

$$\left\{\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \\ 2r\frac{\partial r}{\partial y} &= 2y \\ 2r\frac{\partial r}{\partial z} &= 2z \end{aligned}\right] \Rightarrow \left[\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} \\ \frac{\partial r}{\partial y} &= \frac{y}{r} \\ \frac{\partial r}{\partial z} &= \frac{z}{r} \end{aligned}\right\}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \hat{i}\left[\frac{\partial}{\partial y}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(r^{n-1}y) - \frac{\partial}{\partial y}(r^{n-1}x)\right]$$

no gaps

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Then, for example, the $\underset{\sim}{i}$ component of $\underset{\sim}{\nabla} \times \underset{\sim}{V}$

is

$$\underset{\sim}{i}\left[\frac{\partial}{\partial y}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}y)\right]$$

$$= \underset{\sim}{i}\left[(n-1)r^{n-2} \cdot \frac{\partial r}{\partial y} \cdot z - (n-1)r^{n-2} \cdot \frac{\partial r}{\partial z} \cdot y\right]$$

$$= \underset{\sim}{i}\left[(n-1)r^{n-2} \cdot \frac{y}{r} \cdot z - (n-1)r^{n-2} \cdot \frac{z}{r} \cdot y\right] = \underset{\sim}{0}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

no gaps

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bot

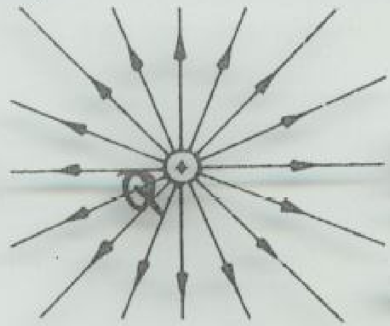
And a similar result is obtained for both the $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components of $\underset{\sim}{\nabla} \times \underset{\sim}{V}$.

So, since $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$,

the central field $\underset{\sim}{V} = r^n \hat{\underset{\sim}{r}}$ is conservative.

Examples of conservative fields

(a) $\underset{\sim}{E}$-field from a <u>static</u> point charge



$$\underset{\sim}{E} = \frac{Q}{4\pi\varepsilon_0} \frac{\hat{\underset{\sim}{r}}}{r^2}$$

inverse-square and radial ($\hat{\underset{\sim}{r}}$)

(b) Any radial inverse square field

i.e. $$\underset{\sim}{V}(\underset{\sim}{r}) = \frac{\eta\, \hat{\underset{\sim}{r}}}{r^2}$$

(where η defines the particular constants of the physical system).

no gaps

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(c) Any central field of the form $\underset{\sim}{V}(\underset{\sim}{r}) = \eta r^n \underset{\sim}{\hat{r}}$.

But note that $\underset{\sim}{V}$ does not need to be radial to be conservative.

One can envisage, for example, non-dissipative mechanical force fields that are not radial.

no gaps

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top

However,

if $\underset{\sim}{V}$ is a conservative vector field

then $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ everywhere

⇒ the field is "IRROTATIONAL"

i.e. it has no vortices / circulation / swirl / etc.

Also note that if $\underset{\sim}{V}$ is conservative

then $\underset{\sim}{V} = \underset{\sim}{\nabla} \phi$ (for some scalar field)

$$\Rightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi)$$

$$= \nabla^2 \phi$$ ← THIS DOES NOT NEED TO BE ZERO.

∴ A conservative field can have sources and sinks of flux but no vortices.

no gaps

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Ex 2 Imagine walking into an electromagnetic theory revision class dealing with the $\underset{\sim}{E}$-field of static charges and a scalar (potential difference) field $V = -\phi$. What's on the board regarding the conservative nature of $\underset{\sim}{E}$?

Ans Something like this ...

$$\boxed{\underset{\sim}{E} = -\underset{\sim}{\nabla} V \iff \underset{\sim}{\nabla} \times \underset{\sim}{E} = \underset{\sim}{0} \iff \oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0 \iff \int_A^B \underset{\sim}{E} \cdot d\underset{\sim}{l} \text{ path independent.}}$$

Recall that there can be sources (+ve charges) and sinks (-ve charges) of the flux of $\underset{\sim}{E}$ over a closed surface (the net charge is inside the surface).

no gaps

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class dealing with the $\underset{\sim}{E}$-field of static charges

...

However, $\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$ means that there cannot be circulation of the field. This is often said as the field having "no vortices"

— This isn't the full story but it gives you an idea of the type of thing to expect.

"a vortex"

