Mathematical Methods and Applications

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MATRICES

- Introduction
  - consistency and number of solutions of 2x2 systems
  - definition of a matrix
  - matrix arithmetic
    * addition and subtraction
    * multiplication by a scalar
    * multiplying matrices

Mathematical Methods and Applications

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- Solution of equations
  - Cramer's rule
  - Laplace expansion of determinants
  - Classification of systems I.

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Introduction

The solution of simultaneous equations is a problem that appears regularly in everyday life and throughout science and engineering. Systems involving, say, just two linear equations are easy to solve and this can be done either graphically or by manipulation of the equations

Ex

$$x + y = 2$$
$$x - y = 0$$

We want to find x and y (the unknowns)

Adding the two equations gives $2x = 2$.

Thus, $x = 1$ and substitution of this value in either equation yields $y = 1$.

double ring blanks

$$x+y=2$$
$$x-y=0$$

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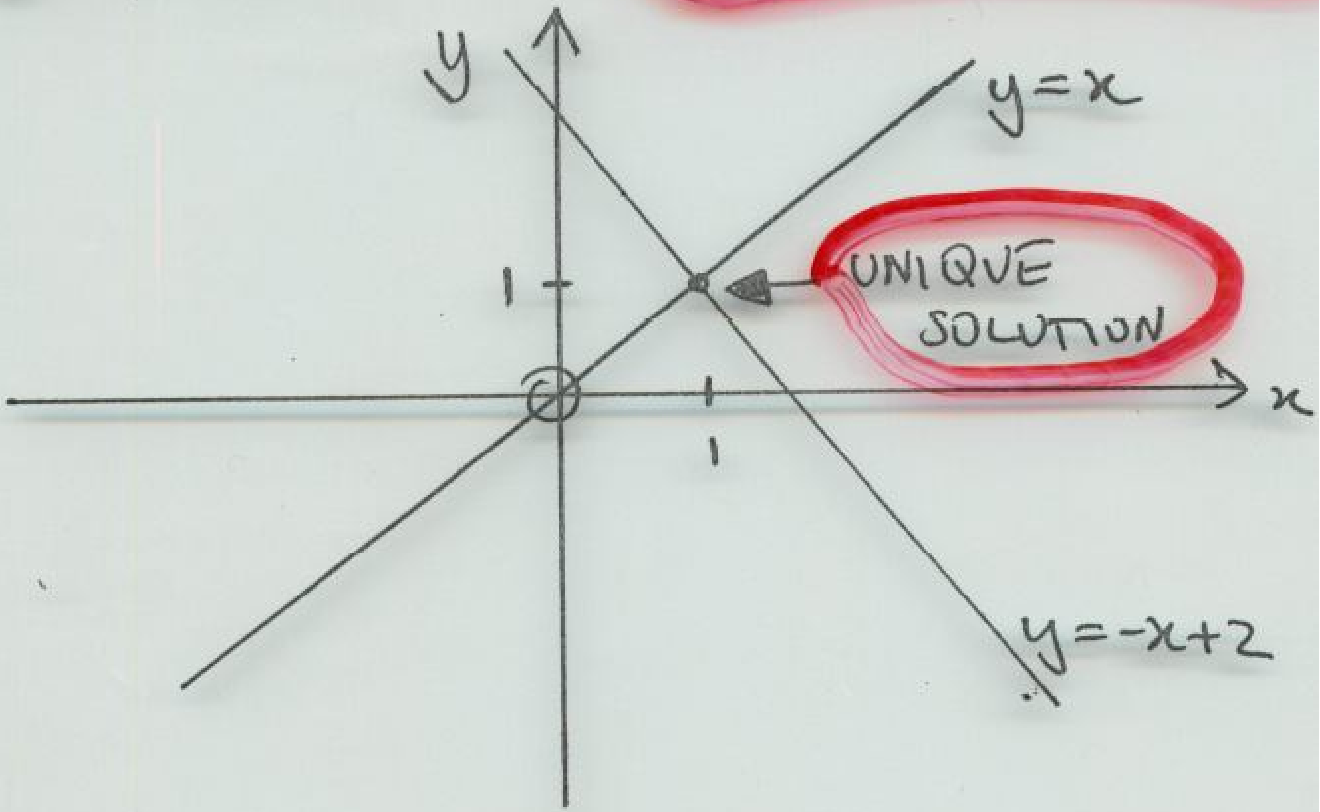
... graphically, we have the lines $y=-x+2$ and $y=x$



i.e. we have 2 equations and 2 unknowns, giving a unique solution.

