Mathematical Methods and Applications

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HANDOUT 8

CONTENTS

- Classification of systems of linear equations

  II. The Rank of a matrix

- Linear Independence of
  - equations
  - vectors
  - functions

Classification of systems of linear equations

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II. The RANK of a matrix

We have now established the terminology, geometry and properties of solutions of 2x2 systems and we have proposed that our findings also apply to all nxn systems. The catch is that, with higher-order systems, we need a means to determine m (the number of independent equations): this is determined by what is called the RANK of the matrix A. Then, by using information about the right-hand side $\underset{\sim}{b}$, one can work out the existence and number of solutions (as was done for the above 2x2 systems)

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Definition

The RANK r of a matrix = the largest non-zero determinant that can be found from its elements (in the order that they appear in the matrix).

In other words, a matrix of rank r contains at least one square sub-matrix of r rows and columns with non-zero determinant while all square sub-matrices of at least (r+1) rows and columns have zero determinant.

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A <u>square sub-matrix</u> of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ would be $\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ or $\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}$

or even $\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}$ or $\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}$. There are a lot to choose from

but we would only need to find <u>one</u> with non-zero determinant.

If A is a square matrix of order n then if the <u>rank of A is n</u> it follows that $|A| \neq 0$ and A is <u>non-singular</u>.

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Some Examples

In each case, we will look for the largest subdeterminant that is non-zero.

Ex1

$$\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$$

A 2x2 square matrix.

The largest square matrix that we can form from this is the matrix itself. So, let's check the determinant of the given 2x2 matrix to see if that is non-zero.

$$\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix} = 4.5 - 2.1 = 20 - 2 = 18 \neq 0$$

$\boxed{\text{i.e. a matrix with 2 rows and columns with non-zero determinant} \Rightarrow \text{matrix has rank } r = 2.}$

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Ex2 $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$ Again, another square matrix.

So check the determinant of this 2x2 matrix.

$$\begin{vmatrix} 6 & 3 \\ 8 & 4 \end{vmatrix} = 6.4 - 3.8 = 24 - 24 = 0. \quad \therefore r < 2.$$

Let's now look for square submatrices.

These are 1x1 matrices i.e. $[6]$, $[3]$, $[8]$, $[4]$.

A consistent definition of the determinant (in terms of the solution of equations) of a 1x1 matrix is the matrix element itself.

$\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$ Ex2

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We now try to find a determinant of one of those submatrices that is non-zero. In this case, they all have non-zero determinant $\therefore$ rank, $r = 1$.

Ex3 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ $\therefore r < 2$

while all 1×1 submatrices have zero element and thus zero determinant, hence $r < 1$.

$\therefore$ rank r of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is 0.

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Ex 4 $\begin{bmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{bmatrix}$ has $\begin{vmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{vmatrix} = 1.(21-30)$

$$-2(12-54)$$

$$+8(20-63)$$

$$= 9 + 84 - 344$$

$$= -269 \neq 0$$

$\therefore$ Matrix has rank $r = 3$.

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Ex5 $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has $\begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 3(12-15)$

$-4(6-12)$

$+5(5-8)$

$= -9+24-15 = 0$

$\therefore$ rank $r < 3$

Now look for a 2x2 submatrix with non-zero determinant.

e.g. test any of $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, $\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}$, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$, $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$, ...

.. we could also test $\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix}$, $\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$, $\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix}$, ...etc!

Ex5 $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ continued ...

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But we only need to find ONE submatrix with non-zero determinant.

For example, $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3.2 - 4.1 = 6 - 4 = 2 \neq 0$

$\therefore$ rank $r = 2$.

We now know how to determine the rank of a single matrix.

Let's try and use this concept to characterise the solution(s) of simultaneous linear equations...

Let's try and use this concept to characterise the solution(s) of simultaneous linear equations...