no gaps

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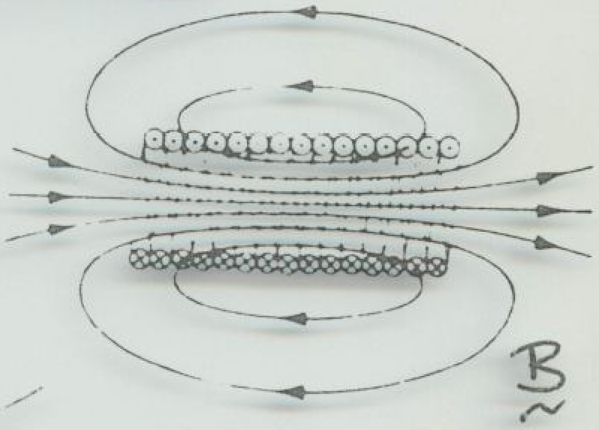
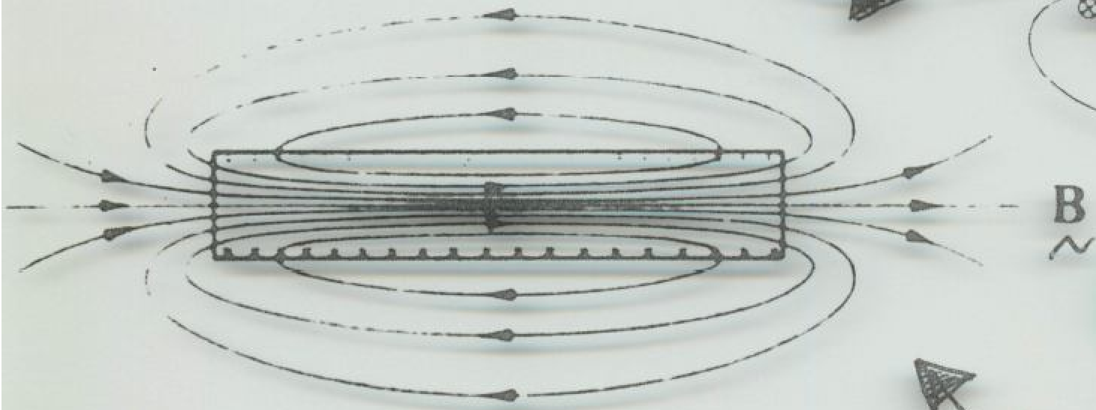
Another special type of vector field:

ZERO DIVERGENCE $\Rightarrow$ "**SOLENOIDAL**"

examples (a) Field of a solenoid!



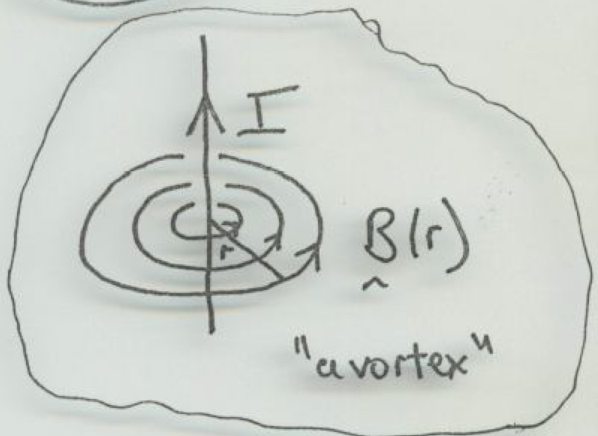


(b) Field of a bar magnet

(c) Field from a current-carrying wire:

$$: \quad \underset{\wedge}{\nabla} \times \underset{\wedge}{B} = \mu_0 \underset{\sim}{J}$$

can be non-zero, can have vortices



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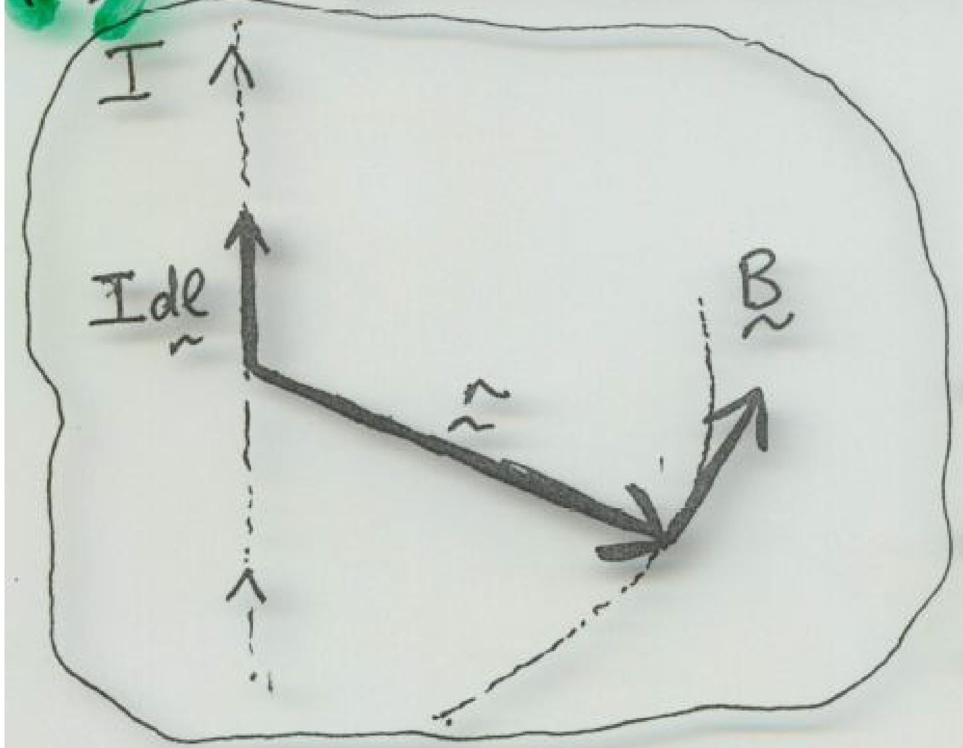
(d) : $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$, but no sources or sinks (no magnetic "monopoles").

