*Maths Methods & Applications (**first semester material**: "Theoretical Physics I") - Dr Graham S McDonald*

3 Sets of Sample Jan/May Exam Questions with Solutions

- <u>HIGH RESOLUTION VERSION</u>-

A. <u>Jan Test Format</u>:

<u>**COMPULSORY**</u> **Section A-type** short questions (worth 40 marks)

+

TWO <u>COMPULSORY</u> Section B-type questions will be included (<u>Topics</u>: Vector Calculus *and* Matrices)

B. <u>May Exam Contributions</u>:

Some **Section A-type short** questions (worth around 13 of the 40 marks of Section A)

+

ONE <u>OPTIONAL</u> Section B-type question (you do 3 from the 6 presented) will be included (<u>Topic</u>: Matrices *or* Differential Equations)

SAMPLE EXAM 1 : Section A = 40 marks + pick two Section B questions (worth 30 marks each)

SECTION A

1. Answer **ALL** parts of the question:

(a) Consider a body that rotates with constant angular velocity $\boldsymbol{\omega} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Calculate the tangential velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ at the point given by the position vector $\mathbf{r} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

(4 Marks)

(b) The position of a moving particle is given by a time-dependent position vector $\mathbf{r}(t)$. Derive an expression for the velocity of the particle, $\mathbf{v}(t) = \dfrac{d\,\mathbf{r}(t)}{dt}$,

when:

$$\mathbf{r}(t) = (2t + 3)\mathbf{i} + (t^2 + 3t)\mathbf{j} + (t^3 + 2t^2)\mathbf{k}.$$

(4 Marks)

(c) A physical quantity is defined over a region of space by the scalar field $\phi(x, y, z)$. Describe what physical property of this field is given by $\mathbf{grad}\phi \ (\equiv \nabla\phi)$.

(6 marks)

(d) For a vector field $\mathbf{A}(x, y, z)$, the divergence theorem can be stated as:

$$\int_V div\mathbf{A} \ dV = \oint_S \mathbf{A} \cdot \mathbf{dS}.$$

Explain the meaning of each of the symbols used in this equation and the interpretation of the quantity $div\mathbf{A}$ that is implied by this theorem.

(8 marks)

(e) Find the matrix products $\mathbf{AB}$ and $\mathbf{BA}$, and hence show that $\mathbf{AB} \neq \mathbf{BA}$, when

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

(4 Marks)

QUESTION 1. IS CONTINUED ON NEXT PAGE

(f) By considering matrix determinants, calculate the *rank* of the matrix:

$$\mathbf{A} = \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}.$$

(4 Marks)

(g) Two *homogeneous* linear equations, with constant coefficients a , b , c and d , take the form:

$$ax + by = 0$$
$$cx + dy = 0$$

Describe the solution(s), x and y , in the cases where the determinant of the coefficient matrix, $\det \mathbf{A}$, satisfies:

(i) $\det \mathbf{A} \neq 0$,

and *(ii)* $\det \mathbf{A} = 0$.

(6 Marks)

(h) Use the *chain rule* to show that $v = f(u)$ is a solution of the partial differential equation:

$$\frac{\partial v}{\partial x} - 5\frac{\partial v}{\partial y} = 0,$$

when $u = y + 5x$ and f is an arbitrary differentiable function.

(4 Marks)

SECTION B

2. A particular scalar field is defined by the equation:

$$\phi(x, y, z) = x\,\mathrm{e}^{y} + y z^{2} + x y z \ .$$

Show that $\nabla\phi$ for this field is given by:

$$\nabla\phi = \left(\mathrm{e}^{y} + yz\right)\mathbf{i} + \left(x\,\mathrm{e}^{y} + z^{2} + xz\right)\mathbf{j} + y(2z + x)\mathbf{k} \ .$$

(7 Marks)

Hence, calculate the *magnitude* of the rate of change of $\phi(x, y, z)$ at the point $(x, y, z) = (2,\ 0, 3)$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(13 Marks)

Use the given form of $\nabla\phi$ (in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$) to prove that the vector field $\mathbf{V}(x, y, z) = \nabla\phi$ is a *conservative field.*

(10 Marks)

3. Answer **BOTH** parts of the question:

(a) Discuss the concept of *vector area* (making reference to the magnitude and to the direction of the cross product of two vectors in your answer).

