*Maths Methods & Applications (**first semester material**: "Theoretical Physics I") - Dr Graham S McDonald*

3 Sets of Sample Jan/May Exam Questions with Solutions

A. Jan Test Format:

COMPULSORY Section A-type short questions (worth 40 marks)

+

TWO COMPULSORY Section B-type questions will be included (Topics: Vector Calculus *and* Matrices)

B. May Exam Contributions:

Some **Section A-type short** questions (worth around 13 of the 40 marks of Section A)

+

ONE OPTIONAL Section B-type question (you do 3 from the 6 presented) will be included (Topic: Matrices *or* Differential Equations)

SAMPLE EXAM 1 : Section A = 40 marks + pick two Section B questions (worth 30 marks each)

SECTION A

1. Answer **ALL** parts of the question:

(a) Consider a body that rotates with constant angular velocity $\boldsymbol{\omega} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Calculate the tangential velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ at the point given by the position vector $\mathbf{r} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

(4 Marks)

(b) The position of a moving particle is given by a time-dependent position vector $\mathbf{r}(t)$. Derive an expression for the velocity of the particle, $\mathbf{v}(t) = \frac{d\,\mathbf{r}(t)}{dt}$,

when:

$$\mathbf{r}(t) = (2t+3)\mathbf{i} + (t^2+3t)\mathbf{j} + (t^3+2t^2)\mathbf{k}.$$

(4 Marks)

(c) A physical quantity is defined over a region of space by the scalar field $\phi(x,y,z)$. Describe what physical property of this field is given by $\mathbf{grad}\phi$ $(\equiv \nabla\phi)$.

(6 marks)

(d) For a vector field $\mathbf{A}(x,y,z)$, the divergence theorem can be stated as:

$$\int_V div\mathbf{A}\; dV = \oint_S \mathbf{A}\cdot d\mathbf{S}.$$

Explain the meaning of each of the symbols used in this equation and the interpretation of the quantity $div\mathbf{A}$ that is implied by this theorem.

(8 marks)

(e) Find the matrix products **AB** and **BA**, and hence show that $\mathbf{AB} \neq \mathbf{BA}$, when

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

(4 Marks)

QUESTION 1. IS CONTINUED ON NEXT PAGE

(f) By considering matrix determinants, calculate the *rank* of the matrix:

$$\mathbf{A} = \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}.$$

(4 Marks)

(g) Two *homogeneous* linear equations, with constant coefficients a, b, c and d, take the form:

$$ax + by = 0$$
$$cx + dy = 0$$

Describe the solution(s), x and y, in the cases where the determinant of the coefficient matrix, $\det\mathbf{A}$, satisfies:

(i) $\det\mathbf{A} \neq 0$,

and *(ii)* $\det\mathbf{A} = 0$.

(6 Marks)

(h) Use the *chain rule* to show that $v = f(u)$ is a solution of the partial differential equation:

$$\frac{\partial v}{\partial x} - 5\frac{\partial v}{\partial y} = 0,$$

when $u = y + 5x$ and f is an arbitrary differentiable function.

(4 Marks)

SECTION B

2. A particular scalar field is defined by the equation:

$$\phi(x,y,z) = x e^{y} + y z^{2} + x y z \ .$$

Show that $\nabla\phi$ for this field is given by:

$$\nabla\phi = \left(e^{y} + yz\right)\mathbf{i} + \left(x e^{y} + z^{2} + xz\right)\mathbf{j} + y(2z + x)\mathbf{k} \ .$$

(7 Marks)

Hence, calculate the *magnitude* of the rate of change of $\phi(x,y,z)$ at the point $(x,y,z) = (2,\, 0, 3)$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(13 Marks)

Use the given form of $\nabla\phi$ (in terms of $\mathbf{i}, \mathbf{j}, \text{and}\, \mathbf{k}$) to prove that the vector field $\mathbf{V}(x,y,z) = \nabla\phi$ is a *conservative field.*

(10 Marks)

3. Answer **BOTH** parts of the question:

(a) Discuss the concept of *vector area* (making reference to the magnitude and to the direction of the cross product of two vectors in your answer).

(6 Marks)

(b) For the vector field $\mathbf{A}(x,y,z) = (3x - 2y)\mathbf{i} + \left(x^{2}z\right)\mathbf{j} + (1 - 2z)\mathbf{k}$, show that:

(i) $\nabla \cdot \mathbf{A} = 1$; and

(ii) $\nabla \times \mathbf{A} = -x^{2}\mathbf{i} + 2(xz + 1)\mathbf{k}$.

(14 Marks)

Consider the same vector field $\mathbf{A}$ over the circular region S that is bounded by the curve $x^{2} + y^{2} = a^{2}$ in the $z = 0$ plane. Adopting the convention that the normal to the surface S is in the direction of the *positive* z-axis, show that:

$$\int_{S} \mathbf{A} \cdot d\mathbf{S} = \pi a^{2} \ .$$

Note: you should use the fact that $\mathbf{A} = (3x - 2y)\mathbf{i} + \mathbf{k}$ in the $z = 0$ plane.

(10 Marks)

4. Answer **BOTH** parts of the question:

(a) If

$$\mathbf{A}=\begin{bmatrix}3 & -2 & 2\\ 1 & 2 & -3\\ 4 & 1 & 2\end{bmatrix},$$

show that the matrix of cofactors of $\mathbf{A}$ is:

$$\mathbf{C}=\begin{bmatrix}7 & -14 & -7\\ 6 & -2 & -11\\ 2 & 11 & 8\end{bmatrix}.$$

Show also that the determinant of $\mathbf{A}$ is $\det(\mathbf{A}) = 35$.

Hence, find the matrix $\mathbf{A}^{-1}$ (that is the *inverse* of the matrix $\mathbf{A}$).

(14 Marks)

(b) Prove that the *eigenvalues* of the matrix $\mathbf{A}=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ are given by $\lambda_1 = -1$ and $\lambda_2 = 5$. Hence, find *either one* of the two linearly independent *eigenvectors* of the matrix $\mathbf{A}$.

(16 Marks)

5. Answer **ALL** parts of the question:

(a) Use the *integrating factor method* to show that the solution of the differential equation:

$$\frac{dx}{dt}+\frac{x}{T}=A\ e^{i\omega t}$$

is

$$x(t)=x(0)\ e^{-t/T}+B\left(e^{i\omega t}-e^{-t/T}\right),$$

where $B=\dfrac{A}{\left(i\omega+1/T\right)}$, and A, T and ω are real constants.

(14 marks)

Identify the transient and the long-term steady-state components of the solution $x(t)$.

Comment on the relative phase of the steady-state component and the driving term, $Ae^{i\omega t}$, in the limit of large damping ($T\to\infty$).

(6 marks)

(b) Use the method of *separation of variables* to prove that a solution of the partial differential equation:

$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial y}+u$$

is given by:

$$u(x,y)=Ke^{c\left(x+\frac{y}{2}\right)}e^{-\frac{y}{2}},$$

where K and c are constants.

(10 Marks)

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SAMPLE EXAM 1 SOLUTIONS

EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	a) $\underset{\sim}{w} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{r} = 2\underset{\sim}{i} - 6\underset{\sim}{j} - 3\underset{\sim}{k}$
	$\underset{\sim}{w} \times \underset{\sim}{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 3 & -1 \\ -6 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix}$
	$= \underset{\sim}{i}[3(-3)-(-1)(-6)] - \underset{\sim}{j}[4(-3)-(-1)2] + \underset{\sim}{k}[4(-6)-3.2]$
4	$= \underset{\sim}{i}(-9-6) - \underset{\sim}{j}(-12+2) + \underset{\sim}{k}(-24-6)$
	$= -15\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\sim}{k}$.
	b) $\underset{\sim}{r}(t) = (2t+3)\underset{\sim}{i} + (t^2+3t)\underset{\sim}{j} + (t^3+2t^2)\underset{\sim}{k}$.
	$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{d}{dt}(2t+3)\underset{\sim}{i} + \frac{d}{dt}(t^2+3t)\underset{\sim}{j} + \frac{d}{dt}(t^3+2t^2)\underset{\sim}{k}$
4	$= 2\underset{\sim}{i} + (2t+3)\underset{\sim}{j} + (3t^2+4t)\underset{\sim}{k}$.
	c) grad ϕ is a vector field that gives direction and magnitude of the maximum rate of change
6	of ϕ with respect to space
	(can also be called "the vector gradient")

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	d) $\int_V \operatorname{div} \underset{\sim}{A}\, dV = \oint_S \underset{\sim}{A} . d\underset{\sim}{S}$ — Volume (integral); vector field; volume element; enclosing (closed) surface (integral); area element $d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\, dS$
	div $\underset{\sim}{A}$ ≡ volume density. R.H.S. ⇒ net flux of $\underset{\sim}{A}$ through S
	$\left(\operatorname{div} A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$ ∴ div A ⇒ net volume density of sources of flux inside S
8	[Sufficient selection to get full marks]
	e) $AB = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4\times1+2\times2 & 4\times1+2\times1 \\ 2\times1+1\times2 & 2\times1+1\times1 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix}$,
4	$BA = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\times4+1\times2 & 1\times2+1\times1 \\ 2\times4+1\times2 & 2\times2+1\times1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 10 & 5 \end{bmatrix}$,
	∴ $AB \neq BA$.