(e) Field from a small current element $I\underset{\sim}{dl}$:



$$\underset{\sim}{B} \propto \frac{I\underset{\sim}{dl} \times \underset{\sim}{\hat{r}}}{r^2}$$

- Inverse-square but <u>not</u> radial
- not conservative
- it <u>is</u> solenoidal

no gaps

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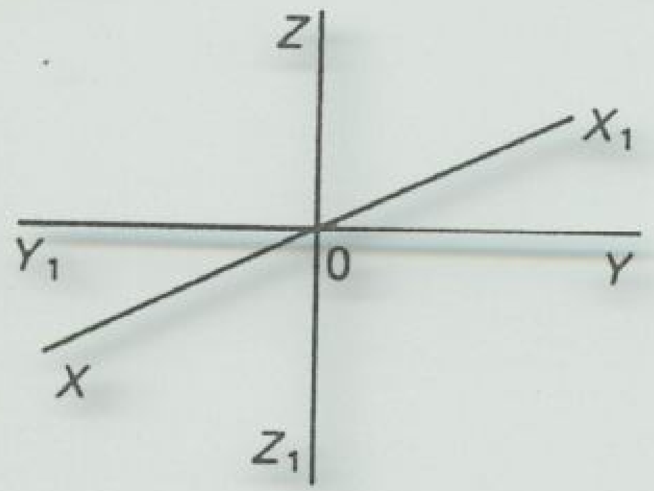
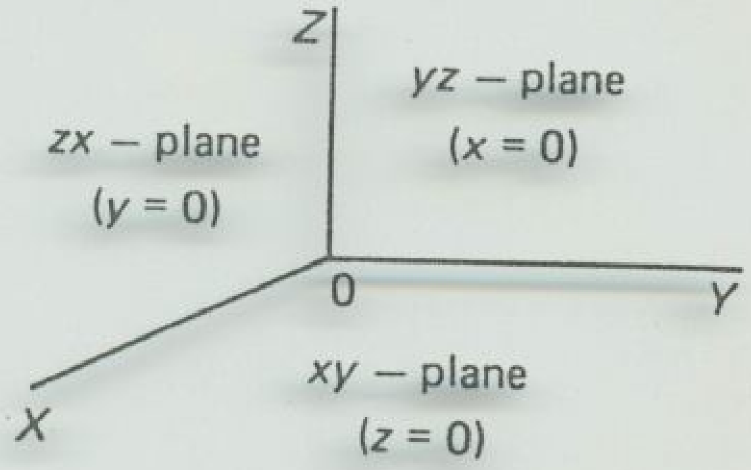
[REFERENCE MATERIAL]

Space coordinate systems

1. *Cartesian coordinates* (x, y, z)—referred to three coordinate axes OX, OY, OZ at right angles to each other. These are arranged in a *right-handed* manner, i.e. turning from OX to OY gives a right-handed screw action in the positive direction of OZ.





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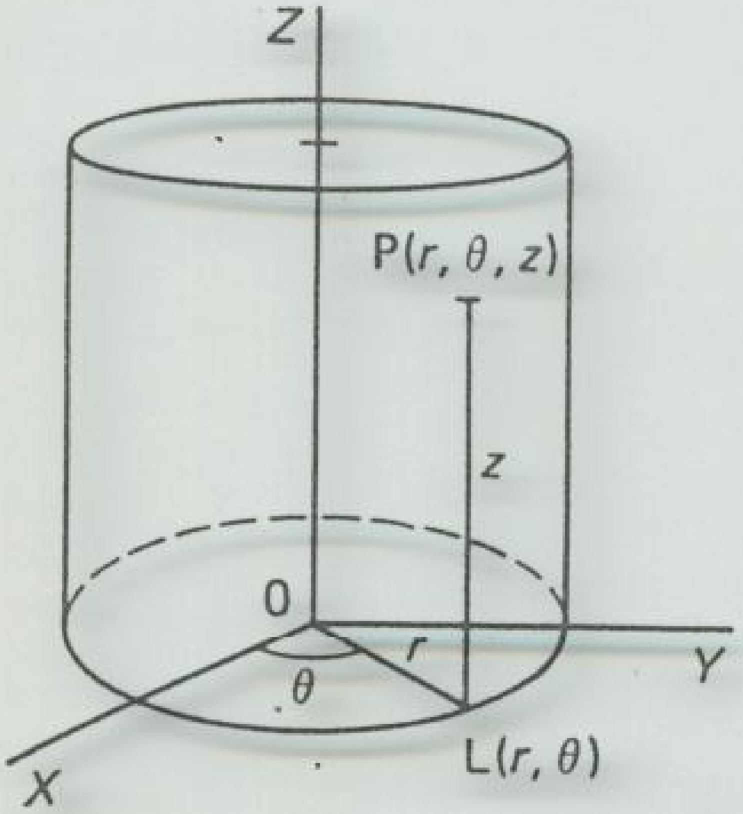
mid

2. *Cylindrical coordinates* (r, θ, z) are useful where an axis of symmetry occurs.



$$x = r\cos\theta; \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta; \qquad \theta = \arctan(y/x)$$

$$z = z; \qquad z = z$$

Any point P is considered as having a position on a cylinder. If L is the projection of P on the *xy*-plane, then (r, θ) are the usual polar coordinates of L. The cylindrical coordinates of P then merely require the addition of the *z*-coordinate.