(6 Marks)

(b) For the vector field $\mathbf{A}(x,y,z)=(3x-2y)\mathbf{i}+(x^2z)\mathbf{j}+(1-2z)\mathbf{k}$, show that:

(i) $\nabla\bullet\mathbf{A}=1$; and

(ii) $\nabla\times\mathbf{A}=-x^2\mathbf{i}+2(xz+1)\mathbf{k}$.

(14 Marks)

Consider the same vector field $\mathbf{A}$ over the circular region S that is bounded by the curve $x^2+y^2=a^2$ in the $z=0$ plane. Adopting the convention that the normal to the surface S is in the direction of the *positive* z-axis, show that:

$$\int_S \mathbf{A}\bullet\mathbf{dS}=\pi a^2 .$$

Note: you should use the fact that $\mathbf{A}=(3x-2y)\mathbf{i}+\mathbf{k}$ in the $z=0$ plane.

(10 Marks)

4. Answer **BOTH** parts of the question:

(a) If

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix},$$

show that the matrix of cofactors of $\mathbf{A}$ is:

$$\mathbf{C} = \begin{bmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{bmatrix}.$$

Show also that the determinant of $\mathbf{A}$ is $\det(\mathbf{A}) = 35$.

Hence, find the matrix $\mathbf{A}^{-1}$ (that is the *inverse* of the matrix **A**).

(14 Marks)

(b) Prove that the *eigenvalues* of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are given by $\lambda_1 = -1$ and $\lambda_2 = 5$. Hence, find *either one* of the two linearly independent *eigenvectors* of the matrix $\mathbf{A}$.

(16 Marks)

5. Answer **ALL** parts of the question:

(a) Use the *integrating factor method* to show that the solution of the differential equation:

$$\frac{dx}{dt}+\frac{x}{T}=A\ e^{i\omega t}$$

is

$$x(t)=x(0)\ e^{-t/T}+B\left(\ e^{i\omega t}-\ e^{-t/T}\ \right),$$

where $B=\dfrac{A}{\left(i\omega+1/T\right)}$, and A, T and ω are real constants.

(14 marks)

Identify the transient and the long-term steady-state components of the solution $x(t)$.

Comment on the relative phase of the steady-state component and the driving term, $A\,e^{i\omega t}$, in the limit of large damping ($T\to\infty$).

(6 marks)

(b) Use the method of *separation of variables* to prove that a solution of the partial differential equation:

$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial y}+u$$

is given by:

$$u(x,y)=Ke^{c\left(x+\frac{y}{2}\right)}e^{-\frac{y}{2}},$$

where K and c are constants.

(10 Marks)

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SAMPLE EXAM 1 SOLUTIONS

EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	a) $\underset{\sim}{w} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{r} = 2\underset{\sim}{i} - 6\underset{\sim}{j} - 3\underset{\sim}{k}$
	$\underset{\sim}{w} \times \underset{\sim}{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 3 & -1 \\ -6 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix}$
	$= \underset{\sim}{i}\left[3(-3) - (-1)(-6)\right] - \underset{\sim}{j}\left[4(-3) - (-1)2\right] + \underset{\sim}{k}\left[4(-6) - 3.2\right]$
4	$= \underset{\sim}{i}\left(-9-6\right) - \underset{\sim}{j}\left(-12+2\right) + \underset{\sim}{k}\left(-24-6\right)$
	$= -15\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\sim}{k}$.
	b) $\underset{\sim}{r}(t) = (2t+3)\underset{\sim}{i} + (t^2+3t)\underset{\sim}{j} + (t^3+2t^2)\underset{\sim}{k}$.
	$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{d}{dt}(2t+3)\underset{\sim}{i} + \frac{d}{dt}(t^2+3t)\underset{\sim}{j} + \frac{d}{dt}(t^3+2t^2)\underset{\sim}{k}$
4	$= 2\underset{\sim}{i} + (2t+3)\underset{\sim}{j} + (3t^2+4t)\underset{\sim}{k}$.
	c) grad ϕ is a vector field that gives direction and magnitude of the maximum rate of change
6	of ϕ with respect to space
	(can also be called "the vector gradient")