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

(f) $A = \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$. $\det A = |A| = \begin{vmatrix} 6 & 3 \\ 8 & 4 \end{vmatrix} = 6.4 - 3.8 = 24 - 24 = 0$

$\therefore \text{rank}(A) < 2$.

Now try to find any 1x1 submatrix with non-zero determinant. Taking any of the above elements gives a non-zero determinant for the corresponding 1x1 matrix. For example, $8 \to [8]$ which has $|8| = 8 \neq 0$.

4 $\Rightarrow \text{rank}(A) = 1$.

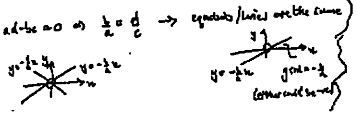
(g) May consider system algebraically, graphically or use the facts directly.

(i) $\det A = 0 \to$ infinite number of solutions, including the trivial solution $(x,y) = (0,0)$.

(ii) $\det A \neq 0 \to$ unique solution, i.e. only the trivial one.

6 e.g. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \Rightarrow ad - bc = 0 \Rightarrow \frac{b}{a} = \frac{d}{c} \to$ equations/lines are the same

$\det A \neq 0 \Rightarrow$



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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

(h) $V = f(y + 5x)$ to be shown to be a solution of: $\frac{\partial V}{\partial x} - 5\frac{\partial V}{\partial y} = 0$

Need to work out $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$. Chain rule can be used, (with $u = y + 5x$).

i.e. $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u} \cdot \frac{\partial u}{\partial x} = 5 \cdot \frac{\partial V}{\partial u}$

$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u} \cdot \frac{\partial u}{\partial y} = 1 \cdot \frac{\partial V}{\partial u}$

Verify solution by substitution

4 $\frac{\partial V}{\partial x} - 5\frac{\partial V}{\partial y} = \left(5 \cdot \frac{\partial V}{\partial u}\right) - 5\left(1 \cdot \frac{\partial V}{\partial u}\right) = 0$, as required

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT

$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$; $\frac{\partial \phi}{\partial x} = e^y + 0 + yz = e^y + yz$.

$\frac{\partial \phi}{\partial y} = xe^y + z^2 + xz$.

$\frac{\partial \phi}{\partial z} = 0 + 2yz + xy = y(2z + x)$.

7 $\therefore \nabla\phi = \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j} + \frac{\partial \phi}{\partial z}\underline{k} = (e^y + yz)\underline{i} + (xe^y + z^2 + xz)\underline{j} + y(2z + x)\underline{k}$.

At point $(2,0,3)$, $\nabla\phi = (e^0 + 0.3)\underline{i} + (2.e^0 + 3^2 + 2.3)\underline{j} + 0.(2.3 + 2)\underline{k}$

$= \underline{i} + 17\underline{j}$.

This is the magnitude and direction of the maximum (spatial) rate of change of ϕ at this point.

For the direction given by $\underline{A} = 3\underline{i} - 2\underline{j} + \underline{k}$, we need to calculate the direction derivative $\frac{d\phi}{ds} = \nabla\phi \cdot \hat{\underline{A}}$, where $\hat{\underline{A}}$ is a unit vector in the direction of $\underline{A}$ (we project the $\nabla\phi$ onto this unit vector).

Here, $|\underline{A}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$

$\therefore \hat{\underline{A}} = \frac{\underline{A}}{|\underline{A}|} = \frac{1}{\sqrt{14}}(3\underline{i} - 2\underline{j} + \underline{k})$.

$\to \frac{d\phi}{ds} = \nabla\phi \cdot \hat{\underline{A}} = (\underline{i} + 17\underline{j}) \cdot \frac{1}{\sqrt{14}}(3\underline{i} - 2\underline{j} + \underline{k})$

13 $= \frac{1}{\sqrt{14}}(3 - 2.17 + 0) = -\frac{31}{\sqrt{14}}$

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EXAMINATION SOLUTION CONTINUATION SHEET 1

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT

$\boxed{\underline{V} = \nabla\phi = (e^y + yz)\underline{i} + (xe^y + z^2 + xz)\underline{j} + y(2z + x)\underline{k}.}$

$\underline{V}$ is conservative $\Leftrightarrow \nabla \times \underline{V} = 0$ [(or) use the reciprocity relations directly i.e. $\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}$, $\frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}$, $\frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}$]

i.e. $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \underline{0}$, where $V_x = e^y + yz$, $V_y = xe^y + z^2 + xz$, $V_z = (2z + x)y$

$\nabla \times \underline{V} = \underline{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \underline{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \underline{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$,

where $\frac{\partial V_z}{\partial y} = 2z + x$, $\frac{\partial V_y}{\partial z} = 2z + x$, $\frac{\partial V_z}{\partial x} = y$, $\frac{\partial V_x}{\partial z} = y$,

$\frac{\partial V_y}{\partial x} = e^y + z$, $\frac{\partial V_x}{\partial y} = e^y + z$.

$\therefore \frac{\partial V_z}{\partial y} = \frac{\partial V_y}{\partial z}$, $\frac{\partial V_z}{\partial x} = \frac{\partial V_x}{\partial z}$, $\frac{\partial V_y}{\partial x} = \frac{\partial V_x}{\partial y}$

$\Rightarrow \nabla \times \underline{V} = 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0} \Rightarrow \underline{V} = \nabla\phi$ conservative (as it must be!)

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EXAMINATION SOLUTION SHEET

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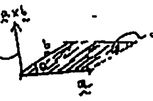
Paper Code:	Question No. 3

SOLUTION

MARKS	TEXT

(a) vector area



direction of $\underset{\sim}{a}\times\underset{\sim}{b}$ gives direction to the normal to this surface i.e. direction of $d\underset{\sim}{S}$

area $= |\underset{\sim}{a}\times\underset{\sim}{b}|$ $= ab\sin\theta$ $\equiv |d\underset{\sim}{S}|$

– could also mention conventions of:

- outward normals for a closed surface
- clockwise sense of bounding curve C for element normals on open surface.

[A SUFFICIENT SELECTION OF ABOVE, UNDER EACH HEADING, FOR FULL MARKS]

6

(b) $\underset{\sim}{A} = (3x-2y)\underset{\sim}{i} + x^2z\underset{\sim}{j} + (1-2z)\underset{\sim}{k} \equiv A_x\underset{\sim}{i} + A_y\underset{\sim}{j} + A_z\underset{\sim}{k}$

(i) $\underset{\sim}{\nabla}.\underset{\sim}{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(3x-2y) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(1-2z)$

$= 3 + 0 - 2 = 1.$

(ii) $\underset{\sim}{\nabla}\times\underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ A_y & A_z \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ A_x & A_z \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ A_x & A_y \end{vmatrix}$

$= \underset{\sim}{i}\left[\frac{\partial}{\partial y}(1-2z) - \frac{\partial}{\partial z}(x^2z)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(1-2z) - \frac{\partial}{\partial z}(3x-2y)\right] + \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(3x-2y)\right]$

$= \underset{\sim}{i}\left[0 - x^2\right] - \underset{\sim}{j}\left[0-0\right] + \underset{\sim}{k}\left[2xz - (-2)\right]$

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SOLUTION

MARKS	TEXT

$= -x^2\underset{\sim}{i} + 2(xz+1)\underset{\sim}{k}.$

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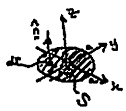

For this circular region, $z=0$

$\therefore \underset{\sim}{A} \to (3x-2y)\underset{\sim}{i} + 0\underset{\sim}{j} + (1-2(0))\underset{\sim}{k}$

i.e. $\underset{\sim}{A} = (3x-2y)\underset{\sim}{i} + \underset{\sim}{k}.$

$\hat{n}$, in the z-direction $\Rightarrow$ unit vector, $\hat{n} = \underset{\sim}{k}$.

i.e. $d\underset{\sim}{S} = \hat{n}\,dS = \underset{\sim}{k}\,dS$ (vector area element).

$\underset{\sim}{A} = (3x-2y)\underset{\sim}{i} + \underset{\sim}{k}$; $d\underset{\sim}{S} = \underset{\sim}{k}\,dS$

For $\int_S \underset{\sim}{A}.d\underset{\sim}{S} = \pi a^2$,

Form $\underset{\sim}{A}.d\underset{\sim}{S} = \left[(3x-2y)\underset{\sim}{i} + \underset{\sim}{k}\right].\underset{\sim}{k}\,dS$

$= (3x-2y)\underset{\sim}{i}.\underset{\sim}{k}\,dS + \underset{\sim}{k}.\underset{\sim}{k}\,dS$

$= dS$ (since $\underset{\sim}{i}.\underset{\sim}{k} = 0$ and $\underset{\sim}{k}.\underset{\sim}{k} = 1$)

Then, $\int_S \underset{\sim}{A}.d\underset{\sim}{S} = \int_S dS = \pi a^2$ (area of the circular disk)

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT

(a)

$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$, sign table $= \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Signed Minors (cofactors): $A_{11} = +\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7$, $A_{12} = -\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} = -(2+12) = -14$, $A_{13} = +\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = 1-8 = -7$, ...

