SUMMARY / OVERVIEW OF ...

Mathematical Methods and Applications

CONTENTS

HANDOUT 10

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no gaps
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- Ordinary Differential Equations
  - Review of 1st and 2nd order linear odes
  - Higher order linear odes
- Partial Differential Equations
  - Arbitrary functions
  - Similarities with the solution of ode's
  - Separation of variables
    - * Finding a solution
    - * Superposition to get the required solution

slides only have selected topics from the Handout

The first section simply reviews first year coverage of ordinary differential equations (ode's). These only have one independent variable. **Some terminology and classifications:**

Definition of a differential equation

– an equation involving derivatives or differentials ...

e.g. 1 $\quad (y'')^2 + 3x = 2(y')^3$ where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ — second order

e.g. 2 $\quad \frac{dy}{dx} + \frac{y}{x} = y^2$ — first order

e.g. 3 $\quad \frac{d^2Q}{dt^2} - 3\frac{dQ}{dt} + 2Q = 4\sin 2t$ — second order

e.g. 4 $\quad \frac{dy}{dx} = \frac{x+y}{x-y}$ or equivalently $(x+y)\,dx + (y-x)\,dy = 0$ — first order

e.g. 5 $\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ — second order

<u>nonlinear</u> (in the dept. variable)

- The highest order of derivative defines the <u>order</u> of the differential equation.

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The **general solution** of an n^{th} order o.d.e. has n arbitrary constants that can take any values.

In an **initial value problem**, one solves an n^{th} order o.d.e. to find the general solution and then applies n **boundary conditions** ("initial values/conditions") to find a **particular solution** that does not have any arbitrary constants.

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Solving O.D.E.'s

Most important methods:

- $$\boxed{\frac{dy}{dx} = f(x)} \rightarrow y = \int f(x)\,dx$$

  by "**direct integration**"

- $$\boxed{\frac{dy}{dx} = f(x)g(y)} \rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

  by "**separation of variables**"

- $$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

  $$\boxed{IF = e^{\int P(x)\,dx}}$$

  $$\boxed{\frac{d}{dx}(IF\ y) = IF\ Q(x).}$$

  $$\boxed{IF\ y = \int IF\ Q(x)\,dx.}$$

  divide by IF

"Integrating Factor Method" for **any** 1st order linear ode's

We also covered two cases where transformation of the dependent variable coverts the equation to a form that can be solved by these previous methods. Namely ...

- $$\boxed{\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}}$$ where M and N are **homogeneous functions** of the **same degree**

Change the dependent variable from y to v where $y = vx$ then

$$\text{LHS} = \frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{and} \quad \text{RHS} = \frac{M(x,y)}{N(x,y)}$$ becomes function of v only.

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Solve the resulting equation by separating the variables v and x, then re-express the solution in terms of x and y.

Note that this method also works for equations of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

- $$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$ **Bernoulli's differential equation**

Change the dependent variable from y to z where $z = y^{1-n}$.
This makes the equation linear and we can use the integrating factor method.

Finally, we looked at <u>Exact Differential Equations</u> ...

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- $\boxed{P(x, y)\, dx + Q(x, y)\, dy = 0}$

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ then the o.d.e. is said to be **exact.**

This means that a function $u(x,y)$ exists such that
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$= P\ \ dx + \ Q\ \ dy = 0\,.$$

Ones solves $\dfrac{\partial u}{\partial x} = P$ and $\dfrac{\partial u}{\partial y} = Q$ to find $u(x,y)$.

Then $du = 0$ gives $u(x,y)=constant$ (this is the general solution of $Pdx + Qdy = 0$).

... before moving on to 2nd order ode's

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0}$$

Second order linear o.d.e. with **constant coefficients** a,b,c

It is called a **homogeneous equation** because the RHS = 0.

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Setting $y = A\,e^{mx}$

gives $\boxed{am^2+bm+c=0}$

(the "auxiliary equation")

Then $m=\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$

gives three different cases ...

i) real different roots m_1 , m_2 and $\boxed{y=A\,e^{m_1x}+B\,e^{m_2x}}$,

OR

ii) real equal roots $m_1 = m_2$ and $\boxed{y=(A+Bx)e^{m_1x}}$,

OR

iii) complex roots

$m_{1,2}=p\pm i\,q$ and $\boxed{y=e^{px}(A\cos qx+B\sin qx)}$.

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)}$$

Second order linear o.d.e. with **constant coefficients** a,b,c
It is **not homogeneous** since RHS is not zero.

Step One

Solve the corresponding homogeneous equation to get $\boxed{y=y_{CF}}$
This is called the "complementary function".

Step Two

The general solution of the full equation is $\boxed{y=y_{CF}+y_{PS}}$.
Where y_{PS} is a particular solution of the full equation.

Find y_{PS} by substituting a trial form into the full equation and equate the coefficients of the functions involved (e.g. e^{2x}, x^2, $\cos x$, etc.).