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Ex
$$x + y = 2$$
$$x + y = 5$$

2 equations and 2 unknowns again.
Subtract the top equation from the bottom one to find

$$x + y = 2$$
$$0.x + 0.y = 3$$

(coefficients are "$\begin{smallmatrix}1 & 1 \\ 0 & 0\end{smallmatrix}$")

Woops! That gives $0=3$. Something's wrong.
The equations are "INCONSISTENT" i.e. we can't have x and y simultaneously satisfying both equations

$$x+y=2$$
$$x+y=5$$

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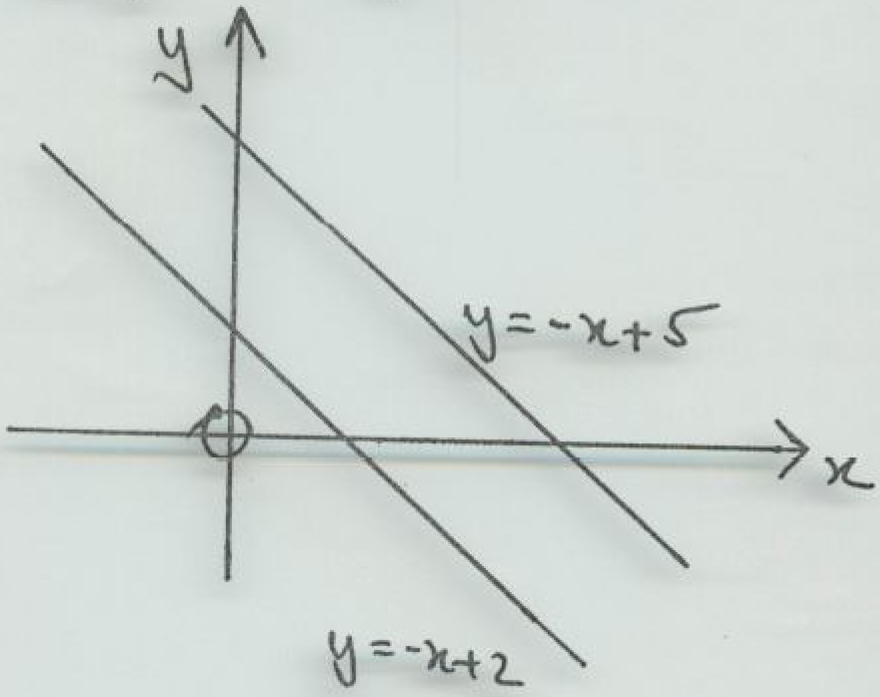
Graphically ...

parallel lines

⇒ they never cross

⇒ no solution



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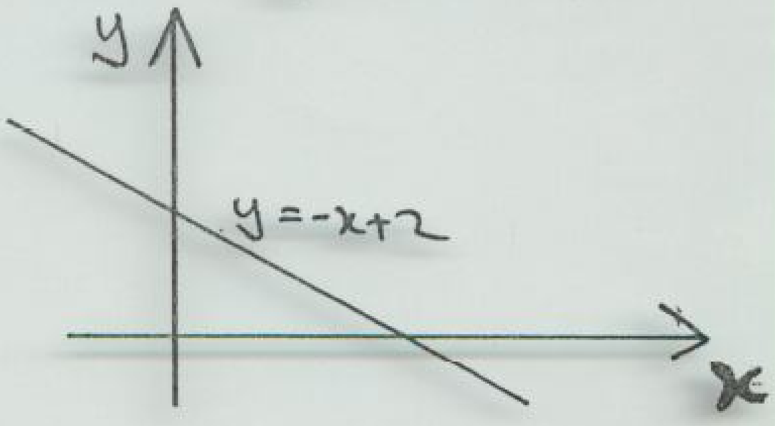
Ex $x+y=2$
$2x+2y=4$

2 equations and 2 unknowns again?

No! The second equation is precisely just twice the first. They are basically the same equation i.e. $x+y=2$ and we really only have 1 equation with 2 unknowns. This is just a line...

→ we have an infinite number of solutions



Ex A "homogeneous system" i.e. the right hand side has zeroes.

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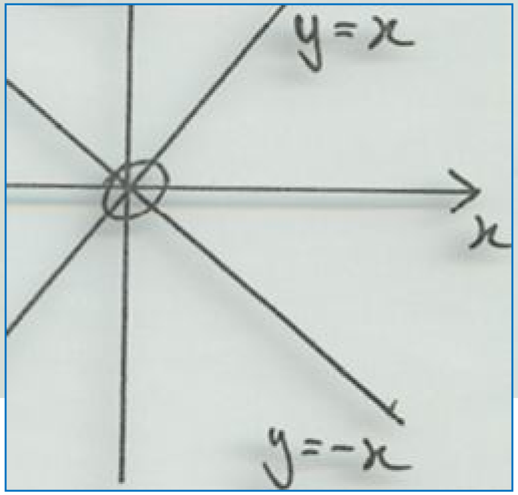
$$x + y = 0$$
$$x - y = 0$$

2 equations and 2 unknowns

— one isn't a multiple of the other, so we do have 2 independent equations

— they are not inconsistent
$y = -x$ and $y = x$ are not parallel lines



BUT, the only solution we can find is the "trivial" or "null" solution $x = y = 0$

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Ex Another "homogeneous" system.

$$x + y = 0$$
$$2x + 2y = 0$$

This time, one equation is a multiple of the other. They are "consistent" – they must be, they are the same equation

Take 2 times the first equation away from the second...

... to give

$$x + y = 0$$
$$0.x + 0.y = 0$$

ie coefficients "$\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$"

$$\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + y &= 0 \\ 0.x + 0.y &= 0 \end{aligned}$$

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The second just says $0=0$ i.e. it doesn't give any information and we are just left with the first equation

$x+y=0$ i.e. $y=-x$.

