A SUMMARY / OVERVIEW OF ...

Mathematical Methods and Applications

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HANDOUT 2

— VECTOR CALCULUS (continued)

- Recap : vector field
  line integral
  conservative fields

- cross product of vectors
  (including matrix determinants)

- triple products of vectors

Let's recap ...

we considered a force that varies in space

i.e. $\underset{\sim}{F}(x, y, z)$ .... vector field

we can specify how this varies in space

$$\underset{\sim}{F} = F_x \underset{\sim}{i} + F_y \underset{\sim}{j} + F_z \underset{\sim}{k}$$

i.e. $\underset{\sim}{F} = (F_x, F_y, F_z)$

where its components F_x, F_y, F_z are functions of position.

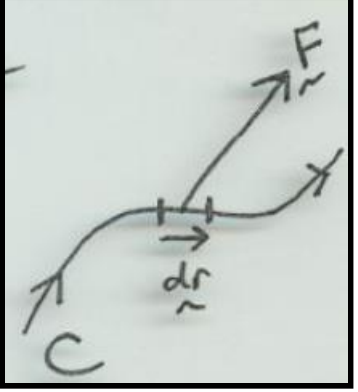
First example was:

$$\underset{\sim}{F} = (y^2, x^2, 0)$$

If ***F*** is constant and the path (length d) is straight then we simply have:

$$W = \underset{\sim}{F} \cdot \underset{\sim}{d}$$

for work done.



For varying ***F*** and curve C, we sum the contributions from elements:

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$$\delta W = \underset{\sim}{F} \cdot \underset{\sim}{dr}$$

$$W = \int_C \underset{\sim}{F} \cdot \underset{\sim}{dr}$$

In Cartesian coordinates:

$$\underset{\sim}{dr} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$$

F*.*dr gives

$$W = \int_C F_x\,dx + F_y\,dy + F_z\,dz$$

$$= \int_{\substack{\text{along}\\ x}} F_x\,dx + \int_{\substack{\text{along}\\ y}} F_y\,dy + \int_{\substack{\text{along}\\ z}} F_z\,dz$$

individual x, y, z contributions

For $\underset{\sim}{F} = (y^2, x^2, 0)$, $F_x = y^2$, $F_y = x^2$, $F_z = 0$.

$$W = \int_{\text{along } x} F_x\,dx + \int_{\text{along } y} F_y\,dy$$

$$W = \int_{\text{along } x} y^2\,dx + \int_{\text{along } y} x^2\,dy$$

This is problematic !!

But, the curve/path information resolves this problem. ☺

Along curve, $y = x^2$ we can set:

$$y = x^2 \rightarrow y^2 = x^4$$

$$y = x^2 \rightarrow x^2 = y$$

Then, we get:
(we *can* now do the integrals)

$$W = \int_{\text{along } x} x^4\,dx + \int_{\text{along } y} y\,dy$$

AND we can put in the limits, in terms of x and y.

Path independence of the integral?

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Mathematically, this hinges on whether we can write

$$dW = \underset{\sim}{F}.\underset{\sim}{dr}$$

Because then

$$\int_A^B \underset{\sim}{F}.\underset{\sim}{dr} = \int_A^B dW = \Big[W\Big]_A^B$$

$$= W_B - W_A$$

only depends on end points, and *not path*

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But what properties of the vector field allows us to write

$$dW = \underset{\sim}{F} \cdot \underset{\sim}{dr} \quad ?$$

i.e. to write

$$F_x\,dx + F_y\,dy + F_z\,dz = dW$$

(an exact differential).