$A_{21} = -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} = +6$, $A_{22} = +\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = -2$, $A_{23} = -\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} = -11$,

$A_{31} = +\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 2$, $A_{32} = -\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = +11$, $A_{33} = +\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 8.$

$\to$ Matrix of cofactors (signed minors) $= \begin{pmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{pmatrix} = C$

Determinant: $\det A = 3A_{11} - 2A_{12} + 2A_{13}$ (along row 1)

$= 21 + 28 - 14 = 35.$

Inverse: $A^{-1} = \frac{1}{\det A}C^T = \frac{1}{35}\begin{pmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{pmatrix}$ (C^T is the transpose of C),

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(b)

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$: Eigenvalues are given by $\det(A-\lambda I) = 0$

where $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$

$= (1-\lambda)(3-\lambda) - 8$

$= 3 - 3\lambda - \lambda + \lambda^2 - 8$

$= \lambda^2 - 4\lambda - 5$

$|A-\lambda I| = 0$ thus gives $\lambda^2 - 4\lambda - 5 = 0$

Solution by inspection: $\left.\begin{aligned}\lambda_1+\lambda_2 &= 4 \\ \lambda_1\lambda_2 &= -5\end{aligned}\right\} \Rightarrow \begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= 5.\end{aligned}$

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT

Let eigenvectors be $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

These satisfy $\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\lambda_1 = -1$ $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{aligned} 2x_1 + 2y_1 &= 0 \\ 4x_1 + 4y_1 &= 0 \end{aligned}$ i.e. $x_1 = -y_1$

$\therefore \underset{\sim}{x}_1 = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, α undetermined scalar.

EITHER OK

$\lambda_2 = 5$ $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{aligned} -4x_2 + 2y_2 &= 0 \\ 4x_2 - 2y_2 &= 0 \end{aligned}$ i.e. $2x_2 = y_2$

$\therefore \underset{\sim}{x}_2 = \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, β an undetermined scalar.

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GS McDONALD

PAPER TITLE:

THEORETICAL PHYSIC I

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S

EXAMINATION SOLUTION SHEET

These boxes MUST be completed

Paper Code:	Question No. 5

SOLUTION

MARKS	TEXT

a) $\boxed{\dfrac{dx}{dt} + \dfrac{x}{\tau} = Ae^{i\Omega t}}$ driving term — DRIVEN / DAMPED DYNAMICS

$(\tau \equiv T;\ \Omega \equiv \omega)$ where τ, A, Ω are constants.

Sol^n We have $\dfrac{dx}{dt} + P(t)x = Q(t)$

where $P(t) = \frac{1}{\tau}$ and $Q(t) = Ae^{i\Omega t}$.

Integrating factor is $e^{\int P(t)dt} = e^{\frac{1}{\tau}\int dt} = e^{t/\tau}$

Multiply to get $e^{t/\tau}\dfrac{dx}{dt} + e^{t/\tau}\dfrac{x}{\tau} = e^{t/\tau}Ae^{i\Omega t}$

i.e. $\dfrac{d}{dt}\left[e^{t/\tau}.x\right] = e^{t/\tau}Ae^{i\Omega t} = Ae^{(i\Omega+\frac{1}{\tau})t}$

Integrate: $\displaystyle\int_0^t \frac{d}{dt}\left[e^{t/\tau}.x\right]dt = \int_0^t Ae^{(i\Omega+\frac{1}{\tau})t}\,dt$

i.e. $e^{t/\tau}.x(t) - x(0) = A\left[\dfrac{1}{(i\Omega+\frac{1}{\tau})}e^{(i\Omega+\frac{1}{\tau})t}\right]_0^t$

$= \dfrac{A}{(i\Omega+\frac{1}{\tau})}\left[e^{(i\Omega+\frac{1}{\tau})t} - 1\right]$

4 — Divide by integrating factor: $\boxed{x(t) = x(0)e^{-t/\tau} + B\left(e^{i\Omega t} - e^{-t/\tau}\right)}$, $B = \dfrac{A}{i\Omega + \frac{1}{\tau}}$

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Theoretical Physics I

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DEPARTMENT OF PHYSICS

CS 1

EXAMINATION SOLUTION CONTINUATION SHEET

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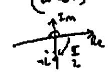
Paper Code:	Question No. 5

SOLUTION

MARKS	TEXT

- Terms involving $e^{-t/\tau}$ decay to zero as $t \to \infty$: transient
- leaves long-term steady-state component $x(t) \to Be^{i\omega t}$
- $B = \dfrac{A}{(i\omega + \frac{1}{\tau})} \to \dfrac{A}{i\omega} = \dfrac{iA}{i.i\omega} = -i\dfrac{A}{\omega} = e^{-i\frac{\pi}{2}}\dfrac{A}{\omega}$ $(\omega = \Omega)$



6 — $\therefore x(t) \to \dfrac{A}{\omega}e^{i(\omega t - \frac{\pi}{2})}$: 90° out of phase with driving term

$\boxed{\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u}$

Set $u = X(x)Y(y)$ and substitute: $Y\dfrac{\partial X}{\partial x} = 2X\dfrac{\partial Y}{\partial y} + XY$

i.e. $\dfrac{1}{X}\dfrac{\partial X}{\partial x} = \dfrac{2}{Y}\dfrac{\partial Y}{\partial y} + 1$ (dividing by XY)

i.e. $\dfrac{1}{X}\dfrac{dX}{dx} = c$ and $\dfrac{2}{Y}\dfrac{dY}{dy} + 1 = c$ (c = separation constant)

$\int \dfrac{dX}{X} = c\int dx$ — $2\int \dfrac{dY}{Y} = (c-1)\int dy$

i.e. $\ln X = cx + A$, $X = e^{cx+A}$ — i.e. $2\ln Y = (c-1)y + B$

i.e. $\ln Y = \frac{(c-1)}{2}y + B'$

i.e. $Y = e^{\frac{(c-1)}{2}y + B'}$

10 — A solution is $u = XY = e^{cx+A}e^{(\frac{c-1}{2})y+B'} = e^{cx+(\frac{c-1}{2})y}e^{A+B'}$

i.e. $u = ke^{c(x+\frac{y}{2}) - \frac{y}{2}}$; where $k = e^{A+B'}$

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SAMPLE EXAM 2

SECTION A

1. Answer ALL parts of the question:

(a) Determine the constant a such that the vectors $\mathbf{A} = 2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ are perpendicular.

(4 Marks)

(b) Consider a body that rotates with constant angular velocity $\boldsymbol{\omega} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Calculate the tangential velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ at the point given by the position vector $\mathbf{r} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$.

(4 Marks)

(c) Define $div\mathbf{V} = \nabla.\mathbf{V}$, where $\mathbf{V} = \mathbf{V}(x, y, z)$ is a vector field. Describe, in words, what properties of the field $\mathbf{V}$ are expressed by its *divergence*.

(5 Marks)

(d) Define $curl\mathbf{V} = \nabla \times \mathbf{V}$, where $\mathbf{V} = \mathbf{V}(x, y, z)$ is a vector field. Describe, in words, what properties of the field $\mathbf{V}$ are expressed by its *curl*.

(5 Marks)

(e) Find the matrix products $\mathbf{AB}$ and $\mathbf{BA}$, and hence show that $\mathbf{AB} \neq \mathbf{BA}$, when

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

(4 Marks)

(f) By considering matrix determinants, determine the *rank* of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(4 Marks)

QUESTION 1. IS CONTINUED ON NEXT PAGE

QUESTION 1 CONTINUED....

(g) The matrix $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has *eigenvalues* given by $\lambda_1 = -2$ and $\lambda_2 = 5$. Hence, verify that:

(i) the product of the eigenvalues of $\mathbf{B}$ is equal to the value of the determinant of $\mathbf{B}$; and

(ii) the sum of the eigenvalues of $\mathbf{B}$ is equal to the sum of the diagonal elements of $\mathbf{B}$.

(4 Marks)

(h) Use the *integrating factor method* to show that the general solution of $\frac{dy}{dx} + py = q$, where p and q are constants, can be written as

$$y(x) = \frac{q}{p} + Ce^{-px},$$

where C is an arbitrary constant.

(5 Marks)

(i) Use the *chain rule* to show that $v = f(u)$ is a solution of the partial differential equation:

$$\frac{\partial v}{\partial x} + \frac{1}{c}\frac{\partial v}{\partial t} = 0,$$

when $u = x - ct$, c is a constant, and f is an arbitrary differentiable function.

(5 Marks)

SECTION B

2. An electrostatic charge distribution gives rise to a scalar potential given by:

$$\phi(x, y, z) = y^2 \sin x + xz^3 + 2z + 4y,$$

where x, y, and z are space coordinates (physical units have been omitted for simplicity). Show that the associated vector field $\mathbf{E} = -\nabla\phi = -\text{grad}\phi$ is given by:

$$\mathbf{E}(x, y, z) = -(y^2 \cos x + z^3)\mathbf{i} - (2y\sin x + 4)\mathbf{j} - (3xz^2 + 2)\mathbf{k}.$$

(10 Marks)

Hence, prove that the original charge distribution density, given by $\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 \text{div}\mathbf{E}$, is:

$$\rho(x, y, z) = \varepsilon_0 (y^2 - 2)\sin x - 6\varepsilon_0 xz.$$

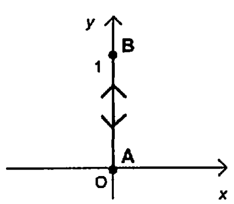
(10 Marks)

For the above vector field $\mathbf{E}(x, y, z)$, verify that $\oint_C \mathbf{E}.d\mathbf{r} = 0$ by considering line elements, $d\mathbf{r}$, along the closed path C (up and down the y-axis, in the $z = 0$ plane) defined as:

(i) from point A at the origin, where $(x, y) = (0,0)$, to point B, where $(x, y) = (0,1)$,

and then

(ii) from point B back to point A (see diagram below).