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For example, adopting the notation of a_{ij} for the ij^{th} element of coefficient matrix A and b_i for the elements of the right hand side vector, a 3×3 system is written as

$$\boxed{\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}}$$

i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or simply

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

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One now defines an AUGMENTED COEFFICIENT MATRIX by including $\underset{\sim}{b}$ as a fourth column; let's denote this new matrix by A_b

i.e. $$A_b = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}.$$

We will now explore the relationship between the ranks of A and A_b and the solution(s) of the equation system.

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This is most easily done by considering 2×2 systems.

For a 2×2 system
$$\boxed{\begin{aligned} ax+by &= e \\ cx+dy &= f \end{aligned}}$$

one has $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A_b = \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$.

$$A_b = \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$$

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To see if A_b is of rank 2, one only needs to test $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\begin{vmatrix} b & e \\ d & f \end{vmatrix}$ and $\begin{vmatrix} a & e \\ c & f \end{vmatrix}$.

If these are all zero then A_b will be of rank 1 if it contains a single non-zero element, otherwise it will be of rank 0.

We will now consider the five 2x2 systems listed on p217 and calculate the rank of A and the rank of A_b.

The results will be tabulated to see if a pattern emerges regarding the solutions of each case ...

$\boxed{n = 2}$ ... two unknowns: x and y

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(i) $\begin{cases} ax + by = e \\ cx + by = f \end{cases} \rightarrow A = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$

$A_b = \begin{bmatrix} a & b & e \\ c & b & f \end{bmatrix}$

Recall that we denote elements with different symbols as distinct and non-zero.

$|A| = ab - bc = b(a-c) \neq 0 \quad \therefore$ A is of rank 2.

Since A_b contains A, A_b is also of rank 2

i.e. $\boxed{\text{rank}(A) = \text{rank}(A_b) = n}$

(where $n=2$ since we are dealing with $n \times n$ i.e. 2×2 systems)

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$\boxed{n = 2}$ ... two unknowns: x and y

(ii) $\begin{aligned} ax + by &= e \\ ax + by &= f \end{aligned}$ $\rightarrow$ $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

$A_b = \begin{bmatrix} a & b & e \\ a & b & f \end{bmatrix}$

$|A| = ab - ba = 0$ $\therefore$ <u>A is of rank 1</u> (elements assumed not all zero)

while $\begin{vmatrix} a & e \\ a & f \end{vmatrix} = af - ea = a(f-e) \neq 0$. $\therefore$ <u>A_b is of rank 2.</u>

i.e. $\boxed{\text{rank}(A) < \text{rank}(A_b) = n}$

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(iii) $\boxed{\begin{aligned} ax + by &= e \\ (ma)x + (mb)y &= me \end{aligned}}$ $\quad n = 2$

$$\rightarrow A = \begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$$

$$|A| = a(mb) - b(ma) = 0$$

$\therefore$ A is of rank 1.

$$A_b = \begin{bmatrix} a & b & e \\ ma & mb & me \end{bmatrix}$$

While $\begin{vmatrix} a & e \\ ma & me \end{vmatrix} = a(me) - e(ma) = 0$

and $\begin{vmatrix} b & e \\ mb & me \end{vmatrix} = b(me) - e(mb) = 0$

Since $|A| = 0$ also, A_b is also of rank 1.

i.e. $\boxed{\text{rank}(A) = \text{rank}(A_b) < n}$

$\boxed{n = 2}$ ... two unknowns: x and y

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(iv)

$$\begin{aligned} ax + by &= 0 \\ cx + by &= 0 \end{aligned} \longrightarrow \quad A = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$$

$$A_b = \begin{bmatrix} a & b & 0 \\ c & b & 0 \end{bmatrix}$$

$|A| = ab - bc = b(a-c) \neq 0 \quad \therefore$ both A and A_b rank 2

i.e. $\boxed{\text{rank}(A) = \text{rank}(A_b) = n}$

And finally …

$\boxed{n = 2}$ (two unknowns: x and y)

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(v) $\boxed{\begin{aligned} ax + by &= 0 \\ (ma)x + (mb)y &= 0 \end{aligned}}$ $\rightarrow$ $A = \begin{bmatrix} a & b \\ ma & mb \end{bmatrix}$

$|A| = a(mb) - b(ma) = 0$

i.e. A is of rank 1.

