Mathematical Methods and Applications

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Properties of determinants

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Suppose we need to work out

$$D = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}.$$

Laplace expansion gives

$$D = (b+c)\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= (b+c)\left[(a+b)(c+a) - bc\right] - a\left[b(a+b) - bc\right] + a\left[bc - c(c+a)\right]$$

= a lot of work and it's getting really messy ....

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If we could show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2 \begin{vmatrix} b & a & 0 \\ 0 & c & b \\ c & 0 & a \end{vmatrix}$$

then we could work it out in a couple of lines.

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$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2\begin{vmatrix} b & a & 0 \\ 0 & c & b \\ c & 0 & a \end{vmatrix}$$

i.e. $D = 2\begin{vmatrix} b & a & 0 \\ 0 & c & b \\ c & 0 & a \end{vmatrix} = 2\left[b.(ca) - a.(-bc) + 0 \right]$

$= \underline{4abc}$

In order to simplify determinants, we need to find the rules of manipulating their elements to ...

- simplify individual elements
- get as many zeroes in there as possible

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Let's play about with them and work out the rules

② Multiplying one row (or column) by a number?

e.g. $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ and $\Delta' = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix}$ (k = constant)

→ Laplace expand using the row (or column) with the k factor

$$\Rightarrow \Delta' = ka \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - kb \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + kc \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

$$= k \left\{ a \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - b \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + c \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} \right\} = \underline{\underline{k \Delta}}$$

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What if a whole row (or column) has only zeroes?

e.g. $\Delta = \begin{vmatrix} 0 & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix}$

→ Laplace expand across the row (or column) with the zeroes

$\Rightarrow \quad \Delta = 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + 0\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} = 0$

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• If two rows are proportional or identical?

e.g. $\Delta = \begin{vmatrix} ka & kb & kc \\ a & b & c \\ d & e & f \end{vmatrix}$

→ Laplace expand along the other row (or column)

$\Rightarrow \Delta = d\left(kbc - kcb\right) - e\left(kac - kca\right) + f\left(kab - kba\right)$

i.e. $\Delta = 0$

If they are identical then just set $k=1$ to show that $\Delta = 0$

If elements of a row (or column) are all the sum of two terms?

e.g. $\Delta = \begin{vmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d & e & f \\ g & h & i \end{vmatrix}$

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$\Rightarrow \Delta = (a_1+a_2)\begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b_1+b_2)\begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c_1+c_2)\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

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$= \left\{ a_1\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - b_1\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + c_1\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} \right\} + \left\{ a_2\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - b_2\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + c_2\begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} \right\}$

i.e. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ d & e & f \\ g & h & i \end{vmatrix}$

Adding a multiple of one row (or column) to another row (or column)?

e.g. $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ and $\Delta' = \begin{vmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{vmatrix}$

Use the previous result (top row has elements as the sum of two terms)

$$\Rightarrow \Delta' = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \underbrace{\begin{vmatrix} kd & ke & kf \\ d & e & f \\ g & h & i \end{vmatrix}}_{\text{one row is a multiple of another} \Rightarrow \text{equals zero}}$$

⇒ determinant remains <u>unchanged</u>.

What if two rows (or two columns) are interchanged?

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$$\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \text{and} \quad \Delta' = \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}. \qquad \Delta' = -a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

(expanding along row 1) $\Delta' = -\Delta$ (expanding along row 2)

So, it follows from the sign table "$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$" that interchanging 2 rows (or columns) gives $\Delta \to -\Delta$.

What if we take the transpose of the matrix?

The transpose is when the rows are taken as columns and the columns are taken as rows : $A \to A^T$ ("A transpose")

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

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e.g. $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ then $A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, etc.

In the 2x2 case $|A| = ad - bc$, $|A^T| = ad - cb = |A|$.

This property can be used to prove the 3x3 case and then that can be used to prove the 4x4 case, and so on.

$$|A| = |A^T|$$

Summarise ...

Properties of determinants

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1) each element of one row (or one column) times K

$\Rightarrow \Delta \to K\Delta$

2) $\Delta = 0$ if (a) all elements in one row (or column) zero

or (b) two rows (or columns) identical or proportional

3) Interchanging two rows (or two columns) $\Rightarrow \Delta \to -\Delta$

4) Δ unchanged if (a) $A \to A^T$

or (b) add multiples of one row (or column) to another

Ex

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$$D = \begin{vmatrix} 4 & 3 & 0 & 1 \\ 9 & 7 & 2 & 3 \\ 4 & 0 & 2 & 1 \\ 3 & -1 & 4 & 0 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 3 & 0 & 4 \\ 3 & 7 & 1 & 9 \\ 1 & 0 & 1 & 4 \\ 0 & -1 & 2 & 3 \end{vmatrix}$$

Interchanging columns 1 and 4
($c_1 \Leftrightarrow c_4 \Rightarrow$ minus sign)

Factor out 2 from column 3

$$= -2 \begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & -1 & 2 & 3 \end{vmatrix}$$

$r_2 \rightarrow r_2 - 3r_1$

$r_3 \rightarrow r_3 - r_1$

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so far ... $D = -2\begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & -1 & 2 & 3 \end{vmatrix}$

$$= -2\begin{vmatrix} -2 & 1 & -3 \\ -3 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$$

laplace expansion down column 1 gives only one term that is non-zero

$$= -2\begin{vmatrix} 1 & 2 & -3 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

factor out -1 from column 1 ⇒ minus sign

$c_1 \leftrightarrow c_2$ (another minus sign) : two minus signs cancel out

$$= -2\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & -3 & 9 \end{vmatrix}$$

$r_2 \to r_2 - r_1$

$r_3 \to r_3 - 2r_1$

$$= -2\begin{vmatrix} 1 & 3 \\ -3 & 9 \end{vmatrix} = -2(9+9)$$

$$= -36.$$

Where a laplace expansion down column 1 gives a single non-zero term

Special types of matrices (I)

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- The transpose, A^T : interchange rows and columns in A
- The complex conjugate, A^* : take the complex conjugate of each element
- Symmetric matrix, $A = A^T$ e.g. $A = \begin{pmatrix} a & b & c \\ b & e & d \\ c & d & f \end{pmatrix}$
- Hermitian matrix, $A = A^\dagger$ (read as "A dagger")

  where $A^\dagger = (A^T)^*$

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- The unit matrix $\underline{I}$ = square matrix full of zeroes except for the main diagonal which has $\underline{1}$'s