$f(x)$	Trial form of y_{PS}
k	C
kx ...	$Cx+D$
kx^2 ...	Cx^2+Dx+E
$k\cos ax$ OR $k\sin ax$	$C\cos ax+D\sin ax$
ke^{ax}	Ce^{ax}
Sum/product of the above	Sum/product of the above
(k, a are given constants)	(C, D, E are constants to be determined)

If the suggested form of y_{PS} already appears in y_{CF} then multiply the trial form of y_{PS} by x until it does not

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Solution of higher order linear differential equations

(with constant coefficients)

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HOMOGENEOUS EQUATIONS (RHS=0)

i.e. $$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = 0$$

Set $y = e^{mx}$

$\Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$

$\Rightarrow$ roots $m_1, m_2, \dots, m_n$

$\Rightarrow$ 3 cases

(i) Roots all real and distinct:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

(ii) Repeated roots (k times)

If m_1 has multiplicity k then its contribution to the solution is

$$y = (c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1}) e^{m_1 x}$$

(iii) Complex roots always appear as conjugate pairs

Each pair of complex roots $p \pm iq$ contributes to the solution: $y = e^{px}(A\cos qx + B\sin qx)$

Continuing with *n*th order, linear o.d.e's with **constant coefficients** ...

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- INHOMOGENEOUS EQUATIONS (RHS $\neq 0$)

General solution
$$\boxed{y = Y_{CF}(x) + Y_{PS}(x)}$$

where $Y_{CF}(x)$ = solution of the homogeneous equation

$Y_{PS}(x)$ = a particular solution of the full equation

To find $Y_{PS}(x)$

- Substitute a trial solution involving unknown constants $C, D, E, \ldots$
- Guess the trial solution from the form of the RHS (as before)

PARTIAL DIFFERENTIAL EQUATIONS

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Some important partial differential equations

Laplace's equation: $\boxed{\nabla^2 u = 0}$

Poisson's equation $\boxed{\nabla^2 u = f(x,y,z)}$

Diffusion or heat flow equation $\boxed{\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}}$

Wave equation $\boxed{\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}}$

Role of arbitrary functions

The arbitrary constants of general solutions of ode's become arbitrary functions in the general solution of p.d.e.'s. Particular solutions then have a particular choice of arbitrary function.

An example ...

$\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = 2x - y}$ general solution $u = x^2 y - \frac{1}{2} x y^2 + F(x) + G(y)$

Here, arbitrary functions F(x) and G(y)

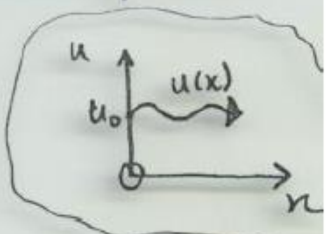
$$\Rightarrow \frac{\partial u}{\partial y} = x^2 - xy + G'(y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2 - xy + G'(y)\right)$$

$$= 2x - y \text{, as required.}$$

Another difference with odes is that initial-value problems

e.g. $\dfrac{du}{dx} = f(x,y)$ with $\underbrace{u(x=0) = u_0}_{\text{initial value}}$



tend to become boundary-value problems:

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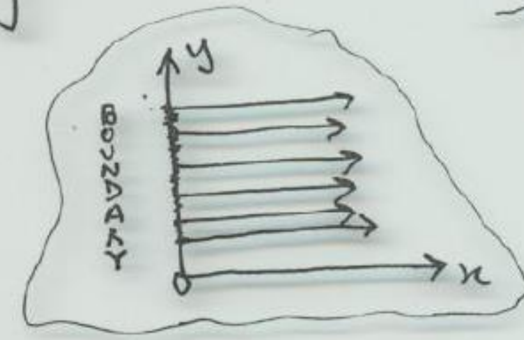
p.d.e's

e.g. $\frac{\partial^2 u}{\partial x \partial y} = f(x,y)$ with $\underbrace{u(x=0,y) = g(y)}_{\text{boundary}}$



i.e. because we have more than one independent variable, boundary conditions are not specified at a point.

We will deal here with linear partial differential equations that have constant coefficients.

e.g. $\boxed{a_0 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial x \partial y} + a_2 \frac{\partial^2 u}{\partial y^2} + a_3 \frac{\partial u}{\partial x} + a_4 \frac{\partial u}{\partial y} + a_5 u = f(x,y)}$

is second order, linear in u and $a_1, a_2, \ldots, a_5$ are constants

If $f(x,y) = 0$ then the equation is <u>homogeneous</u>.

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Solution by direct integration

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Ex

Starting with $\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = 2xy}$ derive the general solution.

Soln

Consider the left-hand side as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ and integrate

with respect to x ... i.e. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 2xy$

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gives $\dfrac{\partial u}{\partial y} = x^2 y + F(y)$

Then, integrate with respect to y ... $\Rightarrow$ $u = \dfrac{x^2 y^2}{2} + \int F(y)\,dy + G(x)$

i.e. $\boxed{u = \dfrac{x^2 y^2}{2} + H(y) + G(x)}$

, where $H(y) = \int F(y)\,dy$

NB

General solution of pde of **order 2** has **2 arbitrary functions**.