So, with one equation and two unknowns, the homogeneous system gives an <u>infinite number of solutions</u>.

This is all easy in the case of 2 equations, but what do we do if there are 3, 4, 5 or more equations?

We need systematic ways of determining ...

- whether solutions exist
- how many exist
- finding them.

In other words, we need to

- write the equations in a way that they can be analysed (MATRICES)
- work out how to answer the above questions (MATRIX THEORY)

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So, what is a matrix?

Consider the system of equations

$$\begin{aligned} x + 2y &= 5 \\ 3x - y &= 8 \end{aligned}$$

The coefficients of x and y "$\begin{matrix} 1 & 2 \\ 3 & -1 \end{matrix}$" form a "matrix".

i.e. a rectangular array of "elements" giving a table of values. To keep things tidy, we put some square or round brackets around the table and give it a name [NOT a value, it's a table of values, just a name for now]

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e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ this is a "2x2" matrix.

we know that the solutions depend on both this and the right hand side of the equations, so let's define another matrix $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$ --- this is a "2x1" matrix

--- it's a (short) column

--- it's like a column vector.

We could call this $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, but because it also represents a vector I might write it as $\boxed{\underset{\sim}{b} \equiv \begin{bmatrix} 5 \\ 8 \end{bmatrix}}$.

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Similarly, let's write the solution to the system of equations (the unknowns) in terms of a matrix $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

Again, for these type of problems, it will usually be just a single column, like a vector.

Thus, I may write $\underset{\sim}{x} \equiv \begin{bmatrix} x \\ y \end{bmatrix}$ to emphasize that this matrix is a single column.

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What other types of matrix could we have?

The coefficients don't need to be numeric. They could be constants $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are real or complex.

They could be expressions, say polynomials $\begin{bmatrix} \alpha & \alpha+1 \\ \alpha^2+2 & \beta^2+1 \end{bmatrix}$.

They could even be other matrices!

Before we can write our system of equations in terms of matrices, we need to know how to manipulate matrices (add, subtract, multiply, etc.)

... the rules of the game $\rightarrow$

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Matrix arithmetic

(a) Matrix addition

Let $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{pmatrix}$.

then $A+B = \begin{pmatrix} 2+3 & 1-5 & 4+1 \\ -3+2 & 0+1 & 2+3 \end{pmatrix}$

$= \begin{pmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{pmatrix}$

Here, A and B have 2 rows & 3 columns, i.e. they are "2x3", read as "2 by 3", matrices. If they were of different order then one simply could <u>not</u> add them.

Easy! But what if they are not the same order?

(b) Matrix subtraction

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad B = \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix}$$

$$A - B = \begin{pmatrix} a-g & b-h & c-i \\ d-j & e-k & f-l \end{pmatrix}$$

Easy! But, again, they must be of the same order!

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(c) Multiplying a matrix by a number (a "scalar")

Let's say "λ times A" where $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$.

$$\boxed{\lambda A = \begin{pmatrix} 2\lambda & \lambda & 4\lambda \\ -3\lambda & 0 & 2\lambda \end{pmatrix}}$$

i.e. we multiply every single element by λ

e.g. $\lambda = 2$ gives $\boxed{\lambda A = \begin{pmatrix} 4 & 2 & 8 \\ -6 & 0 & 4 \end{pmatrix}}$.

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(d) Multiplying one matrix by another (more involved!)

- when can we do it?
- how do we do it?

Ex $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$

a 2x3 matrix   a 3x2 matrix

Then, $AB = \begin{bmatrix} (2)(3)+(1)(2)+(4)(4) & (2)(5)+(1)(-1)+(4)(2) \\ (-3)(3)+(0)(2)+(2)(4) & (-3)(5)+(0)(-1)+(2)(2) \end{bmatrix}$

i.e. $AB = \begin{pmatrix} 24 & 17 \\ -1 & -11 \end{pmatrix}$ ...... a 2x2 matrix

What have we done?

- To get row 1, column 1 element of AB (ie 24)

we multiplied corresponding elements of ...

row 1 of A and column 1 of B

-- it was like a scalar product of vectors

i.e. $(2, 1, 4)$ times $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ gave $(2)(3) + (1)(2) + (4)(4) = 24$

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To get row 1, column 1 element of AB

$(2, 1, 4)$ times $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ gave

row 1 of A

column 1 of B

$(2)(3)$
$+$
$(1)(2)$
$+$
$(4)(4) = 24$

Compare with with vectors $\underset{\sim}{u} = (u_1, u_2, u_3)$

$\underset{\sim}{v} = (v_1, v_2, v_3)$

then $\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ = "a number" / "a scalar"

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- To get row 2 , column 1 element of AB, we did the same operation on row 2 and column 1 of matrices A and B, respectively
- Similarly, row 1 of A and column 2 of B gives the element in row 1 and column 2 of AB.
- and similarly for the row 2, column 2 element of AB

$(2, 1, 4)$ times $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ gave

row 1 of A

column 1 of B

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But these operations like dot products are only possible if the number of elements in the rows of A EQUALS the number of elements in the columns of B.