For a function $W(x,y,z)$, the exact differential dW is defined as

$$dW = \left(\frac{\partial W}{\partial x}\right)dx + \left(\frac{\partial W}{\partial y}\right)dy + \left(\frac{\partial W}{\partial z}\right)dz .$$

From …

$$F_x\,dx + F_y\,dy + F_z\,dz = dW$$

$$dW = \left(\frac{\partial W}{\partial x}\right)dx + \left(\frac{\partial W}{\partial y}\right)dy + \left(\frac{\partial W}{\partial z}\right)dz$$

So the property of the field that makes it conservative is that we have a function $W(x,y,z)$, the "scalar potential" such that

$$\left\{ F_x = \frac{\partial W}{\partial x}, \quad F_y = \frac{\partial W}{\partial y}, \quad F_z = \frac{\partial W}{\partial z} \right\}.$$

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Conditions for field F to be conservative ?

$$F_x = \frac{\partial W}{\partial x}, \quad F_y = \frac{\partial W}{\partial y}, \quad F_z = \frac{\partial W}{\partial z}$$

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But that's not much use if we don't know what

$W(x, y, z)$ is !

In terms of only $\underset{\sim}{F}(x, y, z)$ we can use the

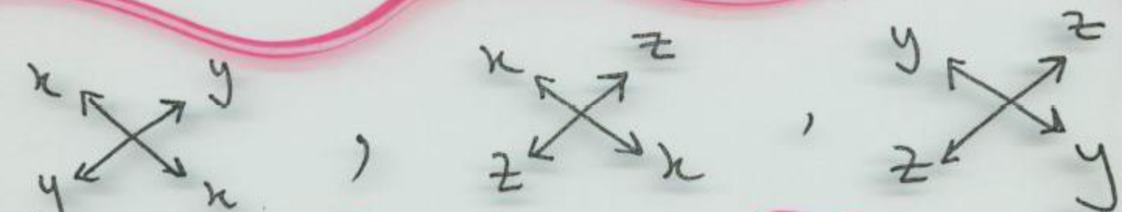
fact that $\boxed{\dfrac{\partial}{\partial y}\left(\dfrac{\partial W}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial y}\right), \text{ etc},}$

to re-express the conditions for the field to be e.g. compare results from $\frac{\partial}{\partial y}$ of F_x condition with $\frac{\partial}{\partial x}$ of F_y

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \quad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}, \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}.$$

The test for ***F*** components labelled **(iii)** earlier.

Notice that pattern here...



This makes the test conditions easier to remember.

So, given a field $\underset{\sim}{F}(x, y, z)$, we can test whether it is conservative by seeing if the above relationships hold.

We then tested a general (radial) inverse square law field

$$\underset{\sim}{V}(\underset{\sim}{r}) = \frac{\eta \hat{\underset{\sim}{r}}}{r^2} = \frac{\eta \underset{\sim}{r}}{r^3}.$$

$$\left(\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} \right)$$

Note We are given $\underset{\sim}{V}(\underset{\sim}{r})$ and not $\underset{\sim}{V}(x,y,z)$ here

so how do we work out $\dfrac{\partial V_x}{\partial y}$, for example?

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$r^2 = x^2 + y^2 + z^2$

Differentiate both sides with respect to y (partially)

$$\rightarrow \quad 2r \frac{\partial r}{\partial y} = 0 + 2y + 0$$

i.e. $2r \dfrac{\partial r}{\partial y} = 2y$ and $\boxed{\dfrac{\partial r}{\partial y} = \dfrac{y}{r}}$.

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Then we can use the ...

Chain Rule ...

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y}$$

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$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Then,

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\eta x}{r^3}\right)$$

$$= \eta x \frac{\partial}{\partial y}\left(\frac{1}{r^3}\right)$$

$$= \eta x \cdot \left(\frac{-3}{r^4}\right)\left(\frac{\partial r}{\partial y}\right)$$

$$= \frac{-3\eta x}{r^4}\left(\frac{y}{r}\right) \quad \text{, from above,}$$

$$= -\frac{3\eta x y}{r^5} \ .$$

CROSS PRODUCT OF VECTORS

The cross product of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is written as

$$\underset{\sim}{a} \times \underset{\sim}{b}$$

where the <u>MAGNITUDE</u> of the cross product is given by

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta ,$$

where θ is the angle between the vectors.