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, etc

The unit matrix plays a role in matrix algebra similar to that played by the number $\underline{1}$ in ordinary algebra

e.g. $\underline{I}A = A\underline{I} = A$

or $\underline{I}\,\underline{I} = \underline{I}$, $\underline{I}^n = \underline{I}$ $(n = 1, 2, 3, \ldots)$

Finding the inverse of a matrix

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The inverse of $n \times n$ matrix A is another $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

- If A does not have an inverse then it is singular
- $II = I \Rightarrow I$ is its own inverse

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Ex

if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

then $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix}$

since $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Ex

if $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ (a diagonal matrix)

then $A^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/\lambda_n \end{bmatrix}$

since $AA^{-1} = \begin{bmatrix} \lambda_1/\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2/\lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n/\lambda_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

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We have verified a couple of examples by direct multiplication but how do we find A^{-1}?

There are actually a number of ways. I will outline two of these

- the formal method

- the row reduction method

The Formal Method

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This can be proved by using Cramer's method, but we will just state the result here.

Recall the terminology used in Laplace expansion ...

... in calculating $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, consider expansion along row 1 and the a_{12} term

i.e the cofactor of a_{ij} is the signed minor of a_{ij}

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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for a_{12}, the cofactor is $A_{ij} = (-1)^{i+j} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

i.e. $A_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Now consider a matrix filled with the cofactors of each element

eg. $C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

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eg. $C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

The inverse,

$$A^{-1} = \frac{1}{|A|} C^T$$

**

C^T is called the adjoint of A.

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- Note that for A^{-1} to exist, A must be square
  - for example $\det A = |A|$ is only defined for square matrices

- Given A^{-1}, we can solve the system $A\underset{\sim}{x} = \underset{\sim}{b}$

i.e. $A^{-1}A\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$

i.e. $I\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$

i.e. $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$

Ex

$$\begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$A \quad \underset{\sim}{x} \quad \underset{\sim}{b}$

Calculate A^{-1} and hence solve for $\underset{\sim}{x}$.

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Outline Ans

The matrix of cofactors for A is found (after calculating all the appropriate 2x2 determinants with their signs given by the sign table) . . .

$$C = \begin{bmatrix} -5 & +4 & -8 \\ +11 & -9 & +17 \\ +6 & -5 & +10 \end{bmatrix}$$

$[\ a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}\ ]$

$|A| = 5\times(-5) + 0\times 11 + 4\times 6 = -1.$ (down 1st column)

$\rightarrow A^{-1} = \dfrac{1}{|A|} C^T$

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$$C = \begin{bmatrix} -5 & +4 & -8 \\ +11 & -9 & +17 \\ +6 & -5 & +10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} C^{T} = -1 \cdot \begin{bmatrix} -5 & 11 & +6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix} = \begin{bmatrix} +5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \underset{\sim}{b} = \begin{bmatrix} +5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} \text{ i.e. } x = 3,\ y = -2,\ z = 3$$

the full working (***without*** the above sign typos !) follows on page 262 ...

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Full Ans

The matrix of cofactors of A, $C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

where $A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} \Rightarrow$

$A_{11} = +\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -2-3 = -5$

$A_{21} = -\begin{vmatrix} 8 & 1 \\ 3 & -1 \end{vmatrix} = -(-8-3) = +11$

$A_{31} = +\begin{vmatrix} 8 & 1 \\ 2 & 1 \end{vmatrix} = 8-2 = 6$

recall sign table $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

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$$A_{12} = -\begin{vmatrix} 0 & 1 \\ 4 & -1 \end{vmatrix} = -(0-4) = +4 \quad , \quad A_{13} = +\begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix} = 0-8 = -8$$

$$A_{22} = +\begin{vmatrix} 5 & 1 \\ 4 & -1 \end{vmatrix} = -5-4 = -9 \quad , \quad A_{23} = -\begin{vmatrix} 5 & 8 \\ 4 & 3 \end{vmatrix} = -(15-32) = +17$$

$$A_{32} = -\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = -(5-0) = -5 \quad . \quad A_{33} = +\begin{vmatrix} 5 & 8 \\ 0 & 2 \end{vmatrix} = 10-0 = 10$$

The matrix of cofactors of A, $C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

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$$\Rightarrow C = \begin{bmatrix} -5 & 4 & -8 \\ 11 & -9 & 17 \\ 6 & -5 & 10 \end{bmatrix}$$

and $C^T = \begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix}$

(swapping rows with columns to get the transpose)

$$A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}.$$

$$C = \begin{bmatrix} -5 & 4 & -8 \\ 11 & -9 & 17 \\ 6 & -5 & 10 \end{bmatrix}$$

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$\det(A) = 5.A_{11} + 0.A_{21} + 4.A_{31}$ (going down 1st column to exploit the zero term)

$= 5.\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + 0.\left\{-\begin{vmatrix} 8 & 1 \\ 3 & -1 \end{vmatrix}\right\} + 4.\begin{vmatrix} 8 & 1 \\ 2 & 1 \end{vmatrix}$ (ie. Laplace expansion)

$= 5.(-2-3) + 0 + 4(8-2)$

$= -25 + 24$. ie. $\det(A) = |A| = -1$

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$$\Rightarrow A^{-1} = \frac{1}{|A|} C^T = -\begin{pmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{pmatrix} = \begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix}$$

$$\Rightarrow \hat{x} = A^{-1}\hat{b} = \begin{pmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10+11-18 \\ -8-9+15 \\ 16+17-30 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$$

i.e. $x = 3$, $y = -2$, $z = 3$

The Row Reduction Method

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≡ "by applying elementary row operations to the identity matrix"

Consider solving $\boxed{A\underset{\sim}{x} = \underset{\sim}{b}}$ for $\underset{\sim}{x}$.