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

d) $\int_V \text{div}\,\underset{\sim}{A}\, dV = \oint_S \underset{\sim}{A}\cdot d\underset{\sim}{S}$

area element $d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\, dS$

Volume (integral) vector field volume element enclosing (closed) surface (integral)

$\text{div}\,\underset{\sim}{A} \equiv$ volume density. R.H.S. $\Rightarrow$ net flux of $\underset{\sim}{A}$ through S

$\left(\text{div}\,A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$ $\therefore$ div A $\Rightarrow$ net volume density of sources of flux inside S

[sufficient selection to get full marks]

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e) $AB = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4\times1+2\times2 & 4\times1+2\times1 \\ 2\times1+1\times2 & 2\times1+1\times1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix},$

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$BA = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\times4+1\times2 & 1\times2+1\times1 \\ 2\times4+1\times2 & 2\times2+1\times1 \end{bmatrix} = \begin{pmatrix} 6 & 3 \\ 10 & 5 \end{pmatrix},$

$\therefore AB \neq BA.$

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

(f) $A = \begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$. $\det A = |A| = \begin{vmatrix} 6 & 3 \\ 8 & 4 \end{vmatrix} = 6.4 - 3.8$
$= 24 - 24 = 0$

$\therefore \text{rank}(A) < 2$.

Now, try to find any 1×1 submatrix with non-zero determinant.
Testing any of the above elements gives a nonzero determinant for the corresponding 1×1 matrix. For example, $8 \to [8]$ which has $|8| = 8 \neq 0$.

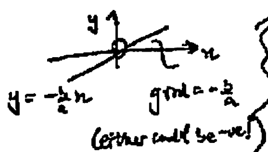
4 $\Rightarrow \text{rank}(A) = 1$.

(g) May consider system algebraically, graphically or use the rank directly.

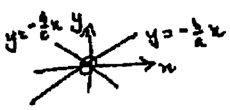
(ii) $\det A = 0 \to$ infinite number of solutions, including the trivial solution $(x,y) = (0,0)$

(i) $\det A \neq 0 \to$ unique solution, i.e. only the trivial one.

e.g. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \Rightarrow ad - bc = 0 \Rightarrow \frac{b}{a} = \frac{d}{c} \to$ equations/lines are the same



6 $\det A \neq 0 \Rightarrow$



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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION	
MARKS	TEXT

a) $v = f(y+5x)$ to be shown to be a solution of:

$$\frac{\partial v}{\partial x} - 5\frac{\partial v}{\partial y} = 0$$

Need to work out $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$. chain rule can be used, (with $u = y+5x$).

ie. $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial x} = 5.\frac{\partial v}{\partial u}$

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial y} = 1.\frac{\partial v}{\partial u}$

Verify solution by substitution

4 $\frac{\partial v}{\partial x} - 5\frac{\partial v}{\partial y} = \left(5.\frac{\partial v}{\partial u}\right) - 5\left(1.\frac{\partial v}{\partial u}\right) = 0$, as required

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS

~~Grad ϕ is a vector field that gives the direction and magnitude of the maximum rate of change of ϕ with respect to space~~ (2)

~~$\phi(x,y,z) = xe^y + yz^2 + xyz$~~

$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$; $\frac{\partial\phi}{\partial x} = e^y + 0 + yz = e^y + yz$.

$\frac{\partial\phi}{\partial y} = xe^y + z^2 + xz$.

$\frac{\partial\phi}{\partial z} = 0 + 2yz + xy = y(2z+x)$.

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$\therefore \underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k} = (e^y + yz)\underset{\sim}{i} + (xe^y + z^2 + xz)\underset{\sim}{j} + y(2z+x)\underset{\sim}{k}$.

At point $(2,0,3)$, $\underset{\sim}{\nabla}\phi = (e^0 + 0.3)\underset{\sim}{i} + (2.e^0 + 3^2 + 2.3)\underset{\sim}{j} + 0.(2.3+2)\underset{\sim}{k}$

$= \underset{\sim}{i} + 17\underset{\sim}{j}$.

This is the magnitude and direction of the maximum (spatial) rate of change of ϕ at this point.

