SUMMARY / OVERVIEW OF ...

HANDOUT 3

VECTOR CALCULUS (continued)

- Evaluating triple products
- Applications of the cross-product
- Differentiation of vectors
  - time variation
  - space variation
  - partial differentiation

and ...

- Grad
  - definition
  - direction derivatives
  - unit normal vectors
- Flux and solid angle

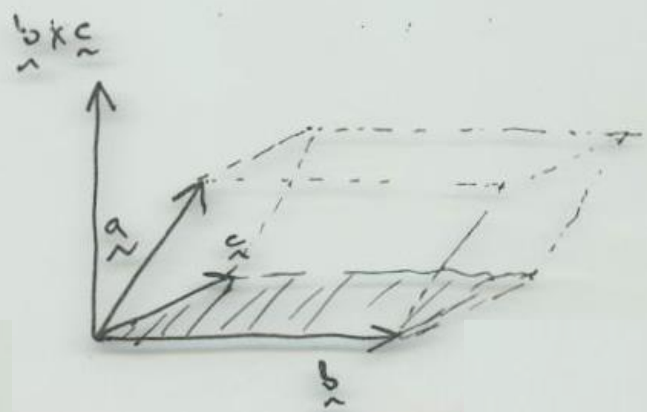
H3
p63
bot

SCALAR TRIPLE PRODUCT

$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) \equiv \pm$ volume of parallelipiped with edges $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$

?

When does the volume of the parallelipiped equal zero?



$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = 0$

$\rightarrow$ when $\underset{\sim}{a}$ lies in the plane of $\underset{\sim}{b}$ and $\underset{\sim}{c}$

i.e. $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are "COPLANAR"

REVIEW OF FUNDAMENTAL CONCEPTS (part three) ...

Applications of the cross-product p66-p68

1. Set of scales (moments and torques): $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$
2. Rotation of a rigid body: $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$

Time-varying Vectors

$$\underset{\sim}{v}(t) = \left(v_1(t) \,,\; v_2(t) \,,\; v_3(t) \right)$$

$$\frac{d}{dt}\underset{\sim}{v} = \frac{dv_1}{dt}\underset{\sim}{i} + \frac{dv_2}{dt}\underset{\sim}{j} + \frac{dv_3}{dt}\underset{\sim}{k}$$ p70

[tangential to the curve swept out by ***v***(t)]

Spatial Derivative of **v**(x) = [v_1 (x) , v_2 (x) , v_3 (x)]

$$\frac{d\underset{\sim}{v}}{dx} = \frac{dv_1}{dx}\underset{\sim}{i} + \frac{dv_2}{dx}\underset{\sim}{j} + \frac{dv_3}{dx}\underset{\sim}{k}$$ **p72**

Derivatives of Products (product rules) ...

In fact, if we let $u = t$ (time) or $u = x$ (space), then it straightforward to show that:

$$\frac{d}{du}\left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right) = \underset{\sim}{a} \cdot \frac{d\underset{\sim}{b}}{du} + \frac{d\underset{\sim}{a}}{du} \cdot \underset{\sim}{b}$$

$$\frac{d}{du}\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) = \underset{\sim}{a} \times \frac{d\underset{\sim}{b}}{du} + \frac{d\underset{\sim}{a}}{du} \times \underset{\sim}{b}$$

Examples from Dynamics: circular motion (***r***,***v***) and (***v***,***a***) p73,74

Partial Derivatives of Vectors

2D vector space $\underset{\sim}{V}(x,y) = \left(v_1(x,y) \; , \; v_2(x,y) \right)$

For example, $\dfrac{\partial}{\partial x}\underset{\sim}{V} = \left(\dfrac{\partial V_1}{\partial x} \; , \; \dfrac{\partial V_2}{\partial x} \right)$

$$= \frac{\partial V_1}{\partial x}\underset{\sim}{i} + \frac{\partial V_2}{\partial x}\underset{\sim}{j}$$

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3D Vector Field ***V***(x,y,z) = v_1 ***i*** + v_2 ***j*** + v_3 ***k*** (where components are are functions of space)