(10 Marks)

3. For a vector field $\mathbf{A}(x, y, z)$, Stokes' theorem can be stated as:

$$\int_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r}.$$

Explain the meaning of each of the symbols used in this equation and the interpretation of the quantity curl $\mathbf{A}$ that is implied by this theorem.

(8 marks)

For the particular vector field:

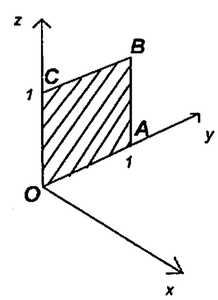
$$\mathbf{A}(x, y, z) = xy\mathbf{i} + (2y - xz)\mathbf{j} + xz\mathbf{k},$$

show that:

$$\text{curl } \mathbf{A} = x\mathbf{i} - z\mathbf{j} - (x + z)\mathbf{k}.$$

(10 marks)

Hence, show that $\int_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = 0$ when S is the square area in the $x = 0$ plane whose corners are O (0,0,0), A (0,1,0), B (0,1,1) and C (0,0,1). This area is illustrated (shaded) in the figure below. In your calculation, assume that $d\mathbf{S}$ points in the *positive* x-direction.



(12 marks)

4. Answer BOTH parts of the question:

(a) Show that the *eigenvalues* of the matrix $A = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix}$ are given by

$\lambda_1 = 4$ and $\lambda_2 = 6$.

(8 Marks)

Hence, find <u>either one</u> of the two linearly independent *eigenvectors* of the matrix A.

(8 Marks)

(b) Use a method of your choice, to find the *inverse* of the matrix B, where

$$B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

(14 Marks)

5. Answer BOTH parts of the question:

(a) An electrical circuit consists of a resistance R and a capacitance C connected in series to a battery of constant voltage V. By considering the voltage dropped across R and C, one arrives at an ordinary differential equation for the charge stored $Q(t)$:

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where t is time. Show that the general solution of this differential equation is given by:

$$Q(t) = Q(0)\,e^{-t/RC} + VC\left(1 - e^{-t/RC}\right).$$

Also find the particular solution when $Q(0) = 0$ and identify the long-term steady-state and transient components of this particular solution. Give a physical interpretation of your results.

(20 Marks)

(b) Use the (partial differential equation) method of *separation of variables* to prove that a solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ is given by $u(x,y) = K\,e^{c(4x+y)}$, where K and c are constants.

(10 Marks)

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PAPER TITLE: Theoretical Physics I

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SAMPLE EXAM 2 SOLUTIONS

EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
4	(a) $\underset{\wedge}{A} = 2\underset{\wedge}{i} + a\underset{\wedge}{j} + \underset{\wedge}{k}$. $\underset{\sim}{B} = 4\underset{\wedge}{i} - 2\underset{\sim}{j} - 2\underset{\wedge}{k}$ $\underset{\wedge}{A}\cdot\underset{\wedge}{B} = (2\underset{\wedge}{i} + a\underset{\wedge}{j} + \underset{\wedge}{k})\cdot(4\underset{\wedge}{i} - 2\underset{\wedge}{j} - 2\underset{\wedge}{k})$ $= 8 - 2a - 2 = 6 - 2a$ Perpendicular when $\underset{\sim}{A}\cdot\underset{\wedge}{B} = 0$ i.e. when $6 - 2a = 0$ i.e. when $a = 3$.
4	(b) $\underset{\wedge}{v} = \underset{\wedge}{w} \times \underset{\wedge}{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\wedge}{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix}$ $= \underset{\sim}{i}\begin{vmatrix} 3 & -1 \\ -6 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underset{\wedge}{k}\begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix}$ $= \underset{\sim}{i}(-9-6) - \underset{\sim}{j}(-12+2) + \underset{\sim}{k}(-24-6)$ $= -15\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\wedge}{k}$.
5	(c) $\text{div}\,\underset{\wedge}{V} = \underset{\wedge}{\nabla}\cdot\underset{\wedge}{V} = \left(\underset{\wedge}{i}\frac{\partial}{\partial x} + \underset{\wedge}{j}\frac{\partial}{\partial y} + \underset{\wedge}{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\underset{\wedge}{i} + V_y\underset{\wedge}{j} + V_z\underset{\wedge}{k}\right)$ $= \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z$, when $\underset{\sim}{V} = (V_x, V_y, V_z)$. div $\underset{\wedge}{V}$ is the net outflow of flux of $\underset{\wedge}{V}$ per unit volume (at a point). Equivalently, it is the volume density of 'sources' and 'sinks' of flux of V.

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Theoretical Physics I

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CS 1

EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
5	(d) curl $\underset{\sim}{V} = \underset{\sim}{\nabla}\times\underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ V_y & V_z \end{vmatrix}$ $-\underset{\sim}{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ V_x & V_z \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ V_x & V_y \end{vmatrix}$ etc. curl $\underset{\sim}{V}$ gives the 'circulation' of the field $\underset{\sim}{V}$ about each point. In other words, it gives the degree (and orientation) of 'twist / swirl / rotation / vorticity' of the field.
4	(e) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$. $BA = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix} \neq AB$ (here, or in general)
4	(f) $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = (0\times 0) - (1\times 0) = 0 - 0 = 0$ $\therefore$ rank < 2 while submatrix $[1]$ can be found with non-zero determinant $(\lvert 1 \rvert \neq 0)$. $\therefore$ rank $= \underline{1}$.

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DEPARTMENT OF PHYSICS

CS 2

Theoretical Physics I

EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	(g) $B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ $\lambda_1 = -2,\ \lambda_2 = 5$ (given eigenvalues)
	(i) $\det B = \lvert B \rvert = (2\times1 - 3\times4) = 2 - 12 = -10$
	while $\lambda_1\lambda_2 = (-2)\times(5) = -10$; (an example of a general result).
	(ii) $\lambda_1 + \lambda_2 = -2 + 5 = 3$,
	while "trace of B" is sum of diagonal elements.
4	Here, trace(B) $= 2 + 1 = 3$, also; (another general result).
	(h) $\frac{dy}{dx} + py = q$ (p, q constants): of standard form: $\frac{dy}{dx} + P(x)y = Q(x)$
	where $P(x) = p$, $Q(x) = q$. Integrating factor $= e^{\int P dx} = e^{\int p dx} = e^{\int p dx} = e^{px}$.
	Multiply equation: $e^{px}\frac{dy}{dx} + e^{px}py = e^{px}q$ to get: $\frac{d}{dx}\left(e^{px}y\right) = e^{px}q$
	Integrate: $e^{px}y = \int e^{px}q\,dx + C \Rightarrow e^{px}y = q\int e^{px}dx + C$
5	$\Rightarrow e^{px}y = \frac{q}{p}e^{px} + C \Rightarrow y = \frac{q}{p} + Ce^{-px}$

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CS 3

Theoretical Physics I

EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	(i) $\boxed{\frac{\partial V}{\partial x} + \frac{1}{c}\frac{\partial V}{\partial t} = 0}$. Verify $V = f(u)$ is solution where $u = x - ct$.
	$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\cdot\frac{\partial u}{\partial x}$ (chain rule)
	$= \frac{\partial V}{\partial u}\cdot 1$ $\left[\text{since } \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x - ct)\right]$.
	$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial u}\cdot\frac{\partial u}{\partial t}$ (chain rule)
	$= \frac{\partial V}{\partial u}\cdot(-c)$
	$\Rightarrow$ Substitute into equation (Left-hand-side):
	$\frac{\partial V}{\partial u}\cdot 1 + \frac{1}{c}\cdot\frac{\partial V}{\partial u}\cdot(-c) = \frac{\partial V}{\partial u} - \frac{\partial V}{\partial u} = 0$ = RHS of equation
	$\therefore$ we have proved this is a solution,
5	without needing to ~~specific~~ specify the function "f".