This system is equivalent to $\boxed{A\underset{\sim}{x} = I\underset{\sim}{b}}$

i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \underline{1} & 0 & \cdots & 0 \\ 0 & \underline{1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \underline{1} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

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$$\boxed{A\underset{\sim}{x} = I\underset{\sim}{b}}$$

If A^{-1} exists, we can apply elementary row operations to both the left and right hand side of this equation to reduce it to

$$\underset{\sim}{x} = A^{-1} I \underset{\sim}{b}$$

or, equivalently, $\boxed{I\underset{\sim}{x} = A^{-1}\underset{\sim}{b}}$.

$$A\underset{\sim}{x} = I\underset{\sim}{b}$$

$$\underset{\sim}{x} = A^{-1}I\underset{\sim}{b}$$

$$I\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

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Consider each elementary row operation as premultiplying the system with an appropriate matrix E_k ($k=1,\ldots,m$ for m elementary row operations).

We then have...

$$E_m E_{m-1} \ldots E_2 E_1 A \underset{\sim}{x} = E_m E_{m-1} \ldots E_2 E_1 I \underset{\sim}{b}$$

in order to reduce the system to...

$$I\underset{\sim}{x} = A^{-1}\underset{\sim}{b}.$$

i.e. by transforming $A \rightarrow I$ we get

$$A^{-1} = E_m E_{m-1} \ldots E_2 E_1 I$$ **

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$$E_m E_{m-1} \cdots E_2 E_1 A \underset{\sim}{x} = E_m E_{m-1} \cdots E_2 E_1 I \underset{\sim}{b}$$

$$I \underset{\sim}{x} = A^{-1} \underset{\sim}{b}$$

To simultaneously apply elementary row operations to both the left and right hand sides, it is convenient to form the "combined coefficient matrix": $[A \,\vdots\, I]$.

from left hand side (A)

from right hand side (I)

One then applies $E_1, E_2, \ldots, E_m$ to the combined matrix until it becomes $[I \,\vdots\, A^{-1}]$ then we have found A^{-1}.

Ex Solve the system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 &= 1 \\ 3x_1 - 2x_2 - 4x_3 &= -4 \end{aligned}$$

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Ans Combined coefficient matrix:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{array}\right]$$

The goal is to reduce A to I using elementary row operations

($r_1 \leftrightarrow r_2$)

by interchanging rows 1 and 2 to get the useful "1" in the top left hand corner

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -11 & -10 & 0 & -3 & 1 \end{array}\right]$$

where
$r_2 \rightarrow r_2 - 2r_1$
$r_3 \rightarrow r_3 - 3r_1$
(we are now using that "1")

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -1 & -4 & -2 & 1 & 1 \end{array}\right]$$

where
$r_3 \rightarrow r_3 - 2r_2$
(just to simplify row 3)

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$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -1 & -4 & -2 & 1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 5 & 3 & -1 & 2 & 0 \end{array}\right]$$

$r_2 \rightarrow -r_2$
$r_3 \rightarrow -r_3$ to reduce amount of negative numbers

$r_2 \leftrightarrow r_3$ get the useful "1" in the diagonal

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & -17 & -11 & 7 & 5 \end{array}\right]$$

$r_3 \rightarrow r_3 - 5r_2$ use that "1"

use that "1" to get zero on top now

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -10 & -6 & 4 & 3 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{array}\right]$$

$r_1 \rightarrow r_1 - 3r_2$

$r_3 \rightarrow r_3/(-17)$ get another "1" in the diagonal

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mid 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -10 & -6 & 4 & 3 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/17 & -2/17 & 1/17 \\ 0 & 1 & 0 & -10/17 & 11/17 & 3/17 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{array}\right]$$

$r_1 \to r_1 + 10r_3$

$r_2 \to r_2 - 4r_3$

use the "1" to finish off the column on the left hand side

we now have the form $I \,\vdots\, A^{-1}$

i.e. $$A^{-1} = \frac{1}{17}\begin{bmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ 11 & -7 & -5 \end{bmatrix}.$$

$$2x_1 + x_2 + x_3 = 5$$
$$x_1 + 3x_2 + 2x_3 = 1$$
$$3x_1 - 2x_2 - 4x_3 = -4$$

$$A\underset{\sim}{x} = I\underset{\sim}{b}$$
$$\underset{\sim}{x} = A^{-1}I\underset{\sim}{b}$$
$$I\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

H9
p265
mid 2

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ 11 & -7 & -5 \end{bmatrix}$$

Then,

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ 11 & -7 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 34 \\ -51 \\ 68 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \Rightarrow x_1 = 2,\ x_2 = -3,\ x_3 = 4$$

More special matrices (II)

- An <u>orthogonal</u> matrix has $A^{-1} = A^T$

  i.e. $A^T A = I$

- A <u>unitary</u> matrix has $A^{-1} = (A^T)^*$

  i.e. $(A^T)^* A = I$

Note: a real unitary matrix is orthogonal.

Applications of Matrices

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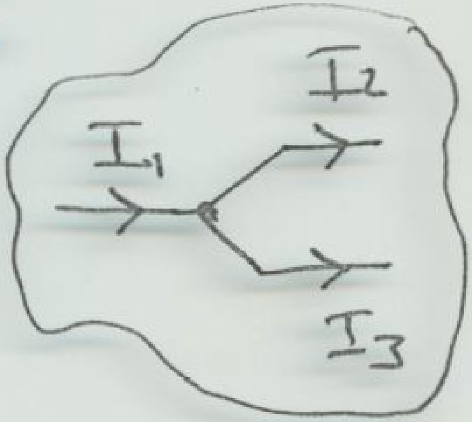
top

① Solution of simultaneous linear equations

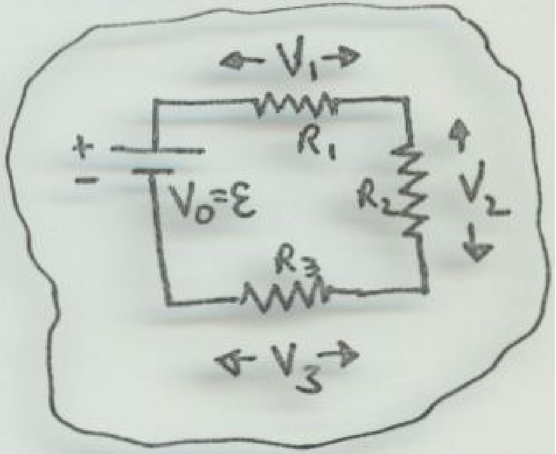
e.g. Kirchoff's laws

- sum of currents into junction = 0



$\Rightarrow I_1 - I_2 - I_3 = 0$

- sum of potential differences around a closed circuit = 0



$\Rightarrow V_0 - V_1 - V_2 - V_3 = 0$

a more complex example, resulting in a larger system of equations, is analysis of a "Wheatstone Bridge" ... (don't worry about detail)