For the direction given by $\underset{\sim}{A} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, we need to calculate the direction derivative $\frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi . \hat{\underset{\sim}{A}}$, where $\hat{\underset{\sim}{A}}$ is a unit vector in the direction of $\underset{\sim}{A}$ (we project the $\nabla\phi$ onto this unit vector).

Here, $|\underset{\sim}{A}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$

$\therefore \hat{\underset{\sim}{A}} = \frac{\underset{\sim}{A}}{|\underset{\sim}{A}|} = \frac{1}{\sqrt{14}}(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$.

$\rightarrow \frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi . \hat{\underset{\sim}{A}} = (\underset{\sim}{i} + 17\underset{\sim}{j}) . \frac{1}{\sqrt{14}}(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

$= \frac{1}{\sqrt{14}}(3 - 2.17 + 0) = -\frac{31}{\sqrt{14}}$.

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Paper Code:	Question No. 2

SOLUTION

MARKS	

$$\boxed{\underset{\sim}{V} = \underset{\sim}{\nabla}\phi = (e^y + yz)\underset{\sim}{i} + (xe^y + z^2 + xz)\underset{\sim}{j} + y(2z+x)\underset{\sim}{k}.}$$

$\underset{\sim}{V}$ is conservative $\Leftrightarrow$ $\underset{\sim}{\nabla}\times\underset{\sim}{V} = 0$

[(OR) use the reciprocity relations directly
i.e. $\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}$, $\frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}$, $\frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}$]

i.e. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \underset{\sim}{0}$, where $V_x = e^y + yz$, $V_y = xe^y + z^2 + xz$, $V_z = (2z+x)y$

$$\underset{\sim}{\nabla}\times\underset{\sim}{V} = \underset{\sim}{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right),$$

Where $\frac{\partial V_z}{\partial y} = 2z+x$, $\frac{\partial V_y}{\partial z} = 2z+x$, $\frac{\partial V_z}{\partial x} = y$, $\frac{\partial V_x}{\partial z} = y$,

$\frac{\partial V_y}{\partial x} = e^y + z$, $\frac{\partial V_x}{\partial y} = e^y + z$.

$\therefore$ $\frac{\partial V_z}{\partial y} = \frac{\partial V_y}{\partial z}$, $\frac{\partial V_z}{\partial x} = \frac{\partial V_x}{\partial z}$, $\frac{\partial V_y}{\partial x} = \frac{\partial V_x}{\partial y}$

$\Rightarrow$ $\underset{\sim}{\nabla}\times\underset{\sim}{V} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} = \underset{\sim}{0}$ $\Rightarrow$ $\underset{\sim}{V} = \underset{\sim}{\nabla}\phi$ conservative

(as it must be!)

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EXAMINATION SOLUTION SHEET

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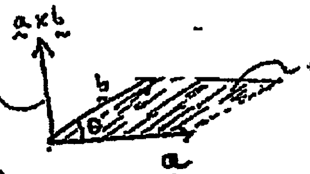
SOLUTION

MARKS	TEXT

(a) $\boxed{\text{vector area}}$



direction of $\underset{\sim}{a}\times\underset{\sim}{b}$ gives direction to the normal to this surface i.e. direction of $d\underset{\sim}{S}$

area $= |\underset{\sim}{a}\times\underset{\sim}{b}|$
$= ab\sin\theta$
$\equiv |d\underset{\sim}{S}|$

— could also mention conventions of:

- outward normals for a closed surface
- clockwise sense of bounding curve C for elements normals on open surface.

6 [A SUFFICIENT SELECTION OF ABOVE, UNDER EACH HEADING, FOR FULL MARKS]

(b) $\underset{\sim}{A} = (3x-2y)\underset{\sim}{i} + x^2z\underset{\sim}{j} + (1-2z)\underset{\sim}{k} \equiv A_x\underset{\sim}{i} + A_y\underset{\sim}{j} + A_z\underset{\sim}{k}$

(i) $\underset{\sim}{\nabla}\cdot\underset{\sim}{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(3x-2y) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(1-2z)$

$= 3 + 0 - 2 = \underline{1}.$

(ii) $\underset{\sim}{\nabla}\times\underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ A_y & A_z \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ A_x & A_z \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ A_x & A_y \end{vmatrix}$