$A_b = \begin{bmatrix} a & b & 0 \\ ma & mb & 0 \end{bmatrix}$

while $\begin{vmatrix} a & 0 \\ ma & 0 \end{vmatrix} = \begin{vmatrix} b & 0 \\ mb & 0 \end{vmatrix} = 0$

$\therefore$ $\boxed{\text{rank}(A) = \text{rank}(A_b) < n}$

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Summarise

(i)	$\text{rank}(A) = \text{rank}(A_b) = n$	: unique solution
(ii)	$\text{rank}(A) < \text{rank}(A_b) = n$	: no solution
(iii)	$\text{rank}(A) = \text{rank}(A_b) < n$	: infinite number of sol$^{\text{n}}$s
(iv)	$\text{rank}(A) = \text{rank}(A_b) = n$	: unique solution
(v)	$\text{rank}(A) = \text{rank}(A_b) < n$	: infinite number of sol$^{\text{n}}$s

→ ONLY 3 DISTINCT CASES

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ONLY 3 DISTINCT CASES

For an nxn system one finds that ...

- rank(A) = rank(A_b) = n ⇒ UNIQUE SOLUTION
- rank(A) = rank(A_b) < n ⇒ INFINITE NUMBER OF SOLUTIONS
- rank(A) < rank(A_b) ⇒ NO SOLUTION

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We also know that ...

- Homogeneous systems are always consistent because they always have at least the trivial solution.
- $\det(A) = 0$ is required for non-trivial solutions of homogeneous systems.
- Inhomogeneous systems only have non-trivial solutions.

The above six results should be memorised; they tell us the character of the solutions without having to work them out!

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What does $\det(A) \equiv |A|$ tell us?

(unique solution, but ...)

Homogeneous systems : $|A| \neq 0 \Rightarrow$ only trivial solution

$|A| = 0 \Rightarrow$ infinite number of solutions

Inhomogeneous systems : $|A| \neq 0 \Rightarrow$ unique non-trivial solution

$|A| = 0 \Rightarrow$ infinite number of solutions when $r(A) = r(A_b) = m < n$

OR

additional possibility: no solution when $r(A) < r(A_b)$

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Note that in Cramer's rule we have

$$x = \frac{\left|\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}\right|}{|A|}, \quad y = \frac{\left|\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}\right|}{|A|}, \text{ etc.}$$

and we cannot find a solution when $|A| = 0$ (using Cramer's rule).

– we cannot divide by zero (when A is singular).

... these are the cases when we <u>don't have</u> either a trivial or non-trivial unique solution ....

some *optional* additional text

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What we have covered so far ...

- consistency and number of solutions of 2x2 systems
  - geometry and algebraic manipulation
- definition of a matrix
- matrix arithmetic
  - addition and subtraction
  - multiplication by a scalar
  - multiplying one matrix by another ("row dot column" rule)
- solutions of equations
  - Cramer's rule
  - Laplace expansion of determinants
  - use of the rank of a matrix

In the very last category, for $n \times n$ systems,
denote rank of A by $r(A)$

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- $r(A) = r(A_b) = n \Rightarrow$ unique solution
  [or simply $r(A) = n$]
  m independent equations $(= n)$
  $|A| \neq 0$ (nonsingular A)

- $r(A) = r(A_b) < n \Rightarrow$ infinite number of solutions
  m independent equations $(< n)$
  $|A| = 0$ (singular A)

PLUS For inhomogeneous systems, we can also have inconsistency
i.e. when $r(A) < r(A_b) \Rightarrow$ no solution
$|A| = 0$ (singular A)

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This is what we will cover now ...

- clarify what we mean by "linear independence"
  - this connects with rank, singularity, vector spaces and matrices in "echelon form"

- properties of determinants
  - these allow us to simplify determinant calculations

- Special types of matrices
  - a brief account of names given to matrices with particular properties

followed by ...