Now, the number of elements in a row = the number of columns of the matrix

and the number of elements in a column = the number of rows of the matrix

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Finally(!), we can multiply A and B if the number of columns of A = the number of rows of B

e.g. $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$

$2 \times \boxed{3 \quad 3} \times 2$

These need to be equal if we want to multiply the matrices. The matrices are said to be "CONFORMABLE"

A **B**

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rows x columns: 2×3 3×2

required equal to form product **AB**

The matrices are said to be "CONFORMABLE"

The order (or "size") of AB is then given by the outer numbers

i.e in this case, AB is a 2×2 matrix.

i.e. the result of multiplying A and B is a $2\times\cancel{3}\ \cancel{3}\times2$ matrix.

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General notation

For a matrix A (as above) one usually denotes the elements as a_{ij} where

$$\boxed{\begin{array}{l} i \text{ is the ROW subscript} \\ j \text{ is the COLUMN subscript} \end{array}}$$

e.g. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$

So, in general, to multiply two matrices
- THINK SCALAR PRODUCT

and
- VISUALISE THIS

to get the ij^{th} element of $AB = C$

to get the ij^{th} element of $AB = C$

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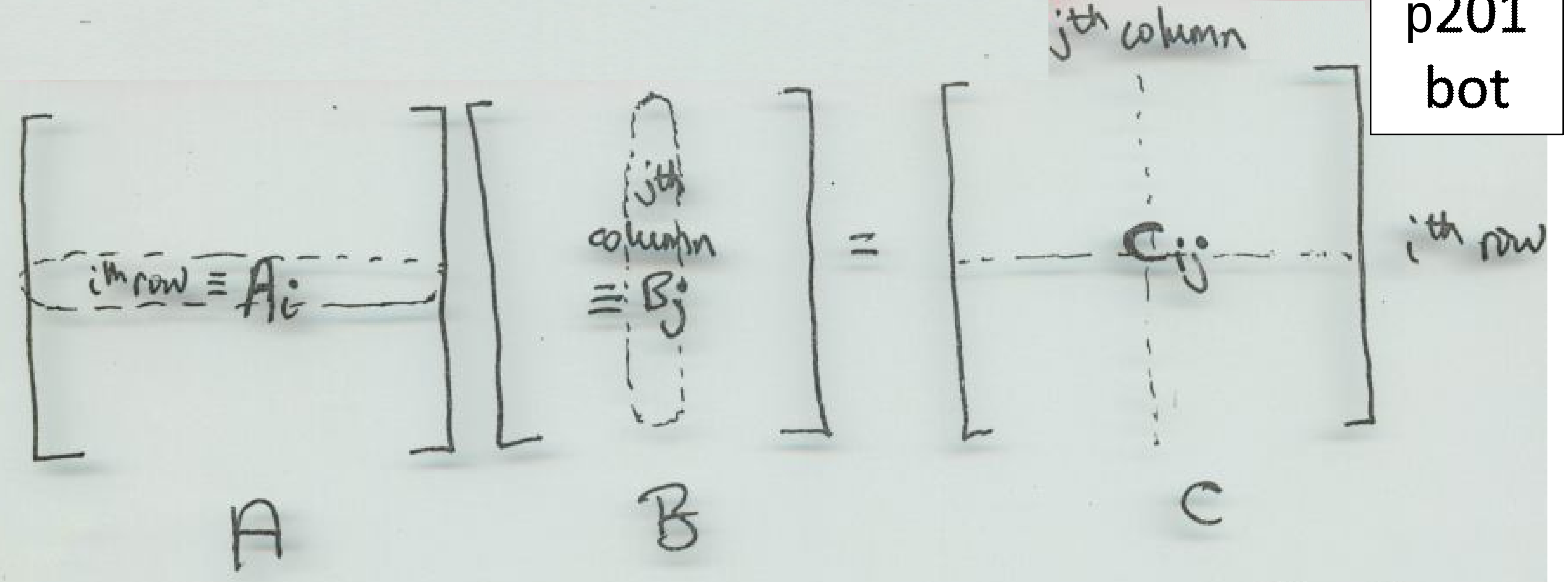

where scalar $c_{ij} \equiv \underset{\sim}{A}_i \cdot \underset{\sim}{B}_j = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$

$= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

... and this is only possible when A and B are conformable

i.e.

$$A \quad B = C$$
$$n \times m \quad m \times p \qquad n \times p$$

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Some multiplication properties

- $A(BC) = (AB)C$ associative
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$

distributive

- $AB \neq BA$ non-commutation

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Another example of "row dot column":

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$2 \times 2 \qquad 2 \times 2$$

conformable

$$AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

..... a 2×2 matrix results

Solutions of equations

Having defined matrix multiplication, we can now express a set of simultaneous linear equations in matrix form.