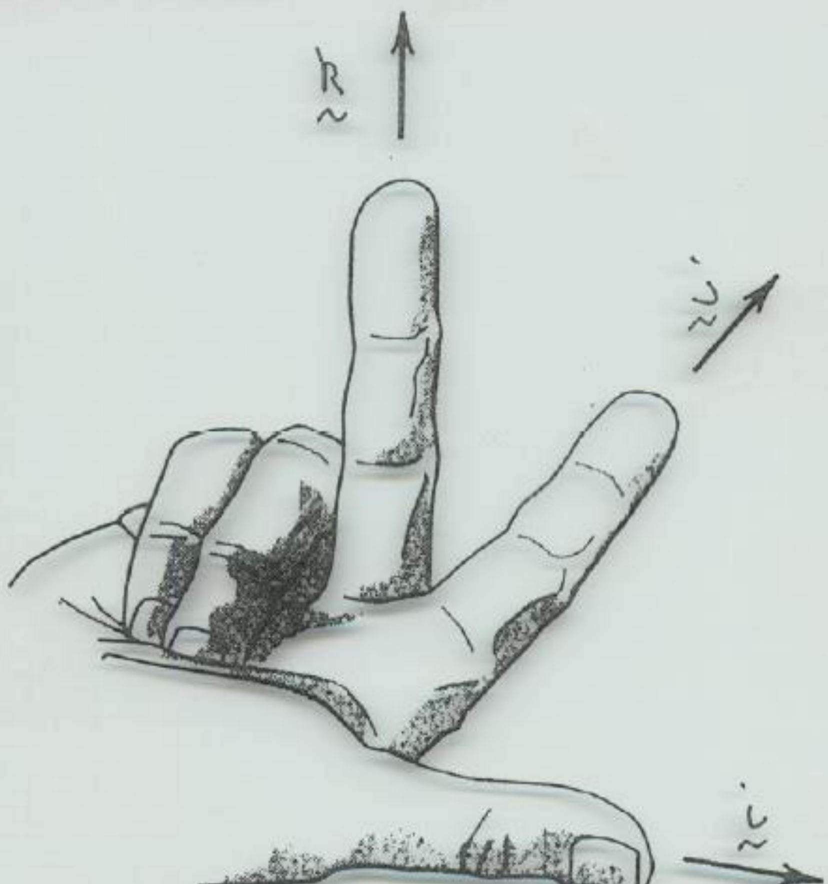
But the cross product is also known as the VECTOR PRODUCT, i.e. $\underset{\sim}{a} \times \underset{\sim}{b}$ is a vector,

and we also $\boxed{\text{define a direction}}$ . . .

$|\hat{i} \times \hat{j}| = 1$, i.e. it's a unit vector, but what direction does this unit vector point in?

We **define** a RIGHT-HANDED SYSTEM such that $\hat{i} \times \underset{\sim}{j} = \underset{\sim}{k}$ (which *is* a unit vector)

THE "RIGHT HAND RULE"



That direction definition leads to the following cumbersome expression for the component form:

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_2 b_3 - a_3 b_2)\,\underset{\sim}{i} + (a_3 b_1 - a_1 b_3)\,\underset{\sim}{j} + (a_1 b_2 - a_2 b_1)\,\underset{\sim}{k}.$$