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bot

$$\rightarrow \quad (R_3+R_4)I_1 \quad -R_3 I_2 \quad -R_4 I_3 = V$$

$$R_3 I_1 - (R_1+R_3+R_5) I_2 \quad +R_5 I_3 = 0$$

$$R_4 I_1 \quad +R_5 I_2 - (R_2+R_4+R_5) I_3 = 0$$

$$\rightarrow \begin{bmatrix} R_3+R_4 & -R_3 & -R_4 \\ R_3 & -(R_1+R_3+R_5) & R_5 \\ R_4 & R_5 & -(R_2+R_4+R_5) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

For example, use Cramer's rule to solve for I_1, I_2, I_3.

- There are many, many more areas of physics where the solution of simultaneous linear equations arises.

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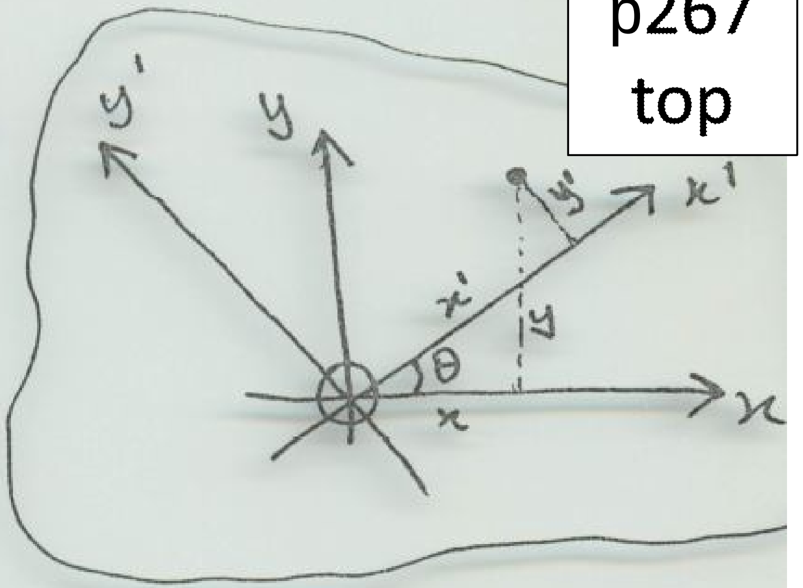
top

② Rotation of a reference frame

Say, by angle θ whereby coordinates $(x, y) \to (x', y')$



Then,

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

i.e.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$A \equiv$ "ROTATION MATRIX"

i.e. $\underset{\sim}{r}' = A\underset{\sim}{r}$ where $\underset{\sim}{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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mid

$$\underset{\sim}{r}' = A\underset{\sim}{r}$$

On geometrical grounds, the inverse matrix will correspond to a rotation of $-\theta$

i.e. $\underset{\sim}{r} = A^{-1}\underset{\sim}{r}'$

where $A^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

You can check this by multiplication i.e. that $AA^{-1} = I$.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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bot

$$\underset{\sim}{r}' = A\underset{\sim}{r}$$

A further rotation ϕ of axes (x', y') to (x'', y'') is then given by

$$\underset{\sim}{r}'' = B\underset{\sim}{r}' = BA\underset{\sim}{r}$$ where $B = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ and $\underset{\sim}{r}'' = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$.

- Note. The combined operation of "A then B" results in a matrix product BA equivalent to a rotation $\theta + \phi$.

③ General linear transformations

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top

One can define more general linear transformations (instead of just rotations) to transform a vector $\underset{\sim}{r}$ in the xyz-system to another $\underset{\sim}{r}'$ in the $x'y'z'$-system ...

i.e. $$\underset{\sim}{r}' = A\underset{\sim}{r}$$ where $$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

i.e.
$$x' = a_1 x + a_2 y + a_3 z$$
$$y' = b_1 x + b_2 y + b_3 z$$
$$z' = c_1 x + c_2 y + c_3 z$$

and $\underset{\sim}{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$, $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\underset{\sim}{r}' = A\underset{\sim}{r}$$

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mid

- One can also combine such transformations as matrix products, say BA, and use the combined transformation matrix rather than operating with A and then with B.

- Of course, to transform from $\underset{\sim}{r}'$ back to $\underset{\sim}{r}$, say, then use the inverse transformation which is equivalent to A^{-1}

$$\underset{\sim}{r}' = A\underset{\sim}{r}$$

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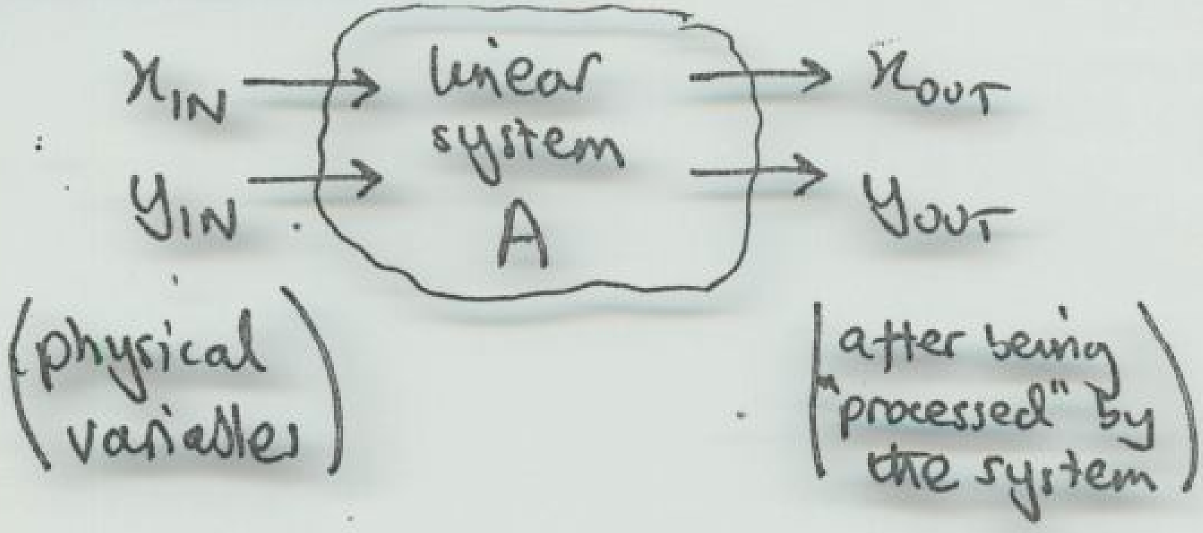
bot