no gaps

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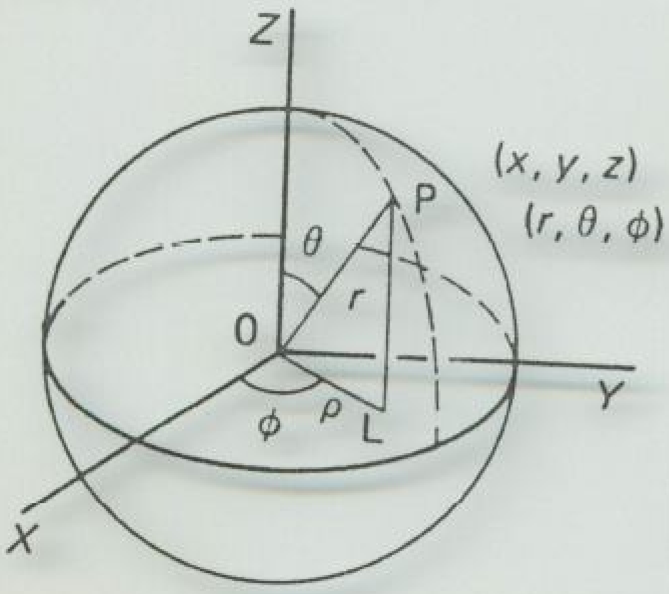
3. *Spherical coordinates* (r, θ, ϕ) are appropriate where a centre of symmetry occurs. The position of a point is considered as being a point on a sphere.



$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\sin\theta\sin\phi \qquad \theta = \arccos(z/r)$$

$$z = r\cos\theta \qquad \phi = \arctan(y/x)$$

(NB $\rho = r\sin\theta$)

r is the distance of P from the origin and is always taken as positive.

L is the projection of P on the xy-plane;
θ is the angle between OP and the positive OZ axis;
ϕ is the angle between OL and the OX axis.

no gaps

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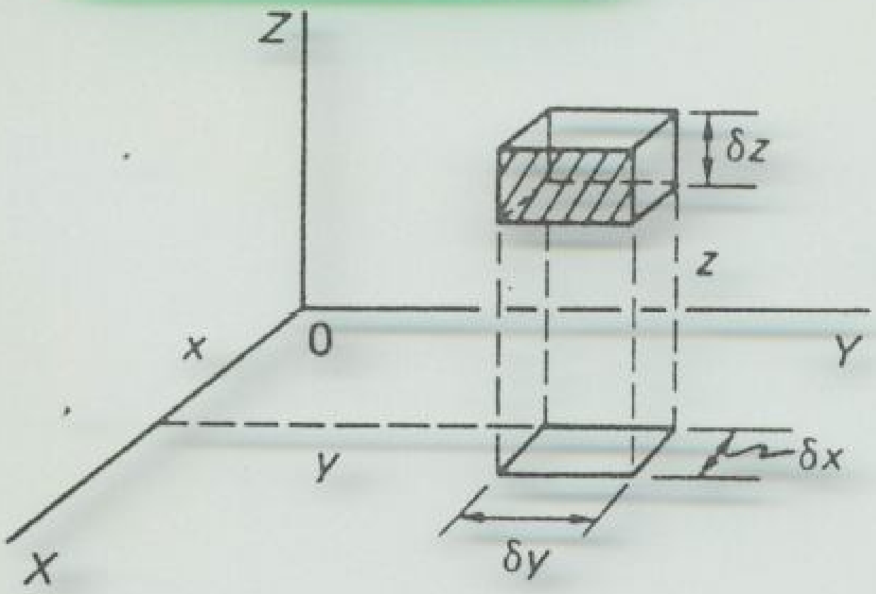
Element of volume in space in the three coordinate systems

1. *Cartesian coordinates*



We have already used this many times.

$$\delta v = \delta x \delta y \delta z$$

no gaps

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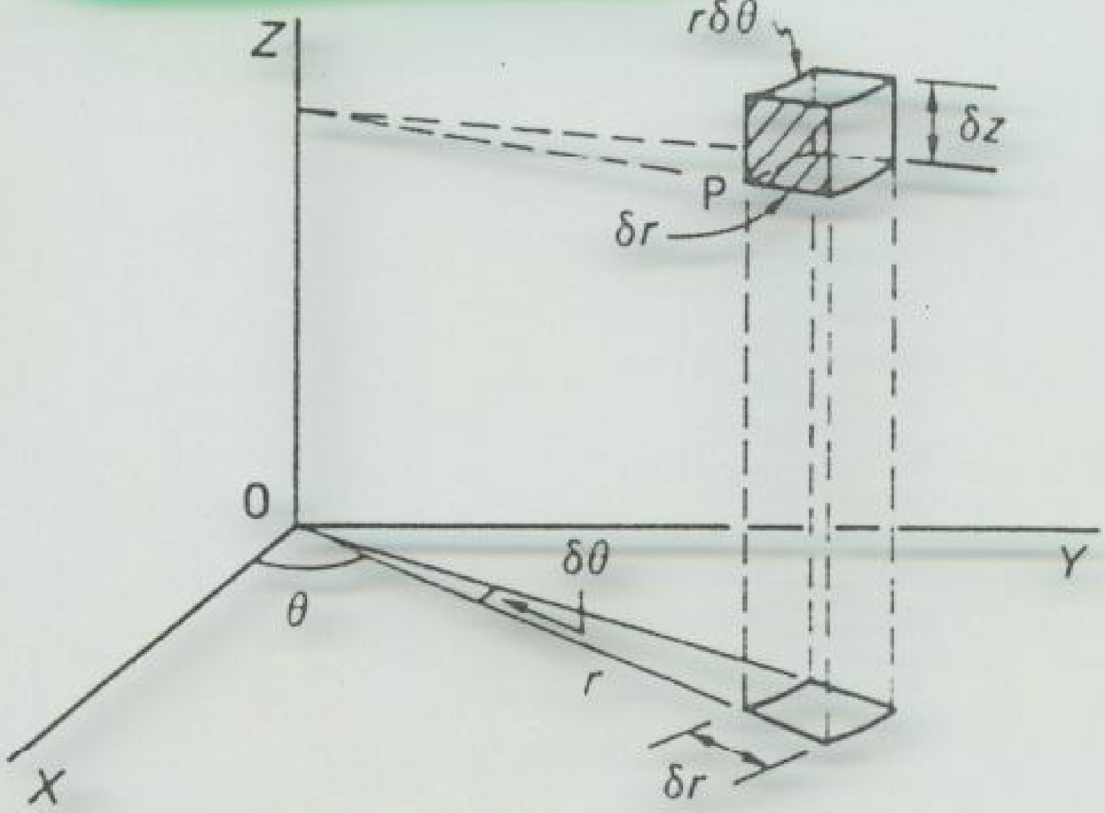
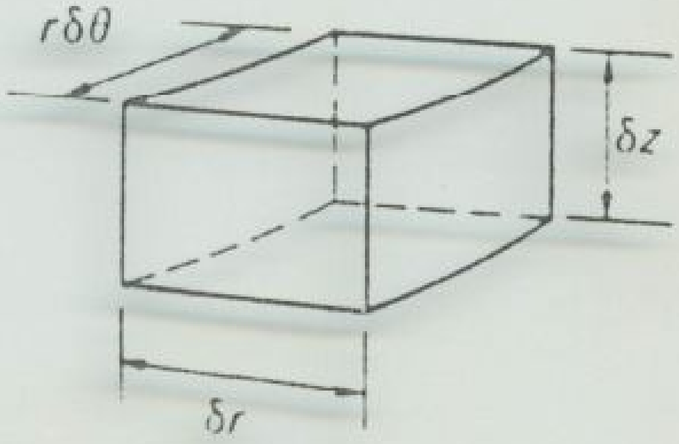
mid