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PAPER TITLE: Theoretical Physics II

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S

EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT
	$\phi = y^2\sin x + xz^3 + 2z + 4y$
	$\underline{\nabla} = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$; $\underline{\nabla}\phi = \underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}$,
	where $\frac{\partial\phi}{\partial x} = y^2\cos x + z^3$, $\frac{\partial\phi}{\partial y} = 2y\sin x + 4$, $\frac{\partial\phi}{\partial z} = 3xz^2 + 2$
	$\Rightarrow \underline{E} = -\underline{\nabla}\phi = -\left\{\underline{i}(y^2\cos x + z^3) + \underline{j}(2y\sin x + 4) + \underline{k}(3xz^2 + 2)\right\}$
10	$= -(y^2\cos x + z^3)\underline{i} - (2y\sin x + 4)\underline{j} - (3xz^2 + 2)\underline{k}$.
	$\rho = \varepsilon_0\underline{\nabla}.\underline{E} = \varepsilon_0\left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right).\left(E_x\underline{i} + E_y\underline{j} + E_z\underline{k}\right)$
	$= \varepsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$
	where $\frac{\partial}{\partial x}E_x = \frac{\partial}{\partial x}\left\{-(y^2\cos x + z^3)\right\} = -(y^2(-\sin x)) = +y^2\sin x$.
	$\frac{\partial}{\partial y}E_y = \frac{\partial}{\partial y}\left\{-(2y\sin x + 4)\right\} = -(2\sin x + 0) = -2\sin x$
	$\frac{\partial}{\partial z}E_z = \frac{\partial}{\partial z}\left\{-(3xz^2 + 2)\right\} = -(6xz + 0) = -6xz$

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DEPARTMENT OF PHYSICS

CS 1

Theoretical Physics I

EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT
	$\therefore\ \rho = \varepsilon_0\underline{\nabla}.\underline{E} = \varepsilon_0\left\{y^2\sin x - 2\sin x - 6xz\right\}$
10	$= \varepsilon_0(y^2 - 2)\sin x - 6\varepsilon_0 xz$.
	$\underline{E} = -(y^2\cos x + z^3)\underline{i} - (2y\sin x + 4)\underline{j} + (3xz^2 + 2)\underline{k}$
	$\underline{E}(z=0) = -y^2\cos x\,\underline{i} - (2y\sin x + 4)\underline{j} + 2\underline{k}$
	Here, $d\underline{r} = dx\,\underline{i} + dy\,\underline{j} = dy\,\underline{j}$ $(dx = dz = 0)$
	$\underline{E}\cdot d\underline{r} = (E_x\underline{i} + E_y\underline{j} + E_z\underline{k}).(dy\,\underline{j})$
	$= E_y\,dy$ $(\underline{j}.\underline{j} = 1,\ \underline{i}.\underline{j} = \underline{k}.\underline{j} = 0)$
	$= -(2y\sin x + 4)\,dy$
	Also, $x = 0 \Rightarrow \underline{E}.d\underline{r} = -4dy$
	$\therefore \oint_C \underline{E}.d\underline{r} = \int_A^B \underline{E}.d\underline{r} + \int_B^A \underline{E}.d\underline{r} = \int_0^1(-4)dy + \int_1^0(-4)dy$
	$= -4[y]_0^1 + (-4).[y]_1^0 = -4 - 4.(-1)$
10	$= -4 + 4 = 0$.

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GS McDONALD — DEPARTMENT OF PHYSICS — **S**

PAPER TITLE: Theoretical Physics I

EXAMINATION SOLUTION SHEET

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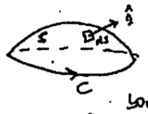
Paper Code:	Question No. 3

SOLUTION

MARKS	TEXT

$$\int_S (\mathrm{curl}\,\underset{\sim}{A})\cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{A}\cdot d\underset{\sim}{r}$$



Surface integral over open surface S of dot product of curl $\underset{\sim}{A}$ $\left(= \left|\begin{smallmatrix}\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{smallmatrix}\right|\right)$ and $d\underset{\sim}{S}$ and closed line/curve integral along bounding curve C (clockwise sense with respect to $\hat{n}$) of dot product of $\underset{\sim}{A}$ and $d\underset{\sim}{r}$ element (along C).

Projection of $\nabla\times\underset{\sim}{A}$ gives surface density that integrates over S to give total circulation of $\underset{\sim}{A}$ around C

curl $\underset{\sim}{A}$ measures circulation/rotation/vorticity ~~at~~ around a point

8   * [Sufficient selection for full marks] *

$$\underset{\sim}{A} = xy\,\underset{\sim}{i} + (2y - xz)\,\underset{\sim}{j} + xz\,\underset{\sim}{k} \equiv A_x\underset{\sim}{i} + A_y\underset{\sim}{j} + A_z\underset{\sim}{k}$$

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ A_y & A_z\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ A_x & A_z\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ A_x & A_y\end{vmatrix}$$

$$= \underset{\sim}{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial}{\partial z}A_y\right) - \underset{\sim}{j}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) + \underset{\sim}{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 3

SOLUTION

MARKS	TEXT

where $\frac{\partial A_z}{\partial y} = \frac{\partial}{\partial y}(xz) = 0$, $\frac{\partial A_y}{\partial z} = \frac{\partial}{\partial z}(2y - xz) = -x$, $\frac{\partial A_z}{\partial x} = \frac{\partial}{\partial x}(xz) = z$,

$\frac{\partial A_x}{\partial z} = \frac{\partial}{\partial z}(xy) = 0$, $\frac{\partial A_y}{\partial x} = \frac{\partial}{\partial x}(2y - xz) = -z$, $\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}(xy) = x$

10    $\therefore\ \underset{\sim}{\nabla}\times\underset{\sim}{A} = \underset{\sim}{i}\left[0 - (-x)\right] - \underset{\sim}{j}\left[z - 0\right] + \underset{\sim}{k}\left[(-z) - x\right] = x\underset{\sim}{i} - z\underset{\sim}{j} - (x+z)\underset{\sim}{k}$

$d\underset{\sim}{S} = dS\,\underset{\sim}{i}$ (+ve x-direction)

i.e. $d\underset{\sim}{S} = dy\,dz\,\underset{\sim}{i}$

AND on S, we have $x = 0$, therefore

$$\underset{\sim}{\nabla}\times\underset{\sim}{A} = 0\underset{\sim}{i} - z\underset{\sim}{j} - (0+z)\underset{\sim}{k} = -z\underset{\sim}{j} - z\underset{\sim}{k}.$$

$$\therefore \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{A})\cdot d\underset{\sim}{S} = \int_S (-z\underset{\sim}{j} - z\underset{\sim}{k})\cdot(\underset{\sim}{i}\,dy\,dz)$$

$$= \int_S \Big(-z\underbrace{\underset{\sim}{j}\cdot\underset{\sim}{i}}_{=0} - z\underbrace{\underset{\sim}{k}\cdot\underset{\sim}{i}}_{=0}\Big)\,dy\,dz$$

12    $= \int_S 0\,.\,dy\,dz = 0.$

[NB Circulation could be calculated instead but this takes longer]

NECESSARY

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EXAMINATION SOLUTION SHEET

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SOLUTION

MARKS

$A = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix}$. Eigenvalues are given by the characteristic equation: $\det(A - \lambda I) = 0$

Here, $\begin{vmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{vmatrix} = 0$

i.e. $(8-\lambda)(2-\lambda) + 8 = 0$

i.e. $16 - 2\lambda - 8\lambda + \lambda^2 + 8 = 0$

i.e. $\lambda^2 - 10\lambda + 24 = 0$

8   i.e. $\left.\begin{matrix}\lambda_1 + \lambda_2 = +10 \\ \lambda_1\lambda_2 = 24\end{matrix}\right\} \Rightarrow \begin{matrix}\lambda_1 = 4 \\ \lambda_2 = 6.\end{matrix}$

Eigenvectors $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{x}_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$:

satisfy $\begin{pmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

FIND **EITHER** EIGENVECTOR...

$\lambda_1 = 4$ $\begin{pmatrix} 8-\lambda_1 & -2 \\ 4 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix}(8-\lambda_1)x_1 - 2y_1 = 0 \\ 4x_1 + (2-\lambda_1)y_1 = 0.\end{matrix}$

i.e. $\begin{matrix}4x_1 - 2y_1 = 0 \\ 4x_1 - 2y_1 = 0\end{matrix}$ $\left(\begin{matrix}\text{since} \\ \lambda_1 = 4\end{matrix}\right)$

i.e. $2x_1 = y_1$.

$\therefore\ \underset{\sim}{x}_1 = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, α undetermined scalar.

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Theoretical Physics I

EXAMINATION SOLUTION CONTINUATION SHEET 2

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SOLUTION

MARKS	OR	TEXT

$\lambda_2 = 6$

$\begin{pmatrix} 8-\lambda_2 & -2 \\ 4 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix}(8-\lambda_2)x_2 - 2y_2 = 0 \\ 4x_2 + (2-\lambda_2)y_2 = 0\end{matrix}$

i.e. $\begin{matrix}2x_2 - 2y_2 = 0 \\ 4x_2 - 4y_2 = 0\end{matrix}$ $\left(\begin{matrix}\text{since} \\ \lambda_2 = 6\end{matrix}\right)$

i.e. $x_2 = y_2$.

$\therefore\ \underset{\sim}{x}_2 = \beta\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

8   where β is an undetermined scalar.

METHOD I (b)

$B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$. Consider $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$ and work on column 1 first.

$\begin{matrix} r_2 \to r_2 + 2r_1 \\ r_3 \to r_3 - r_1 \end{matrix}$ gives $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \end{array}\right]$. Now do column 2, $r_3 \to r_3 + r_2$ gives $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$.

