The Laplacian of Scalar and Vector Fields

The Laplacian, when expressed in Cartesian *x*, *y*, *z* coordinates, is defined as:

$$\nabla^2 = div\ grad = \nabla_{\bullet}\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (\textit{see MMA notes pages 123-137})$$

While it involves the differential vector operator *grad* (which can only be applied to scalar fields), the Laplacian operator itself is actually scalar and it can be applied either:

to a scalar field $\varphi(x,y,z)$, or
to a vector field $\mathbf{V}(x,y,z) = V_x(x,y,z)\mathbf{i} + V_y(x,y,z)\mathbf{j} + V_z(x,y,z)\mathbf{k}$

The above Cartesian form of the Laplacian is applied to a given scalar or vector field as follows.

- The Laplacian of a *scalar field*: $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ (giving a scalar field result)
- The Laplacian of a *vector field* gives another vector field:

$$\nabla^2\mathbf{V}(x,y,z) = \frac{\partial^2\mathbf{V}(x,y,z)}{\partial x^2} + \frac{\partial^2\mathbf{V}(x,y,z)}{\partial y^2} + \frac{\partial^2\mathbf{V}(x,y,z)}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}\left[V_x(x,y,z)\mathbf{i} + V_y(x,y,z)\mathbf{j} + V_z(x,y,z)\mathbf{k}\right]$$
$$+ \frac{\partial^2}{\partial y^2}\left[V_x(x,y,z)\mathbf{i} + V_y(x,y,z)\mathbf{j} + V_z(x,y,z)\mathbf{k}\right]$$
$$+ \frac{\partial^2}{\partial z^2}\left[V_x(x,y,z)\mathbf{i} + V_y(x,y,z)\mathbf{j} + V_z(x,y,z)\mathbf{k}\right]$$

$$= \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)\mathbf{i}$$
$$+ \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)\mathbf{k}$$

$$= \left(\nabla^2 V_x\right)\mathbf{i} + \left(\nabla^2 V_y\right)\mathbf{j} + \left(\nabla^2 V_z\right)\mathbf{k} \qquad (\textit{the result in this final line is the one to use}).$$

Further Worked Examples

There are three worked examples of applying the Laplacian to vector fields on pages 31, 32, and 33 of “Tutorial C” – which is linked to from the MMA course page at:
http://salfordphysics.com/index.php/MMA/

Tutorial C is also directly accessible as:
http://salfordphysics.com/gsmcdonald/TP1_01_Vector%20Calculus_03_Laplacian.pdf

Notes on Tutorial C

p4 EXERCISE 1. Revision of grad, div and curl.

p6 EXERCISE 2 (c).

$$\boxed{\nabla^2\left(x^2+y^2+z^2\right)^{1/2}=\nabla^2 r=\frac{2}{r}}$$

It is useful to use the chain rule:

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2}=\frac{\partial}{\partial u}\left(u^{1/2}\right)\frac{\partial u}{\partial x},\quad \text{where}\quad u=x^2+y^2+z^2$$

and note that calculating $\frac{\partial^2}{\partial x^2}\left(x^2+y^2+z^2\right)^{1/2}$ will need use of the Product Rule for differentiation.

Similar considerations apply for EXERCISE 2 (d).

p6 Quiz (immediately following EXERCISE 2).

$$\boxed{\nabla^2\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)^n=\nabla^2\frac{1}{r^n}=\frac{n(n-1)}{r^{n+2}}}$$

Again, it is useful to use the chain rule:

$$\frac{\partial}{\partial x}r^{-n}=\frac{\partial}{\partial r}\left(r^{-n}\right)\frac{\partial r}{\partial x},\quad \text{where}\quad \frac{\partial r}{\partial x}=\frac{x}{r}$$ (see p42 of Handout 1).

p8 Laplacian of the Product of (Scalar) Fields $u(x,y,z).v(x,y,z)$

In using

$$\boxed{\nabla^2(uv)=\left(\nabla^2 u\right)v+u\left(\nabla^2 v\right)+2\left(\nabla u\right).\left(\nabla v\right)}$$

it can be useful to note that:

$$\nabla^2\left(ax+by+cz\right)=0$$

and that

$$\nabla\left(ax+by+cz\right)=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$$

where a, b, and c are constants.