2. *Cylindrical coordinates*





$$\delta v = r\,\delta\theta\,\delta r\,\delta z$$

$$\therefore\ \delta v = r\,\delta r\,\delta\theta\,\delta z$$

no gaps

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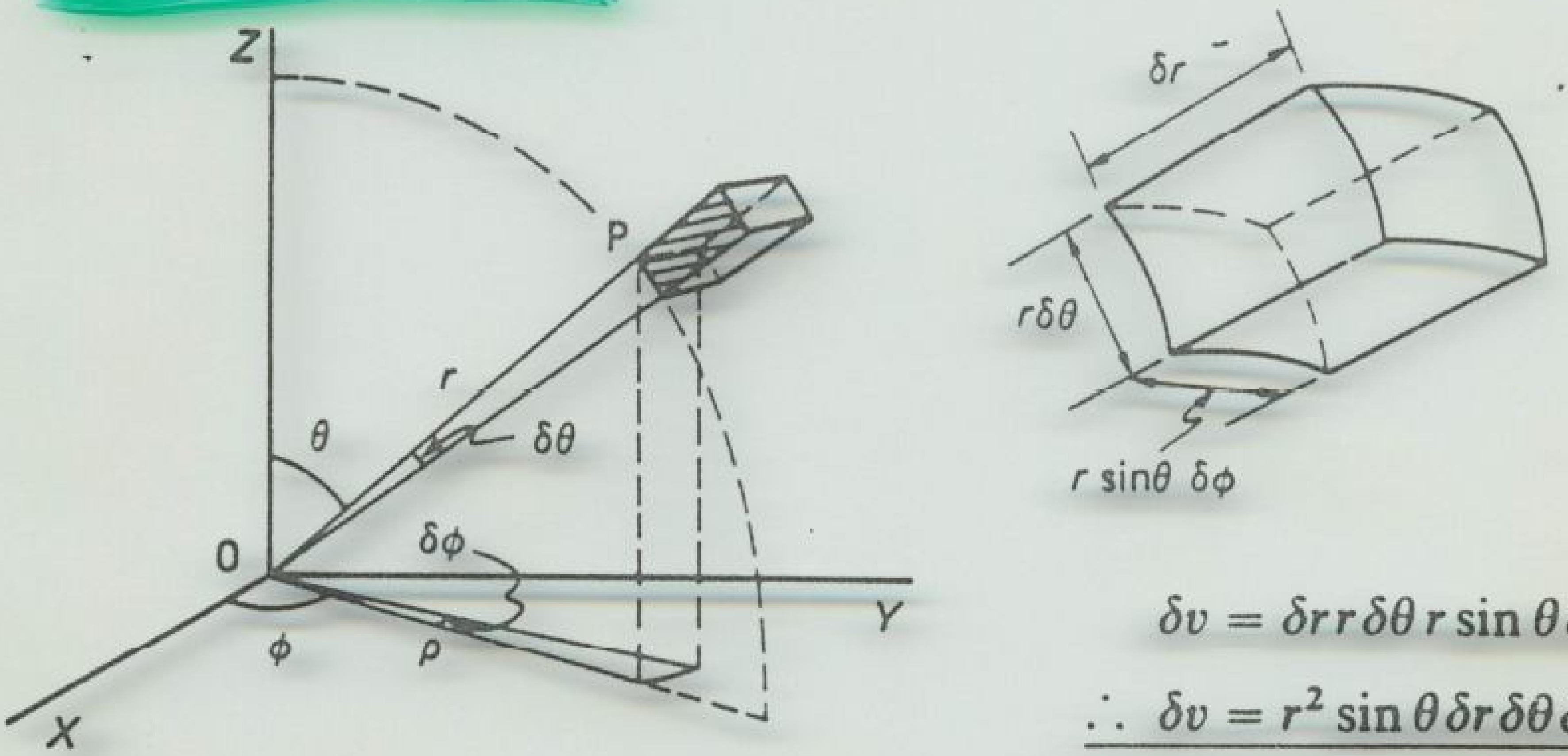
bot