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- finding the inverse of a matrix
  - the formal method
  - the row reduction method
- applications of matrices
  - some general contexts where one encounters matrices
- eigenvalues and eigenvectors
  - a very important aspect of matrices

Linear Independence of

- equations
- vectors
- functions

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Equations

Once again consider the simple 2x2 system:

$$x + y = 2 \quad \text{(i)}$$
$$x - y = 0 \quad \text{(ii)}$$

To solve this system by elimination, one may choose to firstly eliminate x from (ii) by subtracting equation (i) from (ii)

notation: i.e. (ii) → (ii) − (i)

$$\boxed{\begin{array}{ll} x+y=2\,. & \text{(i)} \\ x-y=0\,. & \text{(ii)} \end{array}}$$

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i.e. (ii) $\to$ (ii) $-$ (i) giving $(x-x)\div y-y=0-2$

i.e. $-2y=-2$

i.e. $y=\underline{1}$

We then have

$$\boxed{\begin{array}{ll} x+y=2 & \text{(i)} \\ y=\underline{1}\,. & \text{(ii)} \end{array}}$$

and we can 'back-substitute' to find x

i.e. $\boxed{x+1=2\,,\ \therefore\ x=\underline{1}\,.}$

$$\begin{aligned} x+y &= 2 \\ x-y &= 0 \end{aligned}$$

In terms of matrices, we have

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

i.e. $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$

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If we manipulate the left hand side of equations, then we have to do the same to the right hand side.

So, let's consider the augmented coefficient matrix

$$A_b = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{array}{l} \leftarrow \text{row } 1 \equiv r_1 \\ \leftarrow \text{row } 2 \equiv r_2 \end{array}$$

$$A_b = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

row 1 $\equiv r_1$

row 2 $\equiv r_2$

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To eliminate x from row 2, $r_2 \to r_2 - r_1$

i.e. $\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \end{pmatrix}$ using $r_2 \to r_2 - r_1$

"echelon form"

Now consider solving

$$x + y = 2 \quad \text{(i)}$$
$$2x + 2y = 4 \quad \ldots \text{(ii)}$$

(ii) $\to$ (ii) $-$ 2(i) gives

$$x + y = 2 \quad \text{(i)}$$
$$0 + 0 = 0 \quad \text{(ii)}$$

$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 4 \end{aligned} \Rightarrow \begin{aligned} x + y &= 2 \quad &\text{(i)} \\ 0 + 0 &= 0 \quad &\text{(ii)} \end{aligned}$$

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In terms of matrices, we have

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

and

$$A_b = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2 \end{array}$$

$r_2 \to r_2 - 2r_1$ gives

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{"echelon form"}$$

i.e. a row of zeroes, when
A_b is cast in "echelon form"

Now consider solving

$$x + y = 2 \quad (i)$$
$$x + y = 5 \quad (ii)$$

(ii) → (ii) − (i) gives

$$x + y = 2$$
$$0 = 3$$

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i.e. an inconsistency

i.e. the equations are inconsistent

i.e. the equations cannot be solved because they do not have a solution.

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In terms of matrices,

$$
\begin{aligned}
x + y &= 2 \\
x + y &= 5
\end{aligned}
$$

we have $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

and $A_b = \begin{array}{c} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix} \end{array} \begin{array}{l} \longleftarrow r_1 \\ \longleftarrow r_2 \end{array}$

$$
\begin{aligned}
x + y &= 2 \\
0 &= 3
\end{aligned}
$$

Then, $r_2 \to r_2 - r_1$ gives $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ "echelon form"

Now consider a more involved example ---

$$x + y - 2z = 7 \qquad \text{(i)}$$
$$2x - y - 4z = 2 \qquad \text{(ii)}$$
$$-5x + 4y + 10z = 1 \qquad \text{(iii)}$$
$$3x - y - 6z = 5 \qquad \text{(iv)}$$