For example, $\dfrac{\partial \underset{\sim}{V}}{\partial y} = \dfrac{\partial V_1}{\partial y}\underset{\sim}{i} + \dfrac{\partial V_2}{\partial y}\underset{\sim}{j} + \dfrac{\partial V_3}{\partial y}\underset{\sim}{k}$

p76

We also have **Product Rules** for Partial Differentiation

$$\frac{\partial}{\partial x}(\underset{\sim}{a} \cdot \underset{\sim}{b}) = \underset{\sim}{a} \cdot \frac{\partial \underset{\sim}{b}}{\partial x} + \frac{\partial \underset{\sim}{a}}{\partial x} \cdot \underset{\sim}{b}$$

$$\frac{\partial}{\partial x}(\underset{\sim}{a} \times \underset{\sim}{b}) = \underset{\sim}{a} \times \frac{\partial}{\partial x}\underset{\sim}{b} + \frac{\partial \underset{\sim}{a}}{\partial x} \times \underset{\sim}{b}$$

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GRAD (THE VECTOR GRADIENT OF A SCALAR FUNCTION)

$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x}\underset{\sim}{i} + \frac{\partial \phi}{\partial y}\underset{\sim}{j} + \frac{\partial \phi}{\partial z}\underset{\sim}{k}$$

Note: **different** partial derivatives here !

where $\phi(x,y,z)$ defines a scalar field

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$$

$$\text{grad}\,\phi = \left\{\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right\}\phi$$

grad itself is a **VECTOR DIFFERENTIAL OPERATOR**

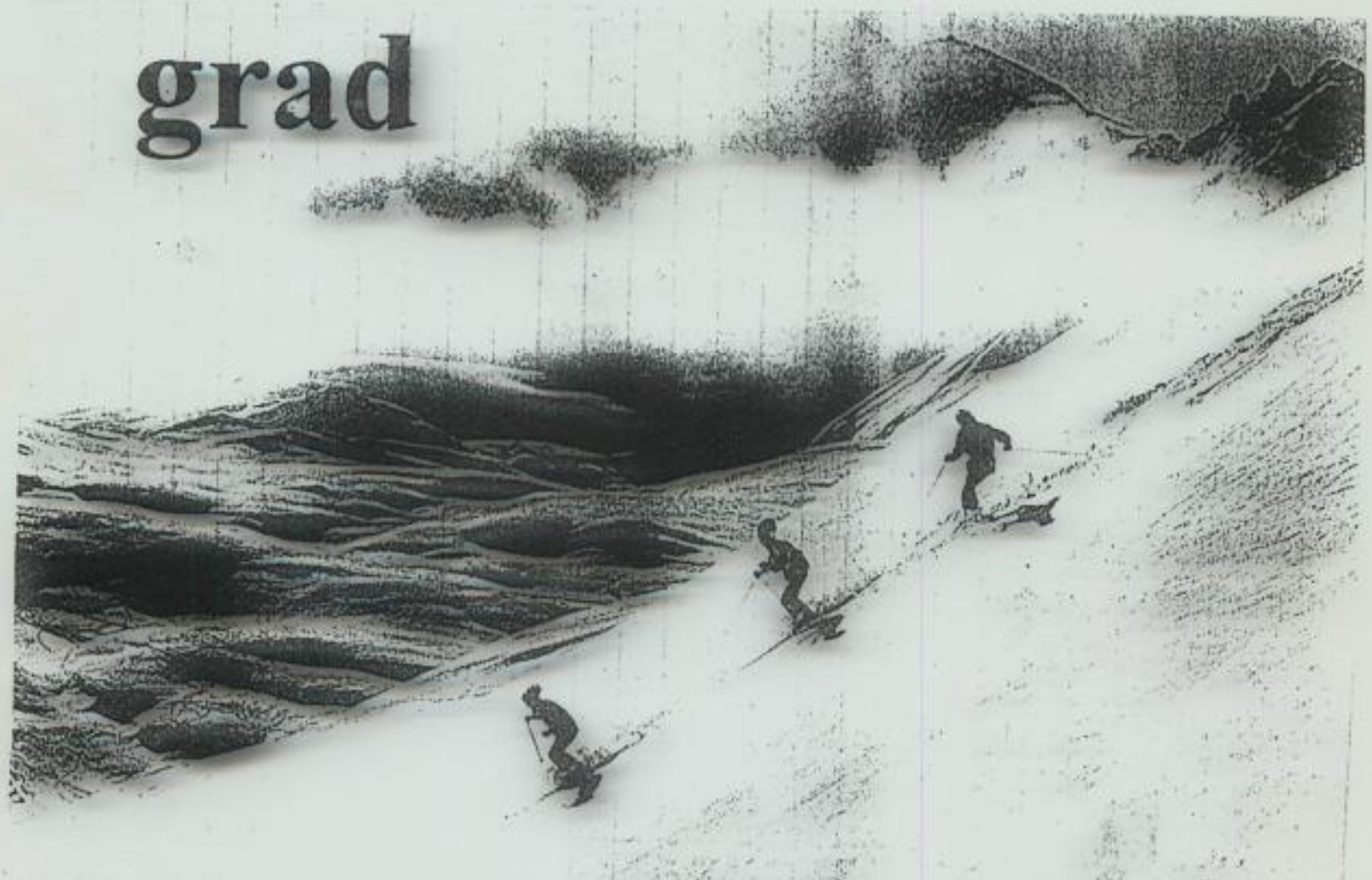
Other notation (called "nabla" or "del" = $\boldsymbol{\nabla}$ = **grad**)