- To transform from $\underset{\sim}{r}''$ back to $\underset{\sim}{r}$, where $\underset{\sim}{r}'' = BA\underset{\sim}{r}$,

then $B^{-1}\underset{\sim}{r}'' = A\underset{\sim}{r}$ and $A^{-1}B^{-1}\underset{\sim}{r}'' = \underset{\sim}{r}$

i.e. $(BA)^{-1} = A^{-1}B^{-1}$ A general result.

4 Systems Theory



linear system :

$$x_{OUT} = a\,x_{IN} + b\,y_{IN}$$

$$y_{OUT} = c\,x_{IN} + d\,y_{IN}$$

i.e. $$\begin{pmatrix} x_{OUT} \\ y_{OUT} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix}$$

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top

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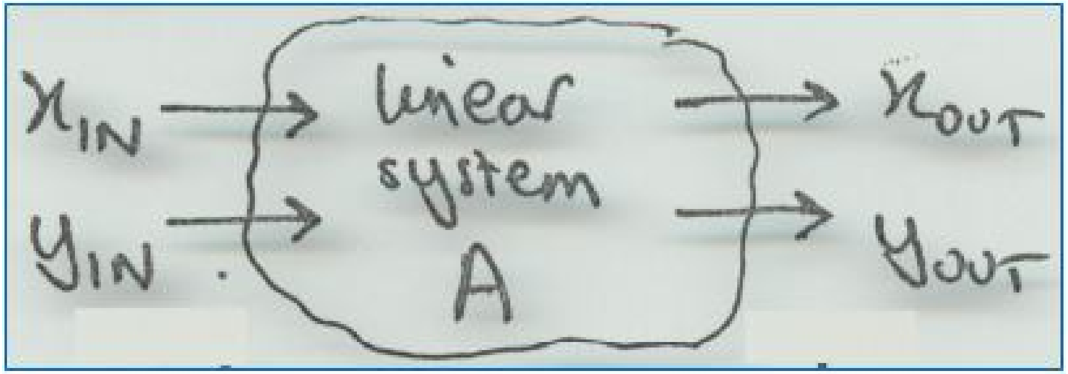
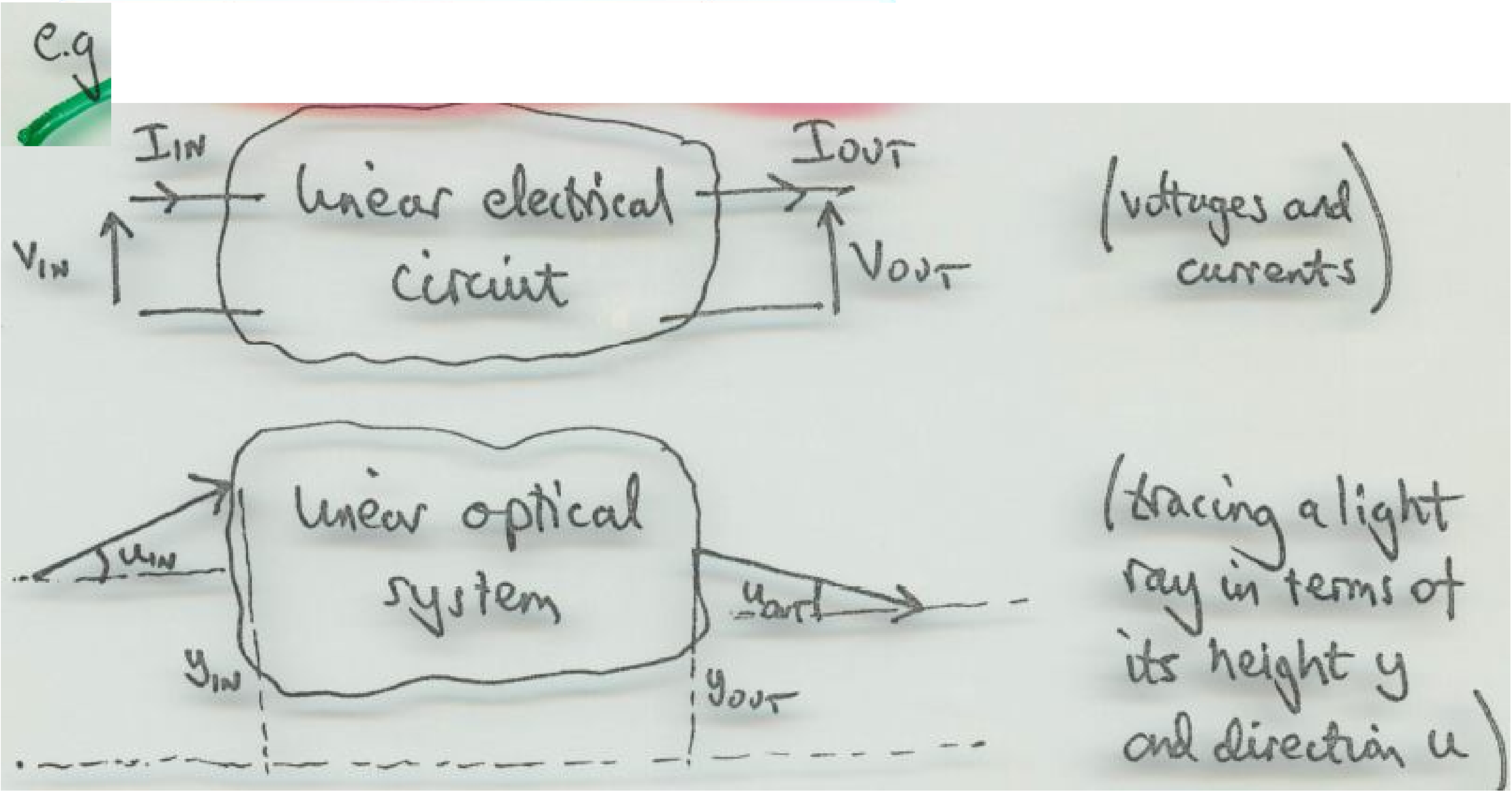
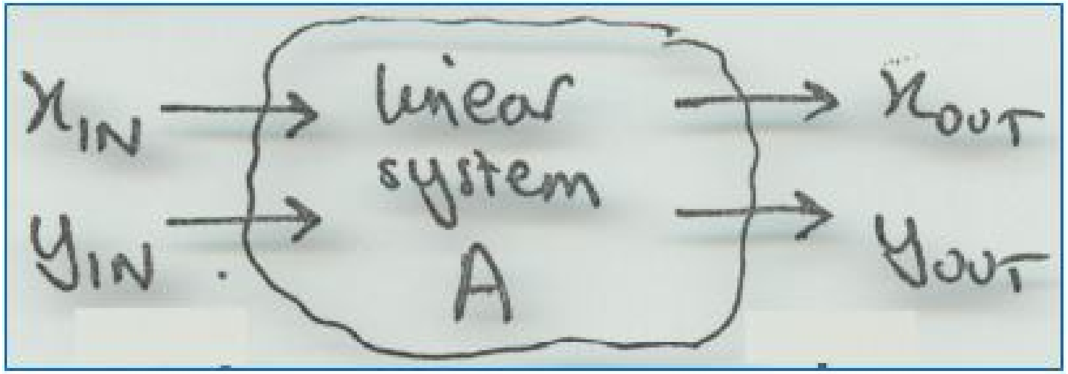
mid