Finally, do column 3: $r_1 \to r_1 + r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ and $r_2 \to r_2 + 2r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ i.e. $B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

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EXAMINATION SOLUTION CONTINUATION SHEET 3

These boxes MUST be completed

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SOLUTION

MARKS

METHOD II (alternative)

Inverse of $B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ by the formal method: $B^{-1} = \frac{1}{|B|} C^T$

here $|B| = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} + 0 - \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}$ (expanding along row 1)

$= 2 - (2-1) = 1.$

Matrix of cofactors $A_{11} = +\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$, $A_{12} = -\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = 4$, $A_{13} = +\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1$

$A_{21} = -\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = 1$, $A_{22} = +\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$, $A_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$

$A_{31} = +\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$, $A_{32} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$, $A_{33} = +\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$

$\therefore B^{-1} = \frac{1}{1} C^T = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$

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...ESSARY

USE A SECOND/THIRD CONTINUATION SHEET IF NECESSARY

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 5

SOLUTION

MARKS

$$\boxed{\frac{dQ}{dt} + \left(\frac{1}{RC}\right)Q = \frac{V}{R}}$$

$IF = e^{\frac{1}{RC}\int dt} = e^{\frac{t}{RC}}$. Now multiply equation

$$e^{t/RC}\frac{dQ}{dt} + \frac{e^{t/RC}}{RC}Q = e^{t/RC}\frac{V}{R}$$

$$\frac{d}{dt}\left[e^{t/RC}Q\right] = e^{t/RC}\cdot\frac{V}{R}$$

$$e^{t/RC}Q = \frac{V}{R}\cdot\frac{1}{\left(\frac{1}{RC}\right)}e^{t/RC} + A$$

i.e. $Q(t) = \frac{V}{R}(RC) + Ae^{-t/RC}$ (multiplying through by $e^{-t/RC}$)

i.e. $Q(t) = VC + Ae^{-t/RC}$

Identify physical character of A

at $t=0$, $Q(0) = VC + A$

$\therefore A = Q(0) - VC$

$\therefore$ General solution is $Q(t) = VC + \left[Q(0) - VC\right]e^{-t/RC}$

i.e. $Q(t) = VC + Q(0)e^{-t/RC} - VCe^{-t/RC}$

i.e. $Q(t) = Q(0)e^{-t/RC} + VC(1 - e^{-t/RC}).$

(+15)

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EXAMINATION SOLUTION CONTINUATION SHEET 1

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Paper Code:	Question No. 5

SOLUTION

MARKS

Particular solution Uncharged at $t=0$ i.e. $Q(0) = 0$ since switch closed

$\therefore Q(t) = VC\left(1 - e^{-t/RC}\right)$

$Q(t) = \underbrace{VC}_{\text{steady-state}} \underbrace{- VCe^{-t/RC}}_{\text{transient}}$ i.e. $\to 0$ as $t \to \infty$

(+5) As $t \to \infty$, all voltage across C since voltage across R is IR

i.e. $\frac{dQ}{dt}R$ and requires time-varying charge.

→ 20 SUBTOTAL

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$$

Separation of variables: $u = X(x)Y(y)$

gives $YX_x = 4XY_y$ (subscript denoting partial derivative)

i.e. $\frac{X_x}{4X} = \frac{Y_y}{Y} = c$ (separation constant)

Each equation can now be treated as an o.d.e. i.e. $\frac{dX}{dx} = 4cX$

and $\frac{dY}{dy} = cY$

Solutions are $X = Ae^{4cx}$, $Y = Be^{cy}$.

10 A solution is thus $\boxed{u = XY = Ke^{c(4x+y)}}$, $K = AB$

SAMPLE EXAM 3

SECTION A

1. Answer ALL parts of the question:

(a) Determine the constant a such that the vectors $\mathbf{F} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = a\mathbf{i} - 7\mathbf{j} - 18\mathbf{k}$ are perpendicular.

(4 Marks)

(b) The position of a moving particle is given by a time-dependent position vector $\mathbf{r}(t)$. Derive an expression for the velocity of the particle, $\mathbf{v}(t) = \frac{d\,\mathbf{r}(t)}{dt}$,

when:

$$\mathbf{r}(t) = e^{-t}\ \mathbf{i} + 2\cos(3t)\ \mathbf{j} + 2\sin(3t)\ \mathbf{k}.$$

(4 Marks)

(c) Describe what is meant by: (a) a *scalar field*; and (b) a *vector field*. With reference to a typical weather forecast map, give one example of each type of field.

(6 Marks)

(d) Describe the property of the vector field $\mathbf{A}$ that $\operatorname{div}\mathbf{A}$ (i.e. $\nabla\cdot\mathbf{A}$) represents (make reference to the divergence theorem and give one, or more, physical examples in your answer).

(5 Marks)

(e) Describe the property of the vector field $\mathbf{A}$ that $\operatorname{curl}\mathbf{A}$ (i.e. $\nabla\times\mathbf{A}$) represents (make reference to Stokes' theorem and give one, or more, physical examples in your answer).

(5 Marks)

QUESTION A IS CONTINUED ON THE NEXT PAGE

(f) By considering matrix determinants, show that the *rank* of the matrix $\mathbf{A}$ is equal to 1 when:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}.$$

(4 Marks)

(g) Relate your answer for part *(c)* to a description of the solution of the following two simultaneous equations:

$$2x + 6y = 1$$
$$3x + 9y = 2.$$

Illustrate your answer with a sketch that includes lines in the *x-y* plane.

(4 Marks)

(h) Two simultaneous linear equations, with constant coefficients a, b, c and d, take the form:

$$ax + by = e$$
$$cx + dy = f$$

where e and f are also constants. Verify that the homogeneous system ($e = f = 0$) always has the trivial solution ($x = y = 0$).

(4 Marks)

(i) Use the *chain rule* to show that $v = f(u)$ is a solution of the partial differential equation:

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0,$$

where $u = y - 3x$ and f is an arbitrary differentiable function.

(4 Marks)

SECTION B

2. A particular scalar field is defined by the equation:

$$\phi(x,y,z) = x^2z + 2xy^2 + yz^2.$$

Show that $\nabla\phi$ for this field is given by:

$$\nabla\phi = \left(2xz + 2y^2\right)\mathbf{i} + \left(4xy + z^2\right)\mathbf{j} + \left(x^2 + 2yz\right)\mathbf{k}.$$

(7 marks)

Hence, calculate the *magnitude* of the rate of change of $\phi(x,y,z)$ at the point $(x,y,z) = (1, 2, -1)$ in the direction of the vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

(13 marks)

Use the given form of $\nabla\phi$ (in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$) to prove that $\nabla\phi$ is a *conservative field*.

(10 marks)

3. Answer BOTH parts of the question:

(a) For a vector field $\mathbf{A}(x,y,z)$, Stokes' theorem can be stated as:

$$\int_S (curl\ \mathbf{A})\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{r}.$$

Explain the meaning of each of the symbols used in this equation and the interpretation of the quantity *curl* $\mathbf{A}$ that is implied by this theorem.

(8 Marks)

(b) The fluid velocity of a particular uniform flow is given by $\mathbf{V}_2 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. An example of a non-uniform flow is given by $\mathbf{V}_3 = 2y\mathbf{i}$. Evaluate *curl* $\mathbf{V}$ for each of the flows $\mathbf{V}_2$ and $\mathbf{V}_3$ and interpret the results. What does *Stokes' theorem* imply regarding the character of the vector fields representing the uniform and the non-uniform flows?

(14 Marks)

The *circulation* of a vector field $\mathbf{V}(x,y,z) = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ around a closed path C, can be written as: $\oint_C \mathbf{V}\cdot d\mathbf{r} = \oint_C V_x\,dx + V_y\,dy + V_z\,dz$. Hence, verify *Stokes' theorem* for the uniform flow $\mathbf{V}_2$ by considering the closed path C in the *x-y* plane (around the four sides of a square) given by:

$$(x,y,z) = (0,0,0) \to (0,1,0) \to (1,1,0) \to (1,0,0) \to (0,0,0).$$

(8 Marks)

4. Answer BOTH parts of the question:

(a) If

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{bmatrix},$$

show that the *matrix of cofactors* of A is:

$$C = \begin{bmatrix} -8 & 7 & -6 \\ 1 & -4 & 2 \\ 5 & -5 & 5 \end{bmatrix}.$$

Show also that the determinant of A is $\det(A) = -5$. Hence, find the matrix A^{-1} (that is the inverse of the matrix A).

(15 Marks)

(b) Prove that the *eigenvalues* of the matrix $B = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ are given by $\lambda_1 = 1$ and $\lambda_2 = 4$. Hence, find *either one* of the two linearly independent *eigenvectors* of the matrix B.

(15 Marks)

5. Answer BOTH parts of the question:

(a) An electrical circuit consists of a resistance R and an inductance L connected in series to a battery of constant voltage V. The voltage dropped across R and L, and the current $I(t)$, are related through the ordinary differential equation:

$$L\frac{dI}{dt} + RI = V,$$

where t is time.

Show that the general solution of this differential equation is given by:

$$I(t) = I(0)\ \exp\left[-\left(\frac{R}{L}\right)t\right] + \frac{V}{R}\left\{1 - \exp\left[-\left(\frac{R}{L}\right)t\right]\right\}.$$

Find the particular solution when $I(0) = 0$ and identify the long-term (steady-state) and transient components of this particular solution.

(18 Marks)

(b) Use the method of *separation of variables* to prove that a solution of the partial differential equation $3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ is given by $u(x, y) = K e^{c(x-3y)}$, where K and c are constants.