$= \underset{\sim}{i}\left[\frac{\partial}{\partial y}(1-2z) - \frac{\partial}{\partial z}(x^2z)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(1-2z) - \frac{\partial}{\partial z}(3x-2y)\right] + \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(3x-2y)\right]$

$= \underset{\sim}{i}\left[0 - x^2\right] - \underset{\sim}{j}\left[0-0\right] + \underset{\sim}{k}\left[2xz - (-2)\right]$

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SOLUTION

MARKS	TEXT

$= -x^2 \underset{\sim}{i} + 2(xz+1)\underset{\sim}{k}.$

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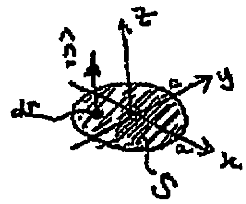
For this circular region, $z=0$



$\therefore \ \underset{\sim}{A} \rightarrow (3x-2y)\underset{\sim}{i} + 0\underset{\sim}{j} + (1-2\cdot 0)\underset{\sim}{k}$

i.e. $\underset{\sim}{A} = (3x-2y)\underset{\sim}{i} + \underset{\sim}{k}.$

$\hat{n}_{S}$ in +ve z-direction $\Rightarrow$ unit vector, $\hat{n} = \underset{\sim}{k}$.

i.e. $d\underset{\sim}{S} = \hat{n}\, dS = \underset{\sim}{k}\, dS$ (vector area element).

$$\underset{\sim}{A} = (3x-2y)\underset{\sim}{i} + \underset{\sim}{k} \quad ; \quad d\underset{\sim}{S} = \underset{\sim}{k}\, dS$$

For $\int_S \underset{\sim}{A}\cdot d\underset{\sim}{S} = \pi a^2$,

Form $\underset{\sim}{A}\cdot d\underset{\sim}{S} = \left[(3x-2y)\underset{\sim}{i} + \underset{\sim}{k}\right]\cdot \underset{\sim}{k}\, dS$

$= (3x-2y)\underset{\sim}{i}\cdot\underset{\sim}{k}\, dS + \underset{\sim}{k}\cdot\underset{\sim}{k}\, dS$

$= dS$ (since $\underset{\sim}{i}\cdot\underset{\sim}{k} = 0$ and $\underset{\sim}{k}\cdot\underset{\sim}{k} = 1$)

Then, $\int_S \underset{\sim}{A}\cdot d\underset{\sim}{S} = \int_S dS = \pi a^2$ (area of the circular disk)

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT

(a)

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}, \quad \text{sign table} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Signed Minors (cofactors):

$$A_{11} = +\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7, \quad A_{12} = -\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} = -(2+12) = -14, \quad A_{13} = +\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = 1-8 = -7,$$

$$A_{21} = -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} = +6, \quad A_{22} = +\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = -2, \quad A_{23} = -\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} = -11,$$

$$A_{31} = +\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 2, \quad A_{32} = -\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = +11, \quad A_{33} = +\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 8.$$

$$\Rightarrow \text{Matrix of cofactors (signed minors)} = \begin{pmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{pmatrix} = C$$

Determinant: $\det A = 3A_{11} - 2A_{12} + 2A_{13}$ (along row 1)

$$= 21 + 28 - 14 = 35.$$

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Inverse: $A^{-1} = \frac{1}{\det A} C^T = \frac{1}{35}\begin{pmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{pmatrix}$ (C^T is the transpose of C)

(b)

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$: Eigenvalues are given by $\det(A - \lambda I) = 0$

where $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)(3-\lambda) - 8$$
$$= 3 - 3\lambda - \lambda + \lambda^2 - 8$$
$$= \lambda^2 - 4\lambda - 5$$

$|A - \lambda I| = 0$ thus gives $\lambda^2 - 4\lambda - 5 = 0$

Solution by inspection: $\left.\begin{matrix} \lambda_1 + \lambda_2 = 4 \\ \lambda_1\lambda_2 = -5 \end{matrix}\right\} \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 5. \end{matrix}$

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT
	Let eigenvectors be $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.
	These satisfy $\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
	$\lambda_1 = -1$ $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix} 2x_1+2y_1=0 \\ 4x_1+4y_1=0 \end{matrix}$ i.e. $x_1 = -y_1$
	$\therefore \underset{\sim}{x}_1 = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, α undetermined scalar.
	EITHER OK
	$\lambda_2 = 5$ $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix} -4x_2+2y_2=0 \\ 4x_2-2y_2=0 \end{matrix}$ i.e. $2x_2 = y_2$
	$\therefore \underset{\sim}{x}_2 = \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, β an undetermined scalar.
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Paper Code:	Question No. 5

SOLUTION

MARKS	TEXT

a) $$\boxed{\frac{dx}{dt} + \frac{x}{\tau} = Ae^{i\Omega t}}$$

$(\tau \equiv T;\ \Omega \equiv \omega)$ driving term

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where τ, A, Ω are constants.