x_{IN}
y_{IN}
linear system A
x_{OUT}
y_{OUT}


e.g.
I_{IN}
V_{IN}
linear electrical circuit
I_{OUT}
V_{OUT}
(voltages and currents)
u_{IN}
y_{IN}
linear optical system
u_{OUT}
y_{OUT}
(tracing a light ray in terms of its height y and direction u)



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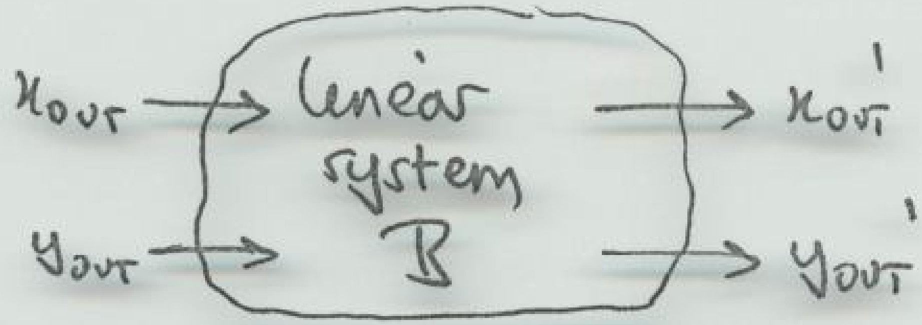
We can then send the output through another linear system $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and get x_{OUT}', y_{OUT}'.

i.e.



where $$\begin{pmatrix} x_{OUT}' \\ y_{OUT}' \end{pmatrix} = B \begin{pmatrix} x_{OUT} \\ y_{OUT} \end{pmatrix} = BA \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix}$$

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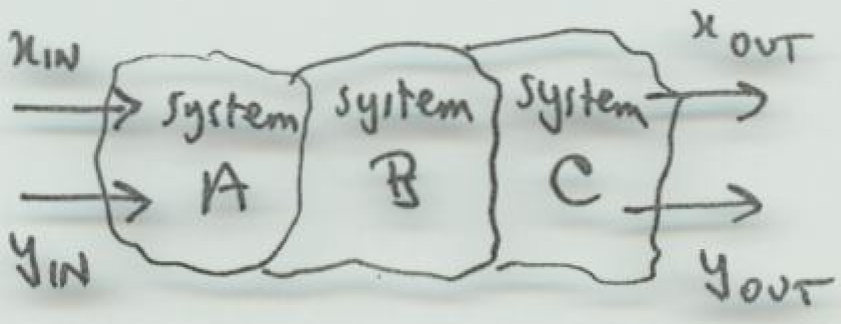
top

where $\begin{pmatrix} x_{OUT}' \\ y_{OUT}' \end{pmatrix} = B \begin{pmatrix} x_{OUT} \\ y_{OUT} \end{pmatrix} = \underset{\uparrow}{\underline{BA}} \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix}$

- systems are "<u>cascaded</u>" to give overall transmission of $\begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix} \rightarrow \begin{pmatrix} x_{OUT}' \\ y_{OUT}' \end{pmatrix}$

And so on...



$$\begin{pmatrix} x_{OUT} \\ y_{OUT} \end{pmatrix} = CBA \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix}$$

$$= D \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix}, \text{ say.}$$

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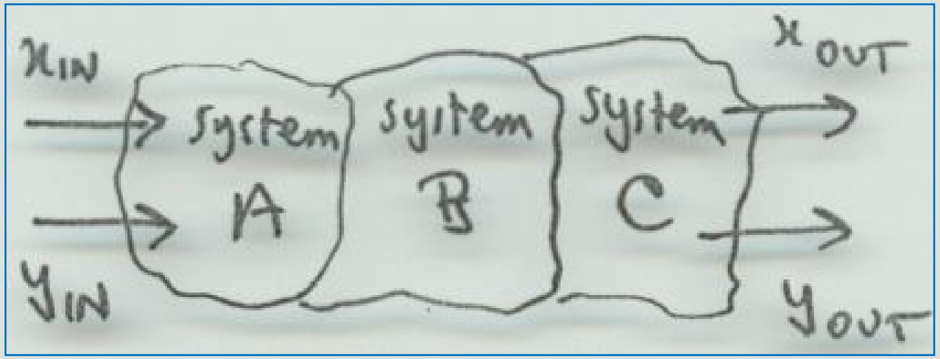
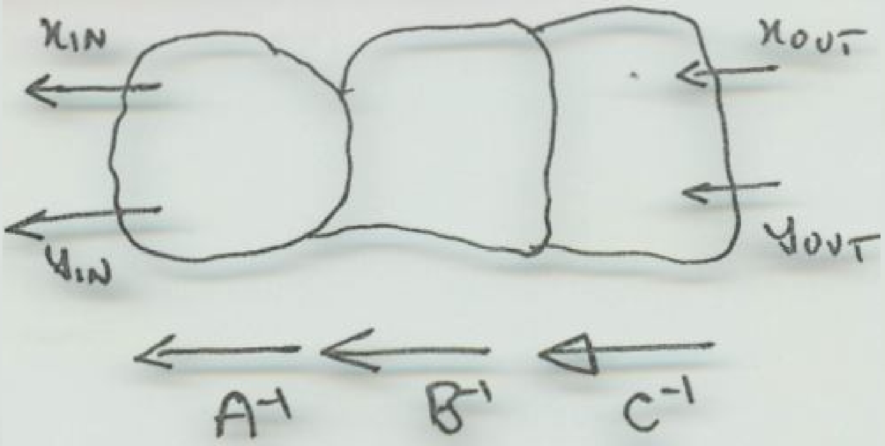
bot