3. *Spherical coordinates*



$$\delta v = \delta r \, r\,\delta\theta \, r \sin\theta \, \delta\phi$$

$$\therefore \; \underline{\delta v = r^2 \sin\theta \, \delta r \, \delta\theta \, \delta\phi}$$

no gaps

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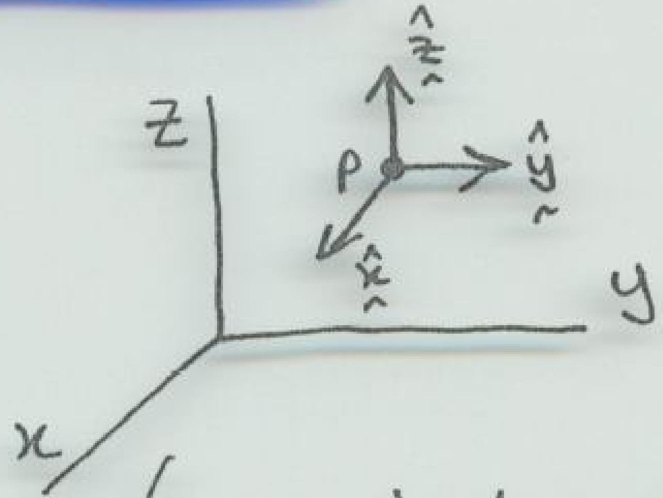
UNIT BASIS VECTORS

Cartesian

$$\underset{\sim}{e}_x \equiv \underset{\sim}{i} \equiv \underset{\sim}{\hat{x}}$$

$$\underset{\sim}{e}_y \equiv \underset{\sim}{j} \equiv \underset{\sim}{\hat{y}}$$

$$\underset{\sim}{e}_z \equiv \underset{\sim}{k} \equiv \underset{\sim}{\hat{z}}$$



(in direction of increasing x, y, z)

no gaps

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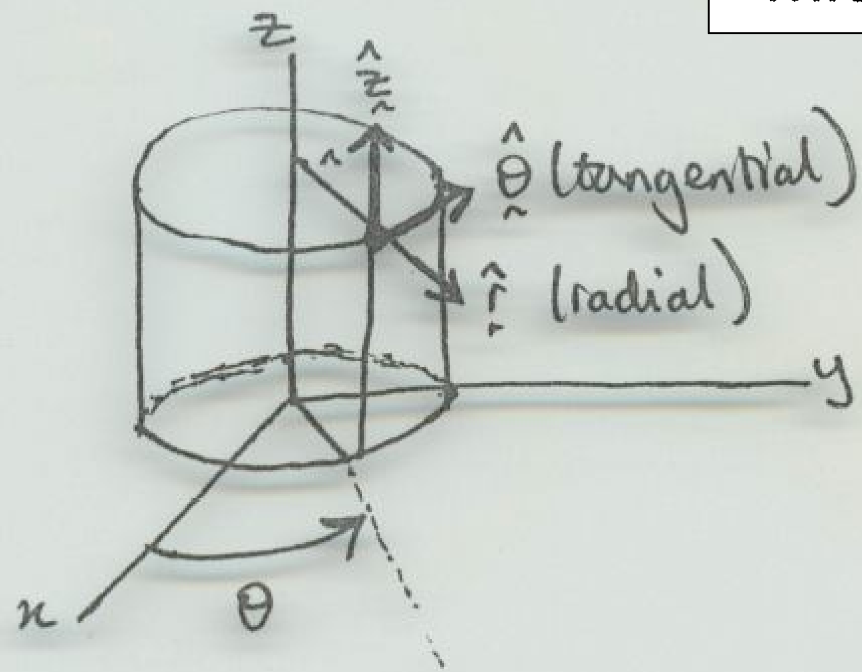
Cylindrical

$$\underset{\sim}{e}_r \equiv \underset{\sim}{\hat{r}}$$

$$\underset{\sim}{e}_\theta \equiv \underset{\sim}{\hat{\theta}}$$

$$\underset{\sim}{e}_z \equiv \underset{\sim}{\hat{z}}$$



(in direction of increasing r, θ, z)