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e.g.
$$ax + by = e$$
$$cx + dy = f$$

i.e.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$
$$(2\times 2) \quad (2\times 1) \rightarrow (2\times 1)$$

I would tend to write this as

$$\boxed{A\,\underset{\sim}{x} = \underset{\sim}{b}}$$

$$\boxed{A\underset{\sim}{x} = \underset{\sim}{b}}$$

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where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv$ coefficient matrix

$\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix} \equiv$ solution vector

$\underset{\sim}{b} = \begin{pmatrix} e \\ f \end{pmatrix} \equiv$ "right hand side"

determining whether the system is homogeneous $\underset{\sim}{b} = \underset{\sim}{0}$ or inhomogeneous $\underset{\sim}{b} \neq \underset{\sim}{0}$.

Let's solve $\boxed{\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}}$ and see what the result is.

multiply the first equation by d and the second by b to get

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$$(ad)x + (bd)y = ed$$
$$(cb)x + (db)y = fb$$

subtract $\quad (ad - bc)x = (ed - fb)$

$$\Rightarrow \boxed{x = \frac{ed - fb}{ad - bc}}$$

Substituting this value of x, $\quad \Rightarrow \boxed{y = \frac{af - ec}{ad - bc}}$

$$ax + by = e$$
$$cx + dy = f$$

$$\Rightarrow \quad \boxed{x = \frac{ed - fb}{ad - bc}} \quad \boxed{y = \frac{af - ec}{ad - bc}}$$

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Now, define a quantity $\boxed{D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$

- note the straight vertical lines, this is <u>not</u> a matrix
  it is simply a number

Then, $\boxed{x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}}$ and $\boxed{y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}}$

$$ax + by = e$$
$$cx + dy = f$$
$$\Rightarrow \quad \boxed{x = \frac{ed - fb}{ad - bc}} \quad \boxed{y = \frac{af - ec}{ad - bc}}$$

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and
$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D}, \quad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- D is a 2×2 determinant defined by the coefficient matrix
- To visualise a 2×2 determinant think of 2 arrows and a subtraction i.e.

$$\text{“}\begin{matrix} a & & b \\ & \searrow & \\ c & & d \end{matrix}\text{”} - \text{“}\begin{matrix} a & & b \\ & \swarrow & \\ c & & d \end{matrix}\text{”} \equiv ad - bc$$

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Ex
$$x + 2y = 5$$
$$3x - y = 8$$
gives $D = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1 - 6 = -7$

Then, $x = \dfrac{\begin{vmatrix} 5 & 2 \\ 8 & -1 \end{vmatrix}}{D}$, $y = \dfrac{\begin{vmatrix} 1 & 5 \\ 3 & 8 \end{vmatrix}}{D}$

ie. $x = 3$, $y = 1$

Can you spot the pattern?

$x = \dfrac{\text{"D with RHS in column 1"}}{D}$, $y = \dfrac{\text{"D with RHS in column 2"}}{D}$

RHS ≡ right hand side → CRAMER'S RULE ←

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This also works for systems of higher order.

Consider 3 simultaneous equations ....

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

typo !

x, y, z are the 3 unknowns and the other symbols denote constants

Then, $x = \dfrac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{D}$, $y = \dfrac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{D}$ and

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$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{D}$$

where now

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

but we need to define 3×3 determinants.

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In the vector calculus section, I gave the particular case

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \left(b_2c_3 - c_2b_3\right) - b_1 \left(a_2c_3 - c_2a_3\right) + c_1 \left(a_2b_3 - b_2a_3\right)$$

i.e. go along row 1 and, in each case, cover up the row and column of that element to find the 2x2 determinant.

In fact, one can use any row or any column, as long as you keep the signs of each term right. The pattern that gives the signs is $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ in the 3x3 case, $\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$ in the 4x4 case,

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \text{ in the } 3\times 3 \text{ case}, \qquad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix} \text{ in the } 4\times 4 \text{ case},$$

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and so on. Going along the top row, for example, introduces a minus sign for the b_1 term. The signs are $(-1)^{i+j}$.

So, one can also use the first column to get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

NB.

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

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Terminology:

- The above process is called the "LAPLACE EXPANSION" or the "LAPLACE DEVELOPMENT"
- The 2x2 determinants that result are called "MINORS"
- The "signed minor" is the minor with the appropriate sign and is called the "COFACTOR" (denoted A_{ij})

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

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e.g. For the <u>a_2 term</u> in the last expansion.

a_2 lies in row 2 $(i=2)$ and column 1 $(j=1)$.

The appropriate sign is $(-1)^{i+j} = (-1)^3 = -1$

The <u>minor</u> in this case is $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ and

the <u>cofactor</u> is $A_{21} = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

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The determinant is then $= \displaystyle\sum_{\substack{\text{along row} \\ \text{(or down column)}}} a_{ij} (-1)^{i+j} (\text{minor of } a_{ij})$

$$= \sum_{\substack{\text{row/} \\ \text{column}}} a_{ij} A_{ij}$$

The determinant is then $= \sum_{\substack{\text{along row} \\ \text{(or down column)}}} a_{ij} (-1)^{i+j} (\text{minor of } a_{ij})$

$$= \sum_{\substack{\text{row/} \\ \text{column}}} a_{ij} A_{ij}$$

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This is a general rule for <u>any</u> order of system. ✱✱

e.g. for a 4×4 determinant, the Laplace expansion gives minors that are 3×3 determinants. These minors can themselves be expanded by Laplace development to give the result in terms of 2×2 determinants.