(12 Marks)

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SAMPLE EXAM 3 SOLUTIONS

EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
4	1. (a) $\underset{\sim}{F} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{r} = a\underset{\sim}{i} - 7\underset{\sim}{j} - 18\underset{\sim}{k}$ $\therefore \underset{\sim}{F}\cdot\underset{\sim}{r} = (2a) + (2).(-7) + (-1).(-18)$ $= 2a - 14 + 18 = 2a + 4$ ie $\underset{\sim}{F}\cdot\underset{\sim}{r} = 0$ when $2a + 4 = 0$ i.e. when $a = -2$.
4	1(b) $\underset{\sim}{r}(t) = e^{-t}\underset{\sim}{i} + 2\cos 3t\,\underset{\sim}{j} + 2\sin 3t\,\underset{\sim}{k} = (r_x(t), r_y(t), r_z(t))$ $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{dr_x}{dt}\underset{\sim}{i} + \frac{dr_y}{dt}\underset{\sim}{j} + \frac{dr_z}{dt}\underset{\sim}{k}$ $\therefore \underset{\sim}{v} = -e^{-t}\underset{\sim}{i} + (-6\sin 3t)\underset{\sim}{j} + 6\cos 3t\,\underset{\sim}{k}$.
6	1(c) Scalar field ≡ Region of space with unique scalar value associated with each point Vector field ≡ as above, with vector value. e.g. scalar: temperature, pressure, vector: wind velocity

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EXAMINATION SOLUTION CONTINUATION SHEET

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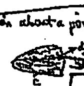
Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
5	1(d) div $\underset{\sim}{A}$ ≡ net volume density of sources and sinks of flux ("net outflow")  $\int_V \underbrace{\mathrm{div}\underset{\sim}{A}}_{\text{volume density}}\, dV = \underbrace{\oint_S \underset{\sim}{A}.d\underset{\sim}{S}}_{\text{net flux through closed surface S}}$ sufficient selection for full marks e.g. ± point charges → E-flux, lack of magnetic monopoles → div $\underset{\sim}{B} = 0$, incompressible fluid with no sources/sinks → div $\underset{\sim}{v} = 0$
5	1(e) curl $\underset{\sim}{A}$ ≡ twist/swirl/rotation/vorticity/circulation about a point  $\int_S \mathrm{curl}\,\underset{\sim}{A}.d\underset{\sim}{S} = \underbrace{\oint_C \underset{\sim}{A}.d\underset{\sim}{r}}_{\text{circulation around C}}$ sufficient selection for full marks e.g. electrostatics : $\underset{\sim}{\nabla}\times\underset{\sim}{E} = \underset{\sim}{0}$ (no vortices, conservative), solenoid, current-carrying wire : loops/vortices in B-field, $\underset{\sim}{\nabla}\times\underset{\sim}{B} \neq \underset{\sim}{0}$, non-uniform fluid flow ("sheared"), tornadoes, hurricanes, plug-holes, fluid vortices...
4	1(f) $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$, $\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 2.9 - 6.3 = 18 - 18 = 0$ $\therefore$ rank(A) < 2 Existence of one 1×1 submatrix with non-zero determinant is sufficient to give A a rank of 1. For 1×1 matrices, determinant = element value. Any of [2], [6], [3], [9] give non-zero determinant, $\therefore$ rank(A) = 1.

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CS2

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

4

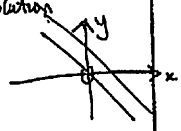
1(g) System is $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$, where $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$ and $|A| = 0$

Augmented coeff-matrix: $A_b = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 9 & 2 \end{bmatrix}$ and, e.g., $\begin{vmatrix} 6 & 1 \\ 9 & 2 \end{vmatrix} \neq 0$

$\Rightarrow \text{rank}(A_b) = 2$.

$\therefore \text{rank}(A) < \text{rank}(A_b) \Rightarrow$ no solution

Sketch: $\left.\begin{array}{l} y = -\frac{1}{3}x + \frac{1}{6} \\ y = -\frac{1}{3}x + \frac{2}{9} \end{array}\right\}$ parallel lines do not cross $\rightarrow$ no solution



(sufficient amount of above for full marks)

4

1(h) $\begin{array}{l} ax + by = e = 0 \\ cx + dy = f = 0 \end{array}$. The trivial solution is $x = y = 0$.

Verify this general result by direct substitution:

$x = 0, y = 0$ gives $\begin{array}{l} a.0 + b.0 = 0 \\ c.0 + d.0 = 0 \end{array}$ ✓ ✓

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CS3

Theoretical Physics I

EXAMINATION SOLUTION CONTINUATION SHEET

These boxes MUST be completed

Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

4

1(i) $v = f(u)$, $u = y - 3x$, $\boxed{\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0}$ (*)

To show this (*),

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial u}\cdot(-3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial u}\cdot(1)$$

$$\therefore \frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\cdot(-3) + 3\frac{\partial v}{\partial u}\cdot(1)$$

$$= -3\frac{\partial v}{\partial u} + 3\frac{\partial v}{\partial u} = 0$$

$\therefore v = f(u)$ is a solution irrespective of the particular form of (arbitrary differentiable) function f.

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT

7

$$\phi(x,y,z) = x^2 z + 2xy^2 + yz^2$$

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

$$\Rightarrow \underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial}{\partial x}(x^2z + 2xy^2 + yz^2) + \underset{\sim}{j}\frac{\partial}{\partial y}(x^2z + 2xy^2 + yz^2) + \underset{\sim}{k}\frac{\partial}{\partial z}(x^2z + 2xy^2 + yz^2)$$

$$= \underset{\sim}{i}(2xz + 2y^2 + 0) + \underset{\sim}{j}(0 + 4xy + z^2) + \underset{\sim}{k}(x^2 + 0 + 2yz)$$

$$\therefore \underset{\sim}{\nabla}\phi = (2xz + 2y^2)\underset{\sim}{i} + (4xy + z^2)\underset{\sim}{j} + (x^2 + 2yz)\underset{\sim}{k}.$$

At point $(1,2,-1)$ $\quad \underset{\sim}{\nabla}\phi = (-2+8)\underset{\sim}{i} + (8+1)\underset{\sim}{j} + (1-4)\underset{\sim}{k}$

$$= 6\hat{i} + 9\hat{j} - 3\hat{k}.$$

Direction derivative $\frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{u}}$, where $\hat{\underset{\sim}{u}}$ is a unit vector in the specified direction

In the direction of $2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$,

$$\hat{\underset{\sim}{u}} = \frac{2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}}{|2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}|}$$

where $|2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}| = (2^2 + 3^2 + (-4)^2)^{\frac{1}{2}}$

$$= (4 + 9 + 16)^{\frac{1}{2}} = \sqrt{29}.$$

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SOLUTION

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$$\therefore \hat{\underset{\sim}{u}} = \frac{1}{\sqrt{29}}(2,3,-4)$$

and $\frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{u}} = (6\underset{\sim}{i} + 9\underset{\sim}{j} - 3\underset{\sim}{k})\cdot\frac{1}{\sqrt{29}}(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k})$

$$= \frac{1}{\sqrt{29}}(12 + 27 + 12)$$

$$= \frac{51}{\sqrt{29}}$$

Let $\underset{\sim}{V} = \underset{\sim}{\nabla}\phi \equiv V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}$,

where $V_x = 2xz + 2y^2$, $V_y = 4xy + z^2$, $V_z = x^2 + 2yz$

$\underset{\sim}{\nabla}\phi$ conservative $\Leftrightarrow \underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$,

where $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$

$$= \underset{\sim}{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \underset{\sim}{j}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \underset{\sim}{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

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$$\underline{\nabla}\times\underline{V} = \underline{i}\left[\frac{\partial}{\partial y}(x^2+2yz) - \frac{\partial}{\partial z}(4xy+z^2)\right] - \underline{j}\left[\frac{\partial}{\partial x}(x^2+2yz) - \frac{\partial}{\partial z}(2xz)\right] + \underline{k}\left[\frac{\partial}{\partial x}(4xy+z^2) - \frac{\partial}{\partial y}(2xz+2y^2)\right]$$

$$= \underline{i}\left[2z-2z\right] - \underline{j}\left[2x-2x\right] + \underline{k}\left[4y-4y\right]$$

$$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0} \quad \text{(for all } x,y,z)$$

10 $\therefore\ \underline{\nabla}\phi = \underline{V}$ is a conservative vector field

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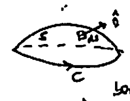
Paper Code:	Question No. 3

SOLUTION

MARKS	TEXT

(a) $\displaystyle\int_S (\text{curl}\,\underline{A})\cdot d\underline{S} = \oint_C \underline{A}\cdot d\underline{r}$



Surface integral over open surface S of dot product of curl $\underline{A}$ $\left(=\begin{vmatrix}\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{vmatrix}\right)$ and $d\underline{S}$ and closed line/curve integral along bounding curve C (clockwise sense with respect to $\hat{\underline{n}}$) of dot product of $\underline{A}$ and $d\underline{r}$ element (along C).

Projection of $\underline{\nabla}\times\underline{A}$ gives surface density that integrates over S to give total circulation of $\underline{A}$ around C

curl $\underline{A}$ measures circulation/rotation/vorticity around a point

8 [sufficient selection for full marks]

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(b) $\underline{V}_2 = 2\underline{i} + 3\underline{j} + \underline{k}$ (a uniform flow)

$$\underline{\nabla}\times\underline{V}_2 = \begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ 2&3&1\end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(0-0) = \underline{0}$$

→ Irrotational field (no circulation/vortices anywhere).

$\underline{V}_3 = 2y\,\underline{i}$

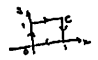
$$\underline{\nabla}\times\underline{V}_3 = \begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\ \partial/\partial x&\partial/\partial y&\partial/\partial z\\ 2y&0&0\end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(0-2) = -2\underline{k}.$$

→ Rotational field (const. circulation/degree of rotation at each point).