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Write this system in matrix form (augmented) and eliminate x from equations (ii) to (iv)

$$A_b = \begin{array}{c} \\ \\ \\ \\ \end{array}\left(\begin{array}{cccc} 1 & 1 & -2 & 7 \\ 2 & -1 & -4 & 2 \\ -5 & 4 & 10 & 1 \\ 3 & -1 & -6 & 5 \end{array}\right)\begin{array}{l} \leftarrow \text{---} \; r_1 \\ \leftarrow \text{---} \; r_2 \\ \leftarrow \text{---} \; r_3 \\ \leftarrow \text{---} \; r_4 \end{array}$$

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$$\begin{array}{cl} \begin{pmatrix} 1 & 1 & -2 & 7 \\ 2 & -1 & -4 & 2 \\ -5 & 4 & 10 & 1 \\ 3 & -1 & -6 & 5 \end{pmatrix} & \begin{array}{l} \leftarrow \text{---} \; r_1 \\ \leftarrow \text{---} \; r_2 \\ \leftarrow \text{---} \; r_3 \\ \leftarrow \text{---} \; r_4 \end{array} \end{array}$$

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This can be reduced to

$$\begin{pmatrix} 1 & 1 & -2 & 7 \\ 0 & -3 & 0 & -12 \\ 0 & 9 & 0 & 36 \\ 0 & -4 & 0 & -16 \end{pmatrix}$$

using

$r_2 \to r_2 - 2r_1$

$r_3 \to r_3 + 5r_1$

$r_4 \to r_4 - 3r_1$

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coefficients of: x y z

towards echelon form

$$\begin{array}{ccccc} x & y & z & & \\ \left(\begin{array}{cccc} 1 & 1 & -2 & 7 \\ 0 & -3 & 0 & -12 \\ 0 & 9 & 0 & 36 \\ 0 & -4 & 0 & -16 \end{array}\right) & & & & \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2 \\ \leftarrow r_3 \\ \leftarrow r_4 \end{array} \end{array}$$

Now eliminate y from the last two equations.

Note that we cannot use multiples of row 1 (ie equation (i)) now, since that would re-introduce x into the last two rows. It is much simpler (and systematic) to use row 2 to eliminate y from the last two equations - - -

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i.e. $\begin{pmatrix} 1 & 1 & -2 & 7 \\ 0 & -3 & 0 & -12 \\ 0 & 9 & 0 & 36 \\ 0 & -4 & 0 & -16 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 7 \\ 0 & -3 & 0 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ "echelon form"

using $r_3 \rightarrow r_3 + 3r_2$

$r_4 \rightarrow r_4 - \frac{4}{3} r_2$

We have now reduced the original set of 4 equations to just 2 equations, namely

$$x + y - 2z = 7$$

and $$-3y = -12$$

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Note Had there been an inconsistency in the equations, then a row would have resulted in

A_b of the form

$$\begin{pmatrix} - & - & - & - \\ 0 & 0 & 0 & b \\ - & - & - & - \end{pmatrix},$$

where $b \neq 0$.

Let's state formally what we have been doing ---

The rules for solving sets of equations by elimination are called the ELEMENTARY ROW OPERATIONS.

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i.e. we can

(i) interchange two rows

(ii) multiply a row by a number

(iii) add a multiple of one row to another row

The systematic way to apply elementary row operations is to reduce the matrix to "echelon form". Echelon means "staircase" here and we want to introduce zeros in the matrix to give the following staircase pattern

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

where $*$ are non-zero elements.

In terms of equations, we would then have something like

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

"echelon form"

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

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and we can solve this system by working 'upwards'

i.e. '<u>back-substitution</u>',

i.e. $z = b_3 / a_{33}$ (we now know z)

then $a_{22} y + a_{23} z = b_2$ (and we can solve to find y)

then $a_{11} x + a_{12} y + a_{13} z = b_1$ (knowing y and z, we can solve this to find x).

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In general, for an nxn system we have...