— it sounds like a lot of work (and it is!) but it works.

Returning to Cramer's rule ...

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Ex $3x - y - z = 2$
$x - 2y - 3z = 0$
$4x + y + 2z = 4$

$$\Rightarrow D = \begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = 2.$$

Then, $$x = \frac{\begin{vmatrix} 2 & -1 & -1 \\ 0 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}}{D} = \frac{2}{2} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -3 \\ 4 & 4 & 2 \end{vmatrix}}{D} = \frac{4}{2} = 2$$

$$z = \frac{\begin{vmatrix} 3 & -1 & 2 \\ 1 & -2 & 0 \\ 4 & 1 & 4 \end{vmatrix}}{D} = \frac{-2}{2} = -1$$

ie solution is

$(x, y, z) = (1, 2, -1)$.

Here is the details of the workings involved in that last example.

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To solve
$$\begin{aligned} 3x - y - z &= 2 \\ x - 2y - 3z &= 0 \\ 4x + y + 2z &= 4 \end{aligned}$$

write the system in matrix form, i.e.

$$\underbrace{\begin{pmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{pmatrix}}_{\text{COEFFICIENT MATRIX}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}}_{\text{"RHS"}}$$

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Determinant of the coefficient matrix, D (expanding along row 1)

$$D = \begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + (-1).\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix}$$

i.e. $D = 3.\left[-4-(-3)\right] + 1\left[2-(-12)\right] - \left[1-(-8)\right]$

i.e. $D = 3\left[-4+3\right] + \left[2+12\right] - \left[1+8\right]$

i.e. $D = -3 + 14 - 9$

$\therefore D = 2$

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"D with RHS in column 1"

$$\begin{vmatrix} 2 & -1 & -1 \\ 0 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} - 0 + 4\begin{vmatrix} -1 & -1 \\ -2 & -3 \end{vmatrix}$$

$$= 2\left[-4-(-3)\right] + 4\left[3-2\right]$$

$$= -2 + 4 = 2.$$

(expanding down column 1 since the zero present simplifies the calculation)

"D with RHS in column 2"

$$\begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -3 \\ 4 & 4 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 0 - 4\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}$$

(expanding down column 2 to exploit the zero, don't forget that the sign table has been used here i.e. $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$)

"D with RHS in column 2"

$$\begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -3 \\ 4 & 4 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 0 - 4\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}$$

$$= -2\left[2-(-12)\right] - 4\left[-9-(-1)\right]$$

$$= -2.(14) - 4.(-8)$$

$$= -28 + 32$$

$$= 4.$$

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"D with RHS in column 3"

$$\begin{vmatrix} 3 & -1 & 2 \\ 1 & -2 & 0 \\ 4 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 0 + 4\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}$$

(using column 3 this time)

$$= 2\left[1-(-8)\right] + 4\left[-6-(-1)\right]$$

$$= 2.(9) + 4.(-5)$$

$$= 18 - 20$$

$$= -2.$$

Then,
$$x = \frac{\text{"D with RHS in column 1"}}{D} = \frac{2}{2} = \underline{1}$$

$$y = \frac{\text{"D with RHS in column 2"}}{D} = \frac{4}{2} = 2$$

$$z = \frac{\text{"D with RHS in column 3"}}{D} = \frac{-2}{2} = -1$$

i.e. the solution is $(x,y,z) = (1,2,-1)$.

Classification of systems of linear equations

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I. Dependence, Consistency, (In)homogeneous, Singularity

- We are now going to introduce some new terminology and classifications regarding systems of simultaneous linear equations.
- The purpose of the classifications is to categorise different systems in terms of the character and existence of solutions.

- Finally, we will end up with general rules that tell us about the solutions of a system without actually finding the solutions themselves.

- These rules are a little abstract. So, in an attempt to provide some insight into their meaning, we will look at the types of 2x2 systems where graphical visualisation of the solutions and algebraic manipulation of the equations is relatively straightforward.

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You will not find this 2x2 development of the terminology and rules in full detail in any books. If you don't like it then fair enough.; you can just go straight to the general rules and how to apply them!

Recall how we started this handout by looking at particular 2x2 systems, their graphical interpretation (in terms of the possible intersection of lines), and the nature of their solutions. Let's try to generalise these ideas...

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In general, 2x2 systems can be written as

$$\boxed{\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}}\quad ,$$

where, unless stated otherwise, we will assume that the constants a, b, c, d and e, f (i.e. those with different symbols) are distinct and non-zero.

- Graphically, these equations can be represented by the lines ....

$$y = -\frac{a}{b}x + \frac{e}{b}$$

$$y = -\frac{c}{d}x + \frac{f}{d}$$

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The equations are called DEPENDENT if one is a multiple of the other. Otherwise, they are called INDEPENDENT.

If our 2x2 system has independent equations then we have m=2 independent equations in the n=2 unknowns (i.e. x and y).

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If the gradients of the lines are unequal, then there will be a solution (the point where the lines cross).