Homogeneous systems

Recall that for ode's one finds the solution of a homogeneous equation by setting $\boxed{y = e^{mx}}$ and then seeking the roots of the resulting characteristic equation, where x is the independent variable.

Now we may have two independent variables, x and t

for example: $\boxed{\dfrac{\partial y}{\partial x} + \dfrac{1}{c}\dfrac{\partial y}{\partial t} = 0}$

$\boxed{\text{Set } y = e^{ax+bt}}$ i.e. $a.e^{ax+bt} + \frac{1}{c}.b.e^{ax+bt} = 0$

i.e. $\left(a + \frac{b}{c}\right)e^{ax+bt} = 0$ i.e. $a + \frac{b}{c} = 0$ i.e. $b = -ac$

$$y = e^{ax+bt} = e^{ax-act} = e^{a(x-ct)},$$

for any a.

This is not the arbitrary function but it suggests an arbitrary function $\boxed{y = F(x-ct)}$.

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$\boxed{y = F(x-ct)}$ solution of $\boxed{\dfrac{\partial y}{\partial x} + \dfrac{1}{c}\dfrac{\partial y}{\partial t} = 0}$??

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Let $u = x - ct$ i.e. $y = F(u)$.

$$\frac{\partial y}{\partial x} = \underbrace{\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}}_{\text{(CHAIN RULE)}} = \frac{\partial F}{\partial u} \quad ; \quad \frac{\partial y}{\partial t} = \underbrace{\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}}_{\text{(CHAIN RULE)}} = \frac{\partial F}{\partial u}\cdot(-c)$$

$$\therefore \quad \frac{\partial y}{\partial x} + \frac{1}{c}\frac{\partial y}{\partial t} = \frac{\partial F}{\partial u} + \frac{1}{c}\cdot(-c)\frac{\partial F}{\partial u} = 0.$$

i.e. this arbitrary function **is** a solution.

- This technique can allow one to quickly determine the general solution of homogeneous partial differential equations.

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The wave equation in one space dimension

Another example ...

i.e. $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

gives $\boxed{u = e^{a(x \pm vt)}, \text{ for any } a.}$

... *two solutions.*

General solution of the 1D wave equation is

$$\boxed{u = F(x+vt) + G(x-vt)},$$

where F and G are arbitrary functions.

system: linear & homogeneous

"*superposition principle*"

Inhomogeneous systems (linear & constant coefficients)

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To solve an inhomogeneous ode for the general solution, we added the general solution of the homogeneous ode to a particular solution of the full equation. $\boxed{y = Y_{cf}(x) + Y_{ps}(x)}$

We can do the same for partial differential equations.

Ex $\boxed{\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = e^{2x+y}}$

Ans Set $u = e^{ax+by}$ in $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = 0$

The general solution of the homogeneous equation can be written as $\boxed{u = F(2x+y) + G(2x-y)}$.

Try $\boxed{u = Ce^{2x+y}}$ as a particular solution and determine C? No. We already have $F(2x+y)$ in the complementary solution. Try $\boxed{u = Cxe^{2x+y} \quad (\text{or } u = Cye^{2x+y})}$.

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Separation of variables (*the p.d.e. technique*!)

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Here we assume that the solution can be expressed as a product of unknown functions of each of the independent variables

$$\boxed{\text{e.g.}\quad u(x,y) = X(x)Y(y)}$$

How do we know that the solution is of this form ?

Generally, the solution we seek is not of this form !

But we can combine separable solutions together to get the desired solution.

Suppose we wish to solve the boundary-value problem $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, $u(0, y) = 8e^{-3y}$

- We assume the solution can be expressed as a product of unknown functions of each of the independent variables i.e. substitute the following into the pde

$$\boxed{u(x, y) = X(x)Y(y)}$$

- Rearrange the result so that the LHS depends only on x and the RHS depends only on y. In this example, we find:

$$\boxed{\frac{1}{4X}\frac{\partial X}{\partial x} = \frac{1}{Y}\frac{\partial Y}{\partial y}}$$

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- Equating LHS and RHS to the "separation constant" c, yields two odes's:

$$\boxed{\frac{dX}{dx} = c4X} \quad \text{and} \quad \boxed{\frac{dY}{dy} = cY}$$

with solutions: $\boxed{X = Ae^{4cx}}$ and $\boxed{Y = Be^{cy}}$

- Reconstruct $u = XY$ and apply the boundary condition(s) to u or to a *sum of solutions of this form*

i.e.

$$\boxed{u = XY = Ke^{c(4x+y)}, \quad \text{where } K = AB}$$

and boundary condition $\boxed{u(0, y) = 8e^{-3y} \equiv Ke^{c(0+y)}}$ yields $\boxed{u(x, y) = 8e^{-3(4x+y)}}$