$$\begin{pmatrix} x_{OUT} \\ y_{OUT} \end{pmatrix} = CBA \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix} = D \begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix}$$

Then, $\boxed{D^{-1} = A^{-1}B^{-1}C^{-1}}$

i.e.



and $$\begin{pmatrix} x_{IN} \\ y_{IN} \end{pmatrix} = A^{-1}B^{-1}C^{-1} \begin{pmatrix} x_{OUT} \\ y_{OUT} \end{pmatrix}$$

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In the above systems examples, the matrices are used to describe a discrete "jump" (in time or space or both) through the system.

I.e. the output is given directly in terms of the input and we do not know how much time it took to go through the system or the details of the "path" in space that was taken.

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mid

- Matrices can also be used to describe continuous spatial or temporal evolution of system parameters (x and y, say).

Consider, for example, temporal evolution (evolution in time t) governed by a pair of ordinary differential equations.

e.g.
$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$

i.e. $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

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bot

i.e. $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $$\boxed{\frac{d\underset{\sim}{u}}{dt} = M\underset{\sim}{u}}$$ where $\underset{\sim}{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, $M = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

This is a very compact way of writing what could be a very large system of equations.

e.g. $\underset{\sim}{u}$ could be an n-dimensional vector where n is large and M an $n \times n$ matrix.

Eigenvalues and Eigenvectors

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top

In the above examples of rotation, general linear transformation, linear systems and differential equations, a matrix is used to describe the discrete or continuous change of a vector.

A very important class of problem that arises in many areas of physics is determining the vectors $\underset{\sim}{x}$ that do not change direction under transformation by matrix A (the eigenvectors of A).

In this case, we have

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

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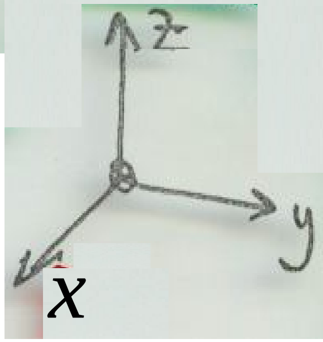
bot

where λ is called the eigenvalue of A corresponding to the eigenvector $\underset{\sim}{x}$. The eigenvalues and eigenvectors of A are also called the characteristic values and vectors of A, respectively.

In physical problems, they can define the characteristic or "normal" modes of a system e.g. the normal modes of vibration. In quantum mechanics, they are often called the "eigenstates".

More complex evolution in the system (when it's not in an eigenstate) can be written in terms of a complete set of eigenvectors. The eigenvectors of a system are linearly independent.

If a solution is three-dimensional, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for example, then all possible solutions of the "system" A (a 3×3 matrix which could be equivalent to a differential equation e.g. $\frac{d}{dt}\underset{\sim}{x} = A\underset{\sim}{x}$) lie somewhere in the three-dimensional "solution space".



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If we have the three linearly independent eigenvectors $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ of A, then any solution in this three-dimensional space can be written as a linear combination of the eigenvectors. They act like "basis vectors" for the system.

→ How do we find the eigenvalues and eigenvectors of

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \quad ?$$

How do we find the eigenvalues and eigenvectors of

$$A \underset{\wedge}{x} = \lambda \underset{\wedge}{x} \quad ?$$

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bot

For an $n \times n$ system, we would have

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}.$$

$$\longleftarrow A \longrightarrow \; \underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$A\underset{\wedge}{x} = \lambda \underset{\wedge}{x}$$

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Recall that this actually means ...

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= \lambda x_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= \lambda x_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= \lambda x_n
\end{aligned}$$

i.e.

$$\begin{aligned}
(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0
\end{aligned}$$

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i.e. $(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$

$a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0$

$\vdots \qquad\qquad \vdots \qquad \vdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0$

i.e.
$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Another way to write this would be in terms of the identity matrix $\underline{I}$:

$$\underline{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{then} \quad \lambda\underline{I} = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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bot

$$\lambda I = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

so

$$A' \underset{\sim}{x} = \underset{\sim}{0}$$

where $A' = A - \lambda I$

$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \Rightarrow A'\underset{\sim}{x} = \underset{\sim}{0}$, where $A' = A - \lambda I$

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$A'\underset{\sim}{x} = \underset{\sim}{0}$ is just a homogeneous system and we thus have only two cases ...

(i) $\det A' \neq 0 \Rightarrow$ only the trivial solution $\underset{\sim}{x} = \underset{\sim}{0}$

--- not very useful. It doesn't tell us anything about the original system A; all homogeneous systems have this solution.

(ii) $\det A' = 0 \Rightarrow$ an infinite number of solutions

--- it is the form of these solutions that give us the eigenvectors (the characteristic modes).

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In other words, we seek eigenvalues λ that give us a singular homogeneous system i.e. we solve

$$\det(A - \lambda I) = 0$$

THE CHARACTERISTIC EQUATION

to find the eigenvalues.

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Ex

A is a 2x2 matrix $\det(A-\lambda I)=0$ gives

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \quad \text{i.e.} \quad (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

This is of the form $\boxed{\lambda^2 + c_1\lambda + c_0 = 0}$ (c_0, c_1 constants).

$\Rightarrow$ two possible eigenvalues λ_1 and λ_2.