If a solution exists, then the equations are called CONSISTENT. Otherwise, they are called INCONSISTENT.

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- One can use the fact of whether a system is HOMOGENEOUS $(e=f=0)$ or INHOMOGENEOUS (either $e \neq 0$ or $f \neq 0$, or both e and f non-zero) to classify and to identify possible solutions.

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- The possibilities are that we have :
  - no solution
  - a unique non-trivial solution
    ($x = y = 0$ is the "trivial solution")
  - a unique but trivial solution
  - an infinite number of solutions.

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- To classify different cases, one can calculate det(A), i.e. $|A|$ = the determinant of the coefficient matrix.

If $|A| = 0$, then A is said to be SINGULAR.

Note that, if we tried to solve the system using Cramer's rule then, $D = |A| = 0$ would give division by zero in the expressions for x and y.

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We will now go through the above six topics (marked as "•") for some particular forms of the general 2x2 system and then try to draw conclusions from our findings.

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The systems that we will consider are:

INHOMOGENEOUS SYSTEMS

Ex 1.
$$ax + by = e$$
$$cx + by = f$$

Ex 2.
$$ax + by = e$$
$$ax + by = f$$

Ex 3.
$$ax + by = e$$
$$(ma)x + (mb)y = (me)$$

[where m is a just a scalar i.e. a number].

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along with the following ...

HOMOGENEOUS SYSTEMS

Ex 4.

$$ax + by = 0$$
$$cx + by = 0$$

Ex 5.

$$ax + by = 0$$
$$(ma)x + (mb)y = 0$$

Ex. 1.

$$\boxed{\begin{aligned} ax + by &= e \\ cx + by &= f \end{aligned}} \Rightarrow \begin{aligned} y &= -\frac{a}{b}x + \frac{e}{b} \\ y &= -\frac{c}{b}x + \frac{f}{b} \end{aligned}$$

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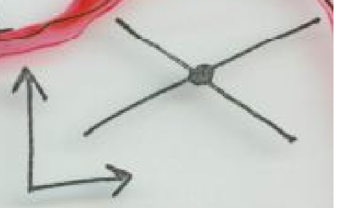
- <u>not</u> a linear multiple (neither $a = c$ nor $e = f$)

  → independent equations, 2 equations in 2 unknowns
- gradients of two lines not equal ⇒ lines cross once

  → unique solution and equations consistent



- equations inhomogeneous ($e \neq 0$ and $f \neq 0$)

  ∴ $y = x = 0$ <u>not</u> a solution
- unique non-trivial solution
- $|A| = ab - bc = b(a-c) \neq 0$, since $a \neq c$.

  A non-singular

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Ex 2. $\boxed{\begin{aligned} ax + by &= e \\ ax + by &= f \end{aligned}}$ $\Rightarrow$ $y = -\frac{a}{b}x + e/b$

$y = -\frac{a}{b}x + f/b$.

x missing (typo!)

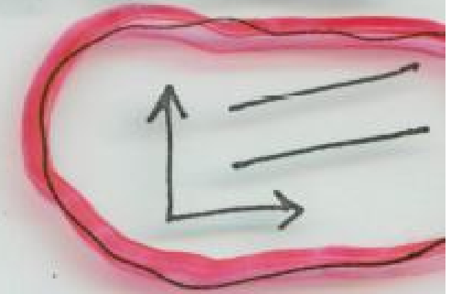
- not a linear multiple since $e \neq f$

→ independent equations, 2 equations in 2 unknowns

- gradients of lines equal but intercepts at $x = 0$ different

→ two lines can never cross and there is no solution:

the equations are inconsistent



- inhomogeneous system with no solution
- $|A| = ab - ba$ and A is singular (i.e. $|A| = 0$)

Ex 3.

$$\boxed{\begin{aligned} ax + by &= e \\ (ma)x + (mb)y &= me \end{aligned}} \Rightarrow \begin{aligned} y &= -\frac{a}{b}x + \frac{e}{b} \\ y &= -\frac{ma}{mb}x + \frac{me}{mb} \end{aligned}$$

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i.e.

$$y = -\frac{a}{b}x + \frac{e}{b}$$

$$y = -\frac{a}{b}x + \frac{e}{b}$$

- One equation is a linear multiple (m) of the other, i.e. the equations are essentially identical and are thus dependent. We only have $\underline{1}$ equation in 2 unknowns

Ex 3.

$$\boxed{\begin{aligned} ax + by &= e \\ (ma)x + (mb)y &= me \end{aligned}}$$

continued ...

$$y = -\frac{a}{b}x + \frac{e}{b}$$

$$y = -\frac{a}{b}x + \frac{e}{b}$$

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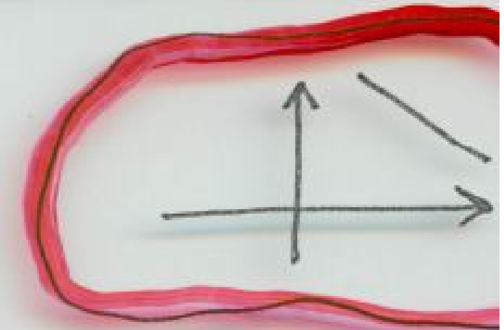
- they are the same equation, any point on this line is a solution, the equations are consistent.