For consistent equations $\quad \text{rank}(A) = \text{rank}(A_b)$

where the number of non-zero rows of A in echelon form equals m, i.e. number of linearly independent equations

and if $\text{rank}(A) = \text{rank}(A_b) = n$

then we have $m = n$ linearly independent equations and a unique solution

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but if rank(A) = rank(A_b) < n

the we have m < n linearly independent equations

and an infinite number of solutions

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For inconsistent equations

$$\boxed{\text{rank}(A) < \text{rank}(A_b)}$$

i.e. the number of non-zero rows of A in echelon form is less than the number of non-zero rows of A_b in echelon form.

next section ...

Linear Independence of

- equations
- vectors
- functions

Vectors

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- $\underset{\sim}{v}_1 = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{v}_2 = \underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{v}_3 = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

  are called linearly <u>dependent</u> because $\underset{\sim}{v}_1 + \underset{\sim}{v}_2 - \underset{\sim}{v}_3 = \underset{\sim}{0}$.

  We can write each as a linear combination of the other two

  e.g. $\underset{\sim}{v}_3 = \underset{\sim}{v}_1 + \underset{\sim}{v}_2$.

- $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are linearly <u>independent</u> because $a\underset{\sim}{i} + b\underset{\sim}{j} \neq \underset{\sim}{0}$

  always (where a and b are numbers not both zero).

  i.e. $a\underset{\sim}{i} + b\underset{\sim}{j} \equiv a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}a\\b\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ only when both a and b are zero.

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In general, $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n$ are linearly <u>dependent</u> if

$a_1\underset{\sim}{v}_1 + a_2\underset{\sim}{v}_2 + \ldots + a_n\underset{\sim}{v}_n = \underset{\sim}{0}$ for some numbers $a_1, a_2, \ldots, a_n$

that are not all zero.

If their linear combination cannot be set to $\underset{\sim}{0}$

without assuming $a_1 = a_2 = \ldots = a_n = 0$ then the vectors

are linearly <u>independent</u>.

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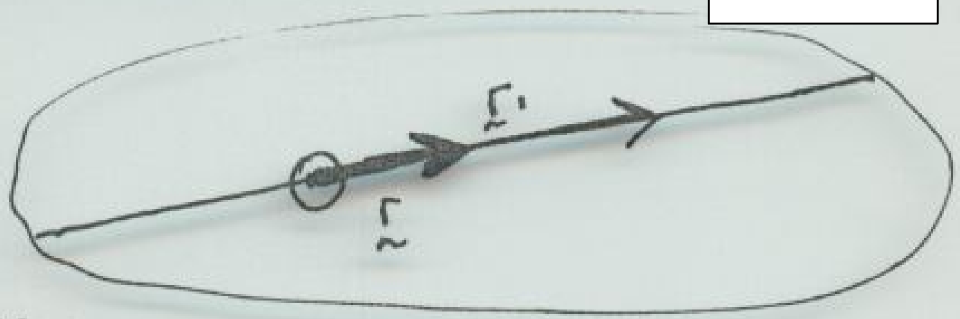
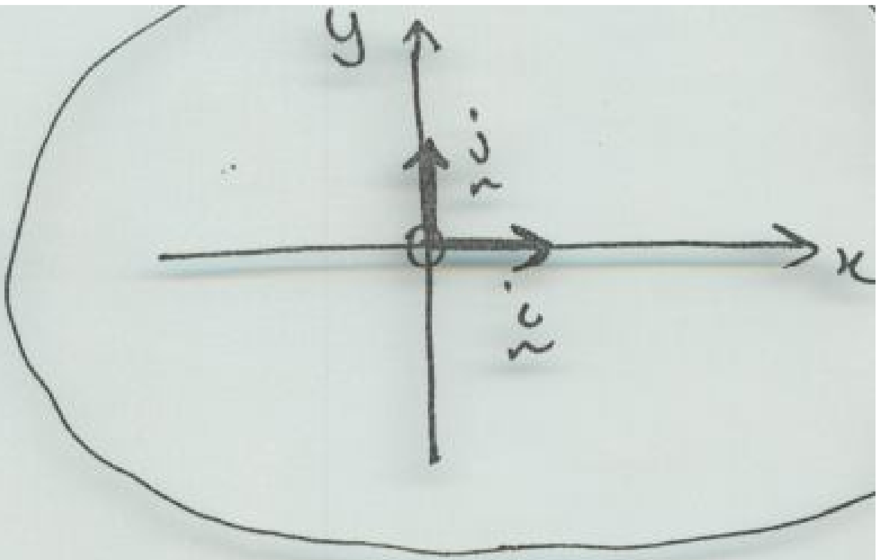
What use are linearly independent vectors?