i.e. $\text{grad}\,\phi = \underset{\sim}{\nabla}\phi$

where $\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$

grad operates on a scalar field to give a vector field

grad

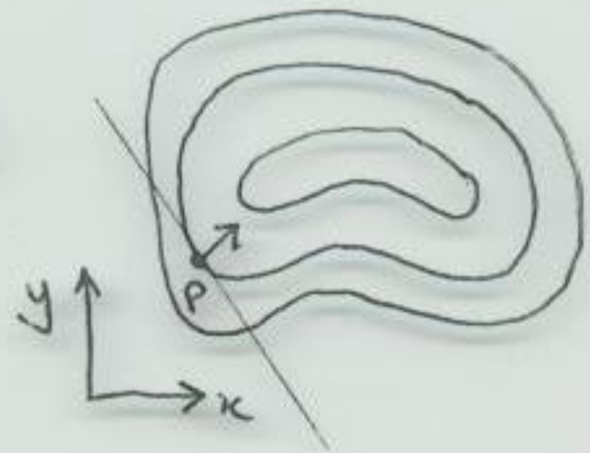


$$\boxed{\text{grad}\,\phi \equiv \underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial \phi}{\partial x} + \underset{\sim}{j}\frac{\partial \phi}{\partial y} + \underset{\sim}{k}\frac{\partial \phi}{\partial z}}$$

→ magnitude <u>and</u> direction of maximum rate of change of ϕ with respect to space

Scalar field ϕ . e.g. height $h(x,y)$, temperature $T(x,y,z)$,
gravitational/electric potential $V(x,y,z)$

$\boxed{2D}$ $\phi(x,y)$



"HILL"

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- contours = lines of constant ϕ
- gradient = max. rate of change of ϕ, $\perp^{ar}$ contour (points "uphill")

grad ϕ is VECTOR SUM OF GRADIENT COMPONENTS (along *x*, *y* and *z*) p81 Example. Ex If scalar field $\phi(x,y,z) = x^2yz^3 + xy^2z^2$ then determine the (vector) gradient, i.e. grad ϕ, at the point $P(1,3,2)$.

Answer. **1.** Calculate: $\boxed{\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}}$

which gives a vector field (**all** the gradient vectors across *x*, *y* and *z*)

$$\therefore \underset{\sim}{\nabla}\phi = (2xyz^3 + y^2z^2)\underset{\sim}{i} + (x^2z^3 + 2xyz^2)\underset{\sim}{j} + (3x^2yz^2 + 2xy^2z)\underset{\sim}{k}$$