- Stokes' theorem $\displaystyle\oint_C \underline{V}\cdot d\underline{r} = \int_S (\underline{\nabla}\times\underline{V})\cdot d\underline{S}$

$\underline{\nabla}\times\underline{V} = \underline{0} \Rightarrow \oint_C \underline{V}\cdot d\underline{r} = 0$ (no circulation around any closed path) (– provided it's a simple path)!

14 also implies path independence of $\int_A^B \underline{V}\cdot d\underline{r}$ and other 'conservative properties'.



$$\oint_C \underline{V}\cdot d\underline{r} = \oint_C V_x\,dx + V_y\,dy + V_z\,dz$$

$$= \oint_C 2\,dx + 3\,dy + 1.\,dz, \text{ using } \underline{V}_2.$$

$$\therefore \oint_C \underline{V}_2\cdot d\underline{r} = \underbrace{\int_0^1 3\,dy}_{(dx=dz=0)} + \underbrace{\int_0^1 2\,dx}_{(dy=dz=0)} + \underbrace{\int_1^0 3\,dy}_{(dx=dz=0)} + \underbrace{\int_1^0 2\,dx}_{(dy=dz=0)}$$

$$= \int_0^1 3\,dy + \int_0^1 2\,dx - \int_0^1 3\,dy - \int_0^1 2\,dx = 0$$

8 (from above, $\underline{\nabla}\times\underline{V}_2 = 0 \Rightarrow \int_S (\underline{\nabla}\times\underline{V})\cdot d\underline{S} = 0$ and we have verified consistency with Stokes' theorem.)

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT

(a)
$$A = \begin{pmatrix}2&-1&-3\\1&2&1\\2&-2&-5\end{pmatrix}$$

cofactors $A_{11} = +\begin{vmatrix}2&1\\-2&-5\end{vmatrix} = -8$, $A_{12} = -\begin{vmatrix}1&1\\2&-5\end{vmatrix} = 7$, $A_{13} = +\begin{vmatrix}1&2\\2&-2\end{vmatrix} = -6$,

$A_{21} = -\begin{vmatrix}-1&-3\\-2&-5\end{vmatrix} = 1$, $A_{22} = +\begin{vmatrix}2&-3\\2&-5\end{vmatrix} = -4$, $A_{23} = -\begin{vmatrix}2&-1\\2&-2\end{vmatrix} = 2$,

$A_{31} = +\begin{vmatrix}-1&-3\\2&1\end{vmatrix} = 5$, $A_{32} = -\begin{vmatrix}2&-3\\1&1\end{vmatrix} = -5$, $A_{33} = +\begin{vmatrix}2&-1\\1&2\end{vmatrix} = 5$.

$\therefore$ Matrix of cofactors, $C = \begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix} = \begin{pmatrix}-8&7&-6\\1&-4&2\\5&-5&5\end{pmatrix}$

e.g. $|A| = 2A_{11} - 1.A_{12} - 3.A_{13}$ (along top row)

$= -16 - 7 + 18 = -5.$

Then, $A^{-1} = \dfrac{1}{|A|}C^T = -\dfrac{1}{5}\begin{pmatrix}-8&1&5\\7&-4&-5\\-6&2&5\end{pmatrix} = \dfrac{1}{5}\begin{pmatrix}8&-1&-5\\-7&4&5\\6&-2&-5\end{pmatrix}.$

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(b) $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$; Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2 = 0$$

i.e. $6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

i.e. $\lambda^2 - 5\lambda + 4 = 0$

By inspection, $\left.\begin{matrix}\lambda_1 + \lambda_2 = +5 \\ \lambda_1\lambda_2 = 4\end{matrix}\right\} \rightarrow \begin{matrix}\lambda_1 = 1 \\ \lambda_2 = 4.\end{matrix}$

Eigenvectors Since $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, these satisfy $A\underset{\sim}{x} - \lambda I \underset{\sim}{x} = \underset{\sim}{0}$

i.e. $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

i.e. $\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Denote $\underset{\sim}{x_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ as the eigenvector associated with λ_1.

Then, $\begin{pmatrix} 3-\lambda_1 & 2 \\ 1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix}(3-\lambda_1)x_1 + 2y_1 = 0 \\ x_1 + (2-\lambda_1)y_1 = 0\end{matrix}$

i.e. $\begin{matrix}(3-1)x_1 + 2y_1 = 0 \\ x_1 + (2-1)y_1 = 0\end{matrix}$ (since $\lambda_1 = 1$)

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i.e. $\left.\begin{matrix}2x_1 + 2y_1 = 0 \\ x_1 + y_1 = 0\end{matrix}\right\} \Rightarrow x_1 = -y_1$

Eigenvectors are only defined in terms of the ratio of the components (to within an undetermined scalar)

$\therefore \underset{\sim}{x_1} = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, α = undetermined scalar.

OR

$$\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Denote $\underset{\sim}{x_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ as the eigenvector associated with $\lambda_2 = 4$.

Then, $\begin{pmatrix} 3-\lambda_2 & 2 \\ 1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix}(3-\lambda_2)x_2 + 2y_2 = 0 \\ x_2 + (2-\lambda_2)y_2 = 0\end{matrix}$

i.e. $\begin{matrix}(3-4)x_2 + 2y_2 = 0 \\ x_2 + (2-4)y_2 = 0\end{matrix}$ (since $\lambda_2 = 4$)

i.e. $\begin{matrix}-x_2 + 2y_2 = 0 \\ x_2 - 2y_2 = 0\end{matrix}$

Both equations imply $x_2 = 2y_2$.

$\therefore \underset{\sim}{x_2} = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where β is an undetermined scalar.

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Paper Code:	Question No. 5

SOLUTION

MARKS	TEXT

a)

Standard form: $L\frac{dI}{dt} + RI = V \rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right)I = \left(\frac{V}{L}\right)$

I.F. $= e^{\int P dt} = e^{\frac{R}{L}\int dt} = e^{\frac{R}{L}t} \Rightarrow e^{\frac{R}{L}t}\frac{dI}{dt} + e^{\frac{R}{L}t}\left(\frac{R}{L}\right)I = e^{\frac{R}{L}t}\left(\frac{V}{L}\right)$

$\Rightarrow \frac{d}{dt}\left[e^{\frac{R}{L}t}.I\right] = e^{\frac{R}{L}t}.\left(\frac{V}{L}\right)$

$\Rightarrow e^{\frac{R}{L}t}.I = \frac{V}{L}\int e^{\frac{R}{L}t}dt + C'$

$= \frac{V}{L}.\left(\frac{L}{R}\right)e^{\frac{R}{L}t} + C = \frac{V}{R}e^{\frac{R}{L}t} + C.$

$\rightarrow \underline{I(t) = V/R + Ce^{-\frac{R}{L}t}}$, after dividing by $e^{\frac{R}{L}t}$.

C can be identified by noting that at $t=0$, $I(0) = \frac{V}{R} + Ce^0 = \frac{V}{R} + C$

$\rightarrow C = I(0) - \frac{V}{R}$. $\rightarrow I(t) = \frac{V}{R} + \left(I(0) - \frac{V}{R}\right)e^{-\frac{R}{L}t} = \frac{V}{R} + I(0)e^{-\frac{R}{L}t} - \frac{V}{R}e^{-\frac{R}{L}t}$

i.e. $I(t) = I(0)e^{-\frac{R}{L}t} + \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$.

$I(t) = I(0)e^{-\frac{R}{L}t} + \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$.

Particular solution when $I(0) = 0$, i.e. switch closed at $t=0 \Rightarrow I=0$ at $t=0$,

$I(t) = 0 + \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$, $I(t) = \underset{\text{(constant/steady-state)}}{\frac{V}{R}} \underbrace{- \frac{V}{R}e^{-\frac{R}{L}t}}_{\text{(time-varying, tends to zero, transient)}}$

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b)

$\boxed{3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0}$ Set $u(x,y) = X(x).Y(y)$ and substitute ....

$3Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 0$, rearrange as functions of x only, y only

$\Rightarrow \frac{3}{X}\frac{\partial X}{\partial x} = -\frac{1}{Y}\frac{\partial Y}{\partial y}$, introduce separation constant (c),

$\rightarrow \frac{3}{X}\frac{\partial X}{\partial x} = c \Rightarrow \boxed{\frac{3}{X}\frac{dX}{dx} = c}$ and $-\frac{1}{Y}\frac{\partial Y}{\partial y} = c \Rightarrow \boxed{\frac{dY}{dy} = -cY}$

i.e. pair of ode's (integrate independently, etc.)

$\frac{dX}{dx} = \frac{c}{3}X$

$\int \frac{dX}{X} = \frac{c}{3}\int dx$ (ode separated)

$\ln X = \frac{c}{3}x + A$

$X = e^{(\frac{c}{3}x + A)}$

$\int \frac{dY}{Y} = -c\int dy$

$\Rightarrow \ln Y = -cy + B$

$\Rightarrow Y = e^{(-cy + B)}$

A solution is then $u = XY = e^{(\frac{c}{3}x + A)}.e^{(-cy + B)}$

$= e^{\frac{c}{3}x - cy}.e^{A+B}$

$= e^{c(\frac{x}{3} - y)}.e^{A+B}$

$= e^{c'(x - 3y)}.e^{A+B}$, $c' = \frac{c}{3}$

$\therefore u(x,y) = ke^{c'(x-3y)}$, $k = e^{A+B}$.

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