Ex 3. $\boxed{\begin{aligned} ax + by &= e \\ (ma)x + (mb)y &= me \end{aligned}}$

continued ...

$$y = -\frac{a}{b}x + \frac{e}{b}$$

$$y = -\frac{a}{b}x + \frac{e}{b}$$

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- we have an inhomogeneous system (not supporting the trivial solution; note that $x=y=0$ requires the RHS constants e and f to be zero). There are an infinite number of solutions (all the points lying on the line).

- $|A| = \begin{vmatrix} a & b \\ ma & mb \end{vmatrix} = a(mb) - b(ma) = 0$

  i.e. A is singular.

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Ex. 4

$$\boxed{\begin{array}{l} ax+by=0 \\ cx+by=0 \end{array}}$$

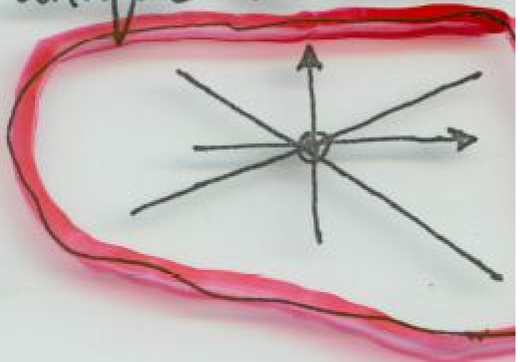
Lines are $y=\frac{-a}{b}x$

$y=-\frac{c}{b}x$

- Equations independent since $a \neq c$: 2 equations in 2 unknowns (not linear multiple of each other)

- $a \neq c \Rightarrow$ different gradients $\Rightarrow$ intersection at a single solution and the equations are consistent.

- Right hand side is null vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: system homogeneous. Lines thus intersect at the origin, giving unique but trivial solution



- $|A| = ab-bc = b(a-c) \neq 0$ since $a \neq c$

i.e. $\det A \neq 0$ and A is non-singular

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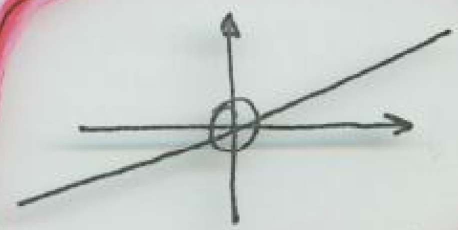
Ex. 5 $\boxed{\begin{aligned} ax+by&=0\\ (ma)x+(mb)y&=0\end{aligned}}$

$y=-\frac{a}{b}x$

$y=-\frac{\cancel{m}a}{\cancel{m}b}x$



- linear multiple, equations dependent : 1 equation in 2 unknowns
- Any point on line is solution : equations consistent
- Homogeneous with an infinite number of solutions (all points on line)
- $|A| = a(mb)-b(ma) = 0$ A singular

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Let's tabulate our findings for these $n \times n = 2 \times 2$ systems with m independent equations

INHOMOGENEOUS SYSTEMS ($RHS \neq \underset{\sim}{0}$)

	$ax+by=e$ $cx+by=f$	$ax+by=e$ $ax+by=f$	$ax+by=e$ $(ma)x+(mb)y=me$
dependent/ independent	indept.	indept.	dept.
m indept. equations, $m=$?	2	2	1
consistent/ inconsistent	consistent	inconsistent	consistent
solutions?	unique non-trivial	no solution	infinite number
$\det(A)=0$ or $\det(A)\neq 0$	$\lvert A\rvert \neq 0$	$\lvert A\rvert = 0$	$\lvert A\rvert = 0$
A singular or non-singular	non-singular	singular	singular

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HOMOGENEOUS SYSTEMS (RHS = $\underset{\sim}{0}$)

	$ax+by=0$ $cx+by=0$	$ax+by=0$ $(ma)x+(mb)y=0$
dependent/ independent	indept.	dependent
m indept. equations, m=?	2	1
consistent/ inconsistent	consistent	consistent
solutions?	unique trivial	infinite number (including trivial)
$\det(A)=0$ or $\det(A)\neq 0$	$\lvert A\rvert \neq 0$	$\lvert A\rvert = 0$
A singular or non-singular	non-singular	singular

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Based on these finding, what might be true for nxn systems (i.e. n equations in n unknowns), where m is the number of independent equations?

nxn unhomogeneous systems

- If A non-singular ($|A| \neq 0$) and $m = n$

then get A UNIQUE NON-TRIVIAL SOLUTION

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If A singular ($|A|=0$) and $m<n$

then get AN INFINITE NUMBER OF SOLUTIONS

If A singular ($|A|=0$) and $m=n$

then get NO SOLUTION (inconsistency)

.---- and for the homogeneous systems ...

nxn homogeneous systems

- Does the trivial solution always exist? → YES
- A non-singular ($|A| \neq 0$) → ONLY THE TRIVIAL SOLUTION EXISTS
- A singular ($|A| = 0$) → AN INFINITE NUMBER OF SOLUTIONS (INCLUDING THE TRIVIAL SOLUTION)