no gaps

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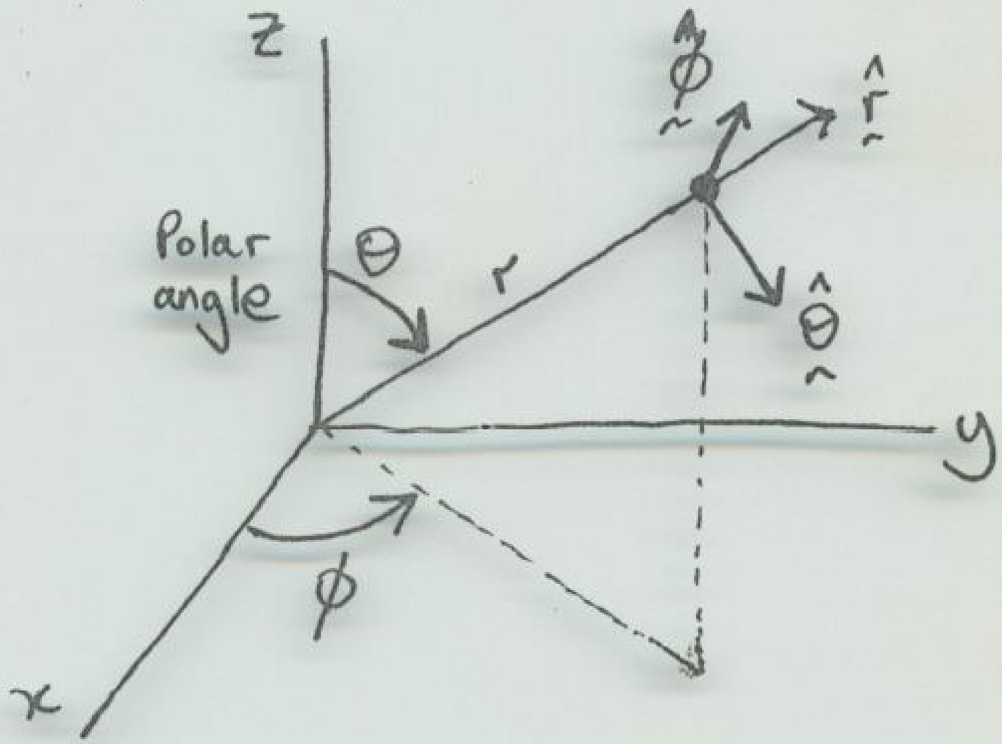
Spherical

$$\underset{\sim}{e}_r \equiv \underset{\sim}{\hat{r}}$$

$$\underset{\sim}{e}_\theta \equiv \underset{\sim}{\hat{\theta}}$$

$$\underset{\sim}{e}_\phi \equiv \underset{\sim}{\hat{\phi}}$$



(in direction of increasing r, θ, ϕ)

no gaps

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SUMMARY – DIFFERENTIAL OPERATORS IN OTHER ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

Below we list the vector differential operators in cylindrical and spherical coordinates. For reference, the corresponding expressions in Cartesian coordinates are also given. f and $\mathbf{A}$ are arbitrary differentiable scalar and vector fields respectively.

- Cartesian coordinates (x, y, z)

$$\mathbf{e}_x \equiv \mathbf{i}\,, \qquad \mathbf{e}_y \equiv \mathbf{j}\,, \qquad \mathbf{e}_z \equiv \mathbf{k}$$

$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$$

$$\mathbf{e}_x \equiv \mathbf{i}\,, \qquad \mathbf{e}_y \equiv \mathbf{j}\,, \qquad \mathbf{e}_z \equiv \mathbf{k}$$

$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$$

no gaps

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mid

Vector components: labelled as 1, 2, & 3

$$(\mathrm{grad}\, f)_1 = \frac{\partial f}{\partial x}$$

$$(\mathrm{grad}\, f)_2 = \frac{\partial f}{\partial y}$$

$$(\mathrm{grad}\, f)_3 = \frac{\partial f}{\partial z}$$

$$\mathrm{div}\,\mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$(\mathrm{curl}\,\mathbf{A})_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}$$

$$(\mathrm{curl}\,\mathbf{A})_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}$$

$$(\mathrm{curl}\,\mathbf{A})_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- cylindrical polar coordinates (r, θ, z)

no gaps

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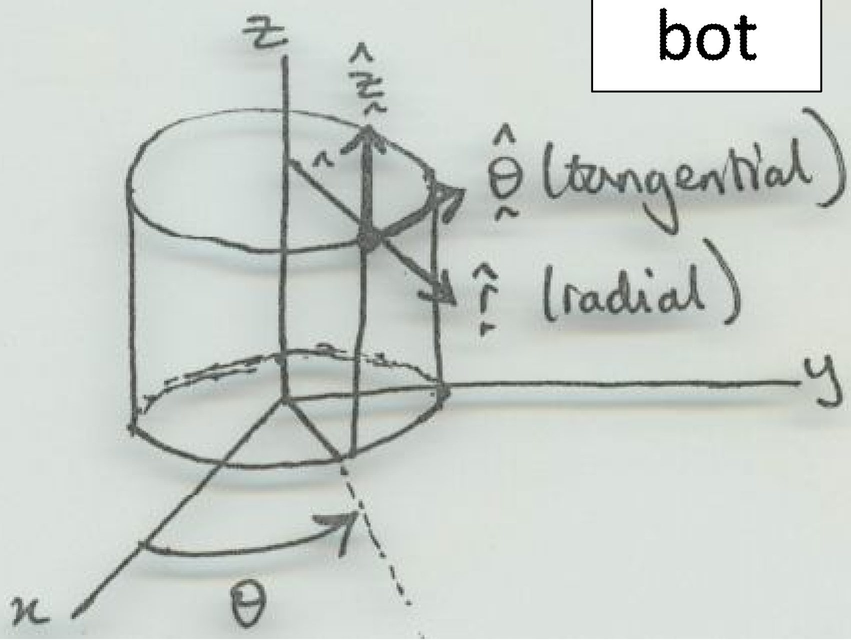
$$\underset{\sim}{e_r} \equiv \underset{\sim}{\hat{r}}$$
$$\underset{\sim}{e_\theta} \equiv \underset{\sim}{\hat{\theta}}$$
$$\underset{\sim}{e_z} \equiv \underset{\sim}{\hat{z}}$$



$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \qquad z = z$$

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\,, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,, \qquad \mathbf{e}_z = \mathbf{k}$$