- 1D space : along a line



a linear multiple of just one vector, say $\underset{\sim}{r}$, can give every position on that line e.g. $\underset{\sim}{r}_1 = 2\underset{\sim}{r}$

- 2D space : the x-y plane



an example of two vectors that are linearly independent is

$\underset{\sim}{i} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \underset{\sim}{j} = \begin{pmatrix}0\\1\end{pmatrix}.$

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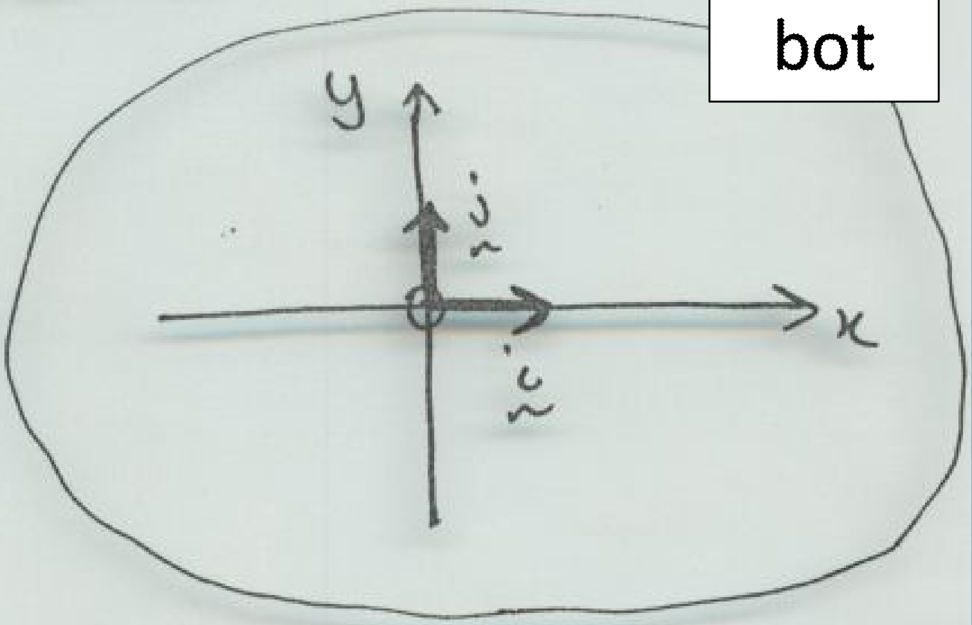
2D space : the x-y plane

an example of two vectors that are linearly independent is

$\underset{\sim}{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$



whereby, <u>any</u> other vector in this 2D space can be written as a linear combination of these i.e. $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$\Rightarrow$ maximum number of linearly independent vectors $= 2$, here.

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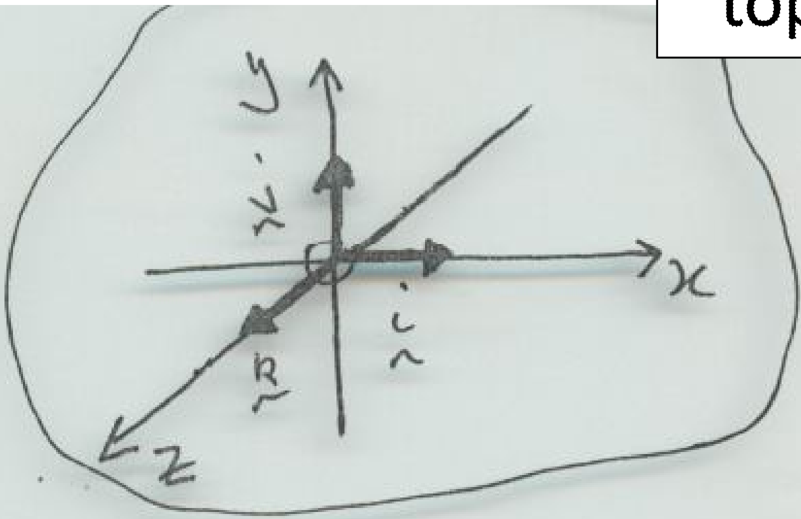
3D space : x, y and z