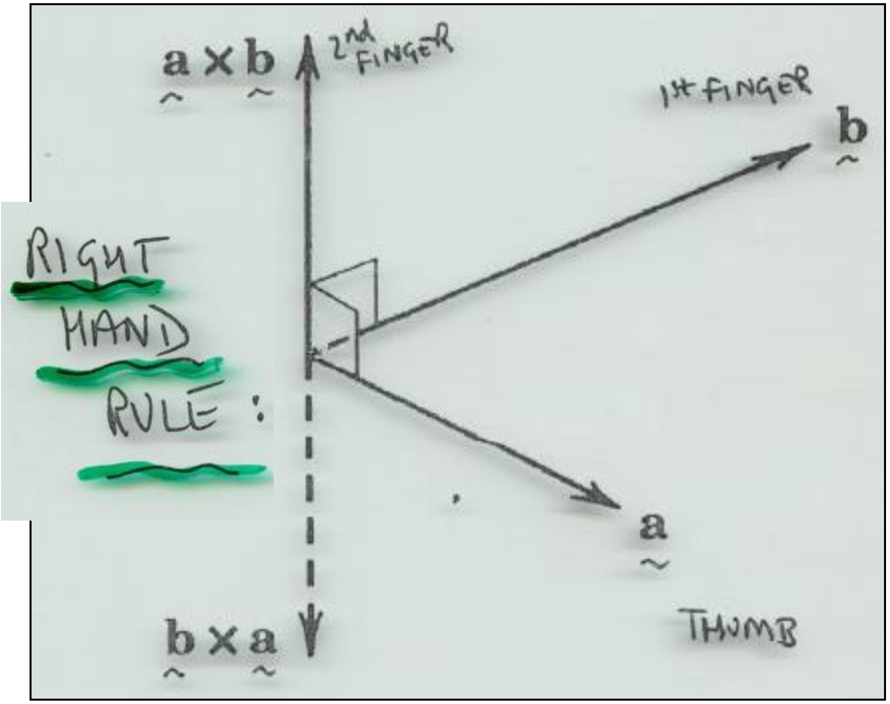
which appears simpler in terms of a matrix determinant definition (more of that later in this course):

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

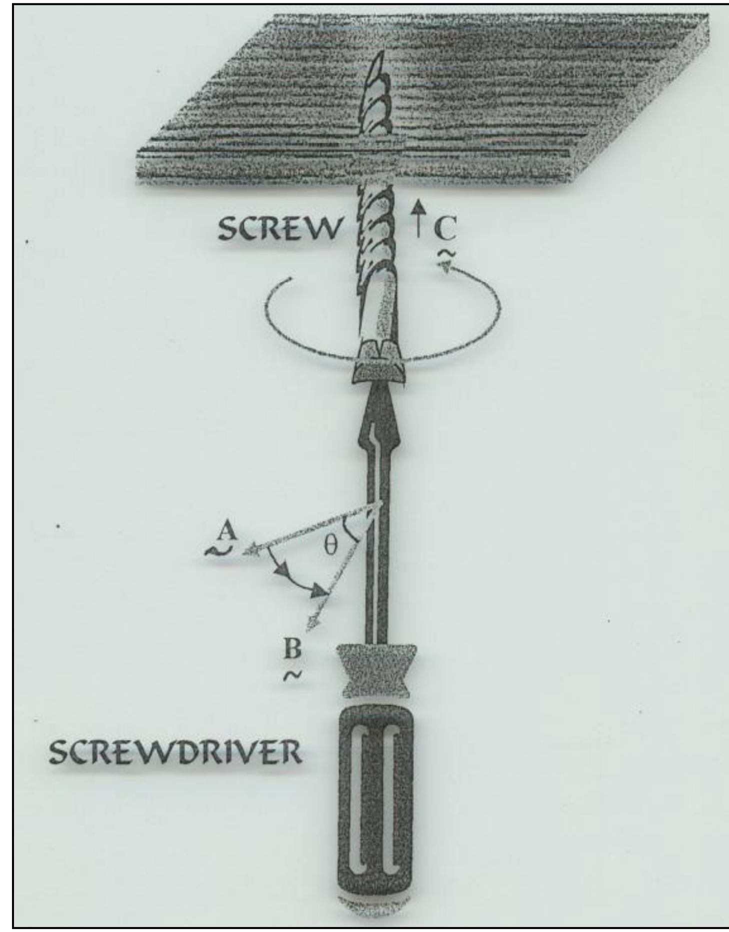
Then ...