2. Substitute for (x,y,z) to get mag. & dir. of greatest rate of spatial change of ϕ at point P.

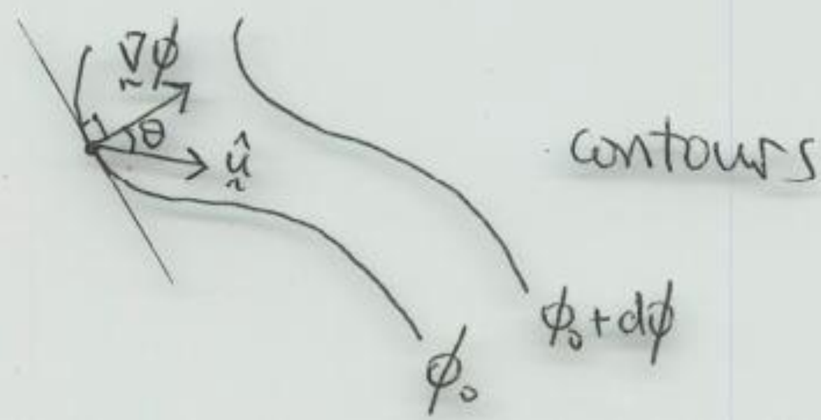
At point $P(1,3,2)$, $x=1$, $y=3$, $z=2$,

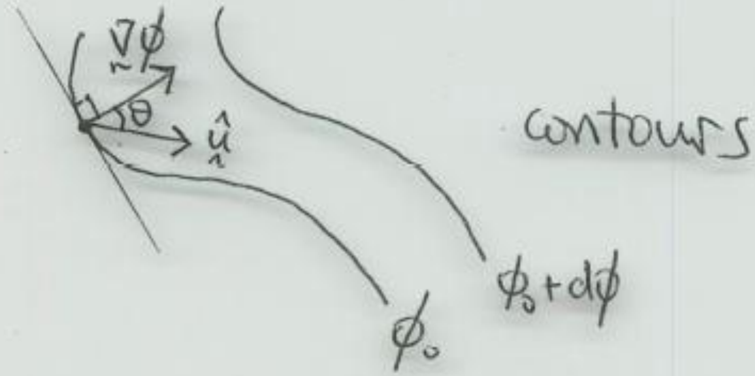
giving $\underset{\sim}{\nabla}\phi = 84\underset{\sim}{i} + 32\underset{\sim}{j} + 72\underset{\sim}{k}$.

gradϕ points in the direction of greatest (positive) change ϕ at any particular point.

How does ϕ change in other directions?

Consider a 2D scalar field $\phi(x,y)$ and two contours along which ϕ has constant values ϕ_0 and $\phi_0 + d\phi$ ...



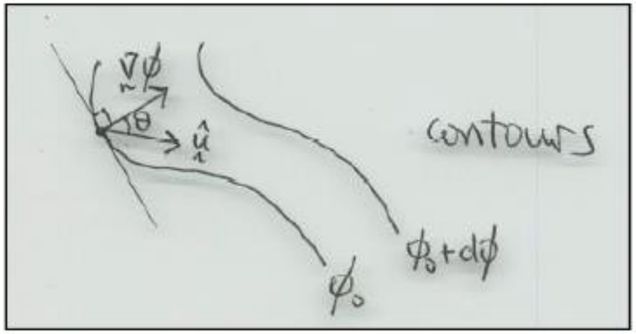


$\hat{u}$ = unit vector in <u>any</u> direction

ds = small distance along $\hat{u}$ (s is a new coordinate: along $\hat{\boldsymbol{u}}$)

$$\boxed{\begin{aligned} \frac{d\phi}{ds} &= \text{rate of change of } \phi \text{ along } \hat{u} = \underline{\text{DIRECTION DERIVATIVE}} \\ &= \nabla\phi \cdot \hat{u} = |\nabla\phi||\hat{u}|\cos\theta \quad \left(\begin{array}{c}\text{component of} \\ \nabla\phi \text{ along } \hat{u}\end{array}\right) \\ \frac{d\phi}{ds} &= |\nabla\phi|\cos\theta, \quad \text{THE PROJECTION OF } \nabla\phi \text{ ALONG } \hat{u}. \end{aligned}}$$

DIRECTION DERIVATIVE



$$\frac{d\phi}{ds} = \text{rate of change of } \phi \text{ along } \hat{u}$$

$$= \nabla\phi \cdot \hat{u} = |\nabla\phi||\hat{u}|\cos\theta$$

$\frac{d\phi}{ds}$ is

- maximum along $\nabla\phi$ (i.e. $\theta = 0$)
- zero along a contour of constant ϕ (i.e. $\theta = 90°$)

Worked and physical examples of ∇ appear in the main text

Physical examples

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Ex A skier goes fastest downhill in the $-\underset{\sim}{\nabla} h$ direction, where $h(x,