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Ex A is a 3x3 matrix

$\det(A-\lambda I)=0$ gives $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$

i.e. $\underbrace{(a_{11}-\lambda)\underbrace{\begin{vmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda \end{vmatrix}}_{\text{quadratic in } \lambda}}_{\text{cubic in } \lambda} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda \end{vmatrix}}_{\text{linear in } \lambda} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22}-\lambda \\ a_{31} & a_{32} \end{vmatrix}}_{\text{linear in } \lambda} = 0$

This is of the form $\boxed{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0}$

$\Rightarrow$ three possible eigenvalues λ_1, λ_2 and λ_3.

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bot

Ex A is an $n \times n$ matrix $\det(A - \lambda I) = 0$ gives

a characteristic equation of the form:

$$\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

$\Rightarrow$ n possible eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

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Given an eigenvalue λ_1, how do we find the corresponding eigenvector $\underset{\sim}{x}_1$?

→ Substitute for λ_1 and solve $A\underset{\sim}{x}_1 = \lambda_1 \underset{\sim}{x}_1$.

Ex $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. Then $|A - \lambda I| = 0$ yields $\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$(2-\lambda)(1-\lambda) - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$\lambda_1 = -2$ and $\lambda_2 = 5$

$\lambda_1 = -2$ and $\lambda_2 = 5$

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$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Now, $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

gives $\left\{ \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e. $\begin{pmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

let’s now substitute each of the values of λ found, trying λ_1 first ...

Requiring $(A-\lambda I)\underset{\sim}{x}=\underset{\sim}{0}$ and using that $A=\begin{bmatrix}2&3\\4&1\end{bmatrix}$

gave us: $\begin{pmatrix}2-\lambda & 3\\ 4 & 1-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

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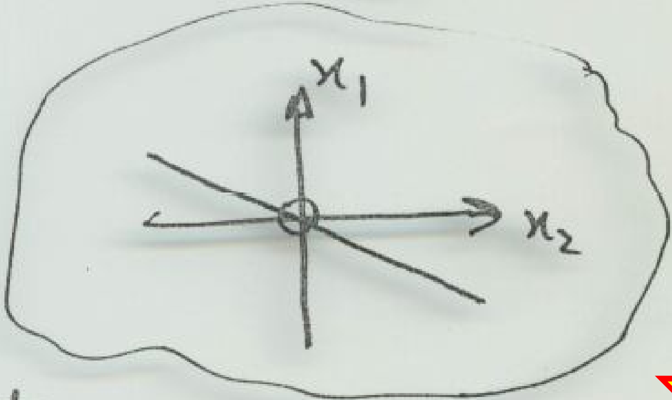
bot

$\lambda_1=-2$ gives $\begin{pmatrix}4 & 3\\ 4 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

Recall that we forced the coefficient matrix to be singular.

i.e. $4x_1+3x_2=0$



This is just a line in the x_1-x_2 plane.

We cannot specify x_1 and x_2 but we know that they are characterised by the gradient

i.e. $x_1/x_2=-3/4$. The eigenvector is then $\underset{\sim}{x}_1=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\alpha\begin{pmatrix}3\\-4\end{pmatrix}$

where α is an undetermined scalar (number)

We found that: $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

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where α is an undetermined scalar

Note that if $A\underset{\sim}{x}_1 = \lambda_1 \underset{\sim}{x}_1$

then $A\alpha \underset{\sim}{x}_1 = \lambda_1 \alpha \underset{\sim}{x}_1$

i.e. if, for λ_1, $\underset{\sim}{x}_1$ is an eigenvector then so is $\alpha \underset{\sim}{x}_1$.

Requiring $(\underset{\sim}{A}-\lambda \underset{\sim}{I})\underset{\sim}{x} = \underset{\sim}{0}$ and using that $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

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gave us: $\begin{pmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

now ...

$\lambda_2 = 5$ gives $\begin{pmatrix} 2-\lambda_2 & 3 \\ 4 & 1-\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e. $\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -3x_1 + 3x_2 = 0$ and $x_1/x_2 = \frac{1}{}$

The second eigenvector is then $\underset{\sim}{x_2} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

where β is another undetermined scalar.

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Another example

Consider the time evolution of variables x and y that is governed by the differential equations

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 2x + y$$

Write $\underset{\wedge}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ to give $\frac{d}{dt}\underset{\wedge}{x} = A\underset{\sim}{x}$

To find the eigenvalues of the system (corresponding to the eigenvectors of the system that preserve their direction as time evolves .... Note that since eigenvectors are <u>only</u> defined in terms of their direction, **when a system is in an eigenstate then it remains in that eigenstate**) ...

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set $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$ and $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \lambda_1 = -1$ and $\lambda_2 = 4$

To find the corresponding eigenvectors .. $\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\lambda_1 = -1$ $\Rightarrow$ $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $3x + 3y = 0$

i.e. $x/y = -1$

and $\underset{\sim}{x}_1 = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, α a scalar

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$\lambda_2 = 4 \Rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e. $-2x + 3y = 0$

i.e. $x/y = 3/2$ and $\underline{\hat{x}_2 = \beta \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$, β a scalar

Now return to the differential equation: $\boxed{\frac{d}{dt}\hat{x} = A\hat{x} = \lambda \hat{x}}$

i.e. $\boxed{\frac{d}{dt}\hat{x} = \lambda \hat{x}}$ i.e. eigenvectors have a time-dependent amplitude of the form $e^{\lambda t}$.

Since it is a 2x2 system, the two eigenvectors form a complete set of linearly independent basis vectors for the solution space.

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Thus, the general solution of system $\frac{d}{dt}\underset{\sim}{x} = A\underset{\sim}{x}$ can be written as a linear combination of these basis vectors i.e. general solution is

$$\underset{\sim}{x} = c_1 e^{\lambda_1 t}\underset{\sim}{x_1} + c_2 e^{\lambda_2 t}\underset{\sim}{x_2}$$

i.e.

$$\boxed{\underset{\sim}{x} = c_1 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{4t}\begin{pmatrix}3\\2\end{pmatrix}}$$

where c_1, c_2 are scalars, and any particular solution of the system $\boxed{\frac{d}{dt}\underset{\sim}{x} = A\underset{\sim}{x}}$ is given by a particular choice of the constants c_1 and c_2.