$$\boxed{\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \underset{\sim}{a} = (a_1, a_2, a_3) \text{ and } \underset{\sim}{b} = (b_1, b_2, b_3).}$$

Visualising the direction of the vector product ...



$$\underset{\sim}{b} \times \underset{\sim}{a} = -\left(\underset{\sim}{a} \times \underset{\sim}{b}\right)$$



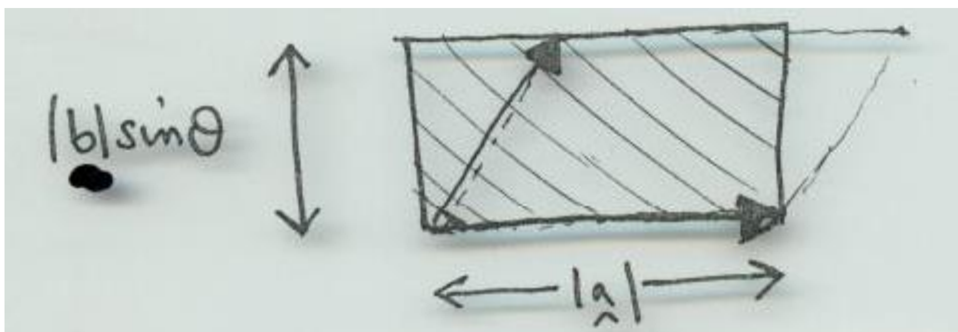
$\underset{\sim}{A}$ ROTATES TOWARDS $\underset{\sim}{B}$ IN A CLOCKWISE SENSE WHEN WE LOOK ALONG $\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B}$.

Visualising the <u>magnitude</u> of the vector product ...

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$$

In the plane of **a** and **b** :

Equivalent area :



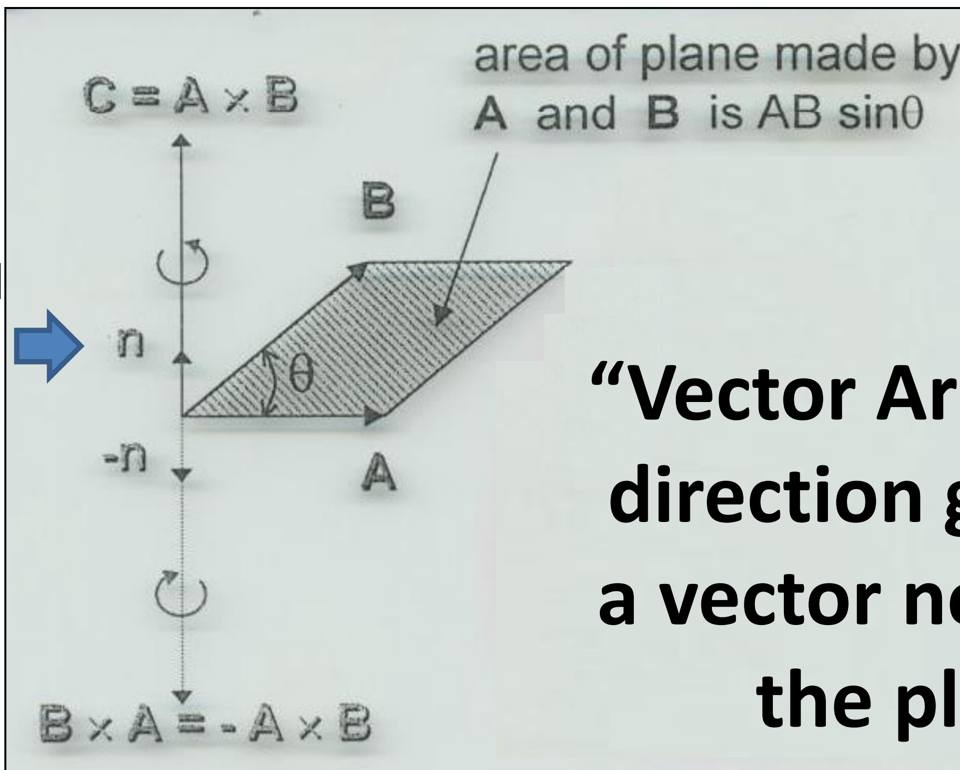
| **a** x **b** | is the area of the parallelogram with “sides” **a** and **b**