$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_z\,\mathbf{e}_z$$

$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_z\,\mathbf{e}_z$$

no gaps

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$$(\operatorname{grad} f)_r = \frac{\partial f}{\partial r}$$

$$(\operatorname{grad} f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$(\operatorname{grad} f)_z = \frac{\partial f}{\partial z}$$

$$\operatorname{div} \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$(\operatorname{curl} \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$

$$(\operatorname{curl} \mathbf{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\operatorname{curl} \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

• spherical polar coordinates (r, θ, ϕ)

no gaps

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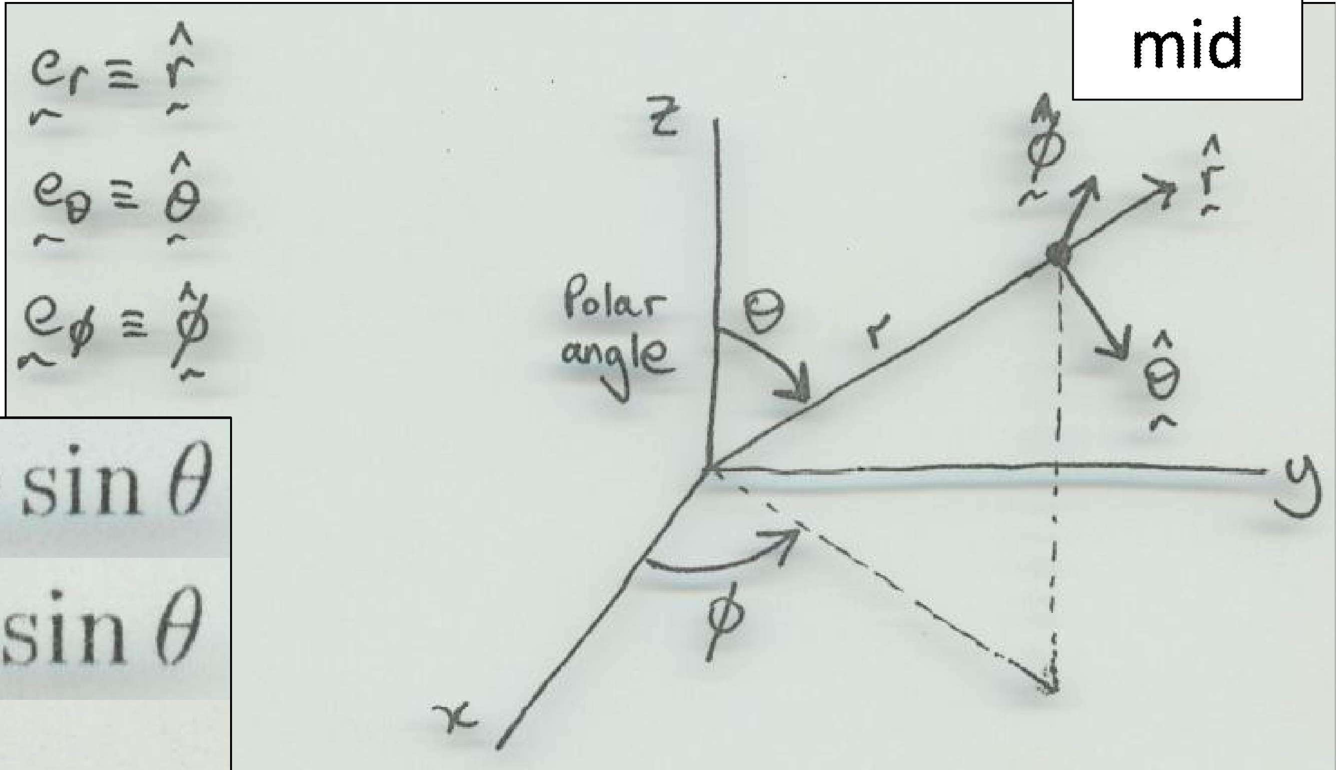
mid



$$
\begin{aligned}
x &= r \cos\phi \sin\theta \\
y &= r \sin\phi \sin\theta \\
z &= r \cos\theta
\end{aligned}
$$

$$\mathbf{A} = A_r\, \mathbf{e}_r + A_\theta\, \mathbf{e}_\theta + A_\phi\, \mathbf{e}_\phi$$

where these basis vectors can also be expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

no gaps

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$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_\phi\,\mathbf{e}_\phi$$

$$\begin{aligned}
(\mathrm{grad}\,f)_r &= \frac{\partial f}{\partial r} \\
(\mathrm{grad}\,f)_\theta &= \frac{1}{r}\frac{\partial f}{\partial \theta} \\
(\mathrm{grad}\,f)_\phi &= \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}
\end{aligned}$$

$$\begin{aligned}
(\mathrm{curl}\,\mathbf{A})_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} \\
(\mathrm{curl}\,\mathbf{A})_\theta &= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) \\
(\mathrm{curl}\,\mathbf{A})_\phi &= \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}
\end{aligned}$$

$$\mathrm{div}\,\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

no gaps

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The power of vectors is that physical laws such as the divergence theorem and Stoke's theorem do not change in different coordinate systems ...

... but one must substitute the appropriate expressions for quantities such as dS and dV. Using the tables for components of vector operations, one finds ...

Finally, if we assemble the components together, we get …

no gaps

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CYLINDRICAL COORDINATES

$$\nabla V \equiv \textbf{grad } V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} \equiv \text{div } \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} \equiv \textbf{curl } \mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}$$

no gaps

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SPHERICAL COORDINATES

$$\nabla V \equiv \textbf{grad } V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} \equiv \text{div } \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla \times \mathbf{A} \equiv \textbf{curl } \mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right)$$

$$+ \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\boldsymbol{\phi}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$