example of three linearly independent vectors is

$$\underset{\sim}{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \underset{\sim}{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \underset{\sim}{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

any other vector in this 3D space can be written as a linear combination of these i.e. $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$\Rightarrow$ maximum number of linearly independent vectors $= 3$, here

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In other words,

$$\boxed{\begin{aligned}&\text{the dimension of the space}\\ =\ &\text{the required number of basis vectors}\\ =\ &\text{the number of linearly independent vectors}\\ &\text{needed to "span" the space}\end{aligned}}$$

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Problems: given a set of vectors, how do we determine if they are linearly independent? If they are not independent, can we determine a smaller set of vectors that are linearly independent?

Solution: in solving sets of equations we take linear combinations of the equations to simply the system. If we do this with vectors then we can see if they can add up to zero.

an example follows ...

OK. So let's put the vectors in rows
and pretend that we are solving a set of equations ...

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An example Given vectors $\underset{\sim}{v}_1 = (1,4,-5)$
$\underset{\sim}{v}_2 = (5,2,1)$
$\underset{\sim}{v}_3 = (2,-1,3)$
$\underset{\sim}{v}_4 = (3,-6,11)$,

are they linearly independent and, if not, can we find a smaller set that is?

Solution Consider the vectors as rows in a matrix

i.e. $$A = \begin{pmatrix} 1 & 4 & -5 \\ 5 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & -6 & 11 \end{pmatrix}$$ (treat them like equations)

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$$A = \begin{pmatrix} 1 & 4 & -5 \\ 5 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & -6 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 & -5 \\ 0 & -18 & 26 \\ 0 & -9 & 13 \\ 0 & -18 & 26 \end{pmatrix}$$

using
$r_2 \to r_2 - 5r_1$
$r_3 \to r_3 - 2r_1$
$r_4 \to r_4 - 3r_1$

(e.g row 2 becomes row 2 minus 5 times row 1)

$$\longrightarrow \begin{pmatrix} 1 & 4 & -5 \\ 0 & -9 & 13 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

using
$r_2 \to r_2 - r_3$
$r_3 \to r_3 + r_2$
$r_4 \to r_4 - r_2$

Note Get zeroes in column 1 first, then do column 2, etc.

$$A = \begin{pmatrix} 1 & 4 & -5 \\ 5 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & -6 & 11 \end{pmatrix}$$

reduced to ...

$$\begin{pmatrix} 1 & 4 & -5 \\ 0 & -9 & 13 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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So ...

- there were only two independent vectors in the original four

  i.e. two could be written as a linear combination of the other (independent) two vectors

- (1, 4, -5) and (0, -9, 13) are two linearly independent vectors

$$A = \begin{pmatrix} 1 & 4 & -5 \\ 5 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & -6 & 11 \end{pmatrix}$$

reduced to ...

$$\begin{pmatrix} 1 & 4 & -5 \\ 0 & -9 & 13 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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- Note that two linearly independent rows in A is equivalent to saying that rank(A) = 2.

i.e. number of non-zero rows of A in echelon form
= rank of A = number of linearly independent row vectors

Linear Independence of

- equations
- vectors
- functions

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Functions

Linear (in)dependence of functions is defined in the same way... Functions $f_1(x), f_2(x), \dots, f_n(x)$ are linearly <u>dependent</u> if some linear combination of them is zero

i.e. $a_1 f_1(x) + a_2 f_2(x) + \dots + a_n f_n(x) = 0$

for some numbers $a_1, a_2, \dots, a_n$ not all zero.