Combining direction and magnitude information: <u>VECTOR AREA</u>

In vector calculus, *area is often treated as a vector*:

magnitude = size of planar area
direction = perpendicular to the plane

normal vector, ***n***



"Vector Area" with direction given by a vector normal to the plane

Element of area:

$$\boldsymbol{dS} = dS\,\hat{\boldsymbol{n}}$$

unit normal vector

TRIPLE PRODUCTS

There are 3 meaningful (i.e. consistent) ways that one can form the product of three vectors...

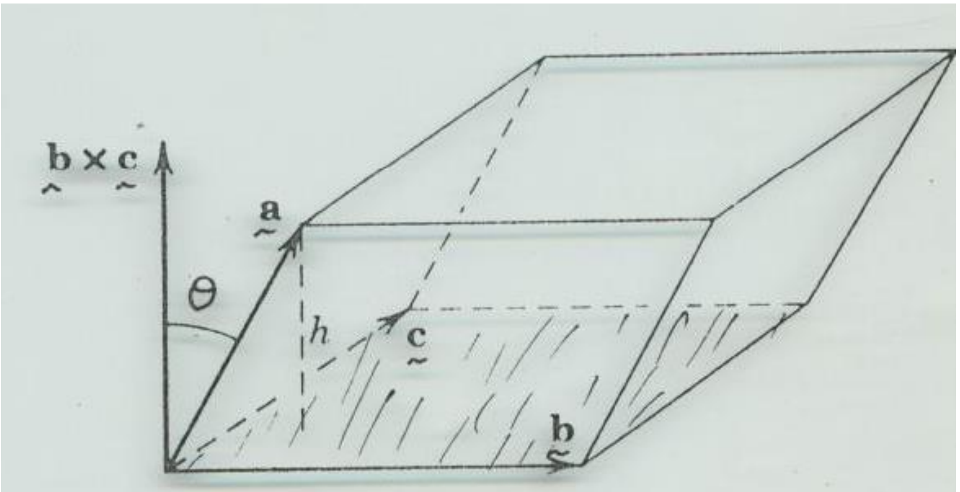
... $(\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$, $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$, $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$

1. $\boxed{(\underset{\sim}{a} \cdot \underset{\sim}{b})\,\underset{\sim}{c} \neq \underset{\sim}{a}\,(\underset{\sim}{b} \cdot \underset{\sim}{c})}$ gives a **vector**

2. $\boxed{\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})}$ "**<u>SCALAR</u> TRIPLE PRODUCT**"

3. $\boxed{\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) \neq (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}}$ "**<u>VECTOR</u> TRIPLE PRODUCT**"

2. $\boxed{\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})}$ "<u>SCALAR</u> **TRIPLE PRODUCT**"



$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) =$

$\pm$ volume of the parallelipiped

with edges $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$

Also ...

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where ...

$\underset{\sim}{a} = (a_1, a_2, a_3)$

$\underset{\sim}{b} = (b_1, b_2, b_3)$

$\underset{\sim}{c} = (c_1, c_2, c_3)$

3. $\boxed{\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) \neq (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}}$ "**<u>VECTOR</u> TRIPLE PRODUCT**"

In fact, there are two useful identities for the vector triple product ---

$$\text{(i)} \quad \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\,\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\,\underset{\sim}{c}$$

$$\text{(ii)} \quad (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c} = (\underset{\sim}{a} \cdot \underset{\sim}{c})\,\underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{c})\,\underset{\sim}{a}$$

--- which demonstrate that

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) \neq (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}$$