Sol^n We have $\frac{dx}{dt} + P(t)x = Q(t)$

where $P(t) = \frac{1}{\tau}$ and $Q(t) = Ae^{i\Omega t}$.

Integrating factor is $e^{\int P(t)dt} = e^{\frac{1}{\tau}\int dt} = e^{t/\tau}$

Multiply to get $e^{t/\tau}\frac{dx}{dt} + e^{t/\tau}\frac{x}{\tau} = e^{t/\tau}Ae^{i\Omega t}$

i.e. $\frac{d}{dt}\left[e^{t/\tau}.x\right] = e^{t/\tau}Ae^{i\Omega t} = Ae^{(i\Omega + \frac{1}{\tau})t}$

Integrate : $\int_0^t \frac{d}{dt}\left[e^{t/\tau}.x\right]dt = \int_0^t Ae^{(i\Omega+\frac{1}{\tau})t}\,dt$

i.e. $e^{t/\tau}.x(t) - x(0) = A\left[\frac{1}{(i\Omega+\frac{1}{\tau})}e^{(i\Omega+\frac{1}{\tau})t}\right]_0^t$

$= \frac{A}{(i\Omega+\frac{1}{\tau})}\left[e^{(i\Omega+\frac{1}{\tau})t} - 1\right]$

4   divide by integrating factor : $\boxed{x(t) = x(0)e^{-t/\tau} + B\left(e^{i\Omega t} - e^{-t/\tau}\right)}$, $B = \frac{A}{i\Omega + \frac{1}{\tau}}$

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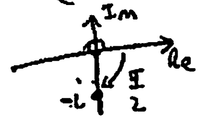
Paper Code:	Question No. 5

SOLUTION

MARKS	TEXT

- Terms involving $e^{-t/\tau}$ decay to zero as $t \to \infty$: transient
- leaves long-term steady-state component $x(t) \to Be^{i\omega t}$
- $B = \frac{A}{\left(i\omega + \frac{1}{\tau}\right)} \to \frac{A}{i\omega} = \frac{iA}{i.i\omega} = -i\frac{A}{\omega} = e^{-i\frac{\pi}{2}}\frac{A}{\omega}$ $(\omega = \Omega)$



6 $\therefore x(t) \to \frac{A}{\omega} e^{i(\omega t - \frac{\pi}{2})}$: 90° out of phase with driving term

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$$\boxed{\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u}$$

Set $u = X(x)Y(y)$ and substitute: $Y\frac{\partial X}{\partial x} = 2X\frac{\partial Y}{\partial y} + XY$

i.e. $\frac{1}{X}\frac{\partial X}{\partial x} = \frac{2}{Y}\frac{\partial Y}{\partial y} + 1$ $\left[\text{dividing by } XY\right]$

i.e. $\frac{1}{X}\frac{dX}{dx} = c$ and $\frac{2}{Y}\frac{dY}{dy} + 1 = c$ (c = separation constant)

$\int \frac{dX}{X} = c\int dx$

i.e. $\ln X = cx + A$, $X = e^{cx+A}$

$2\int \frac{dY}{Y} = (c-1)\int dy$

i.e. $2\ln Y = (c-1)y + B$

i.e. $\ln Y = \frac{(c-1)}{2}y + B'$

i.e. $Y = e^{\frac{(c-1)}{2}y + B'}$

10 A solution is $u = XY = e^{cx+A}e^{\left(\frac{c-1}{2}\right)y + B'} = e^{cx + \left(\frac{c-1}{2}\right)y}e^{A+B'}$

i.e. $u = ke^{c(x+\frac{y}{2}) - y/2}$; where $k = e^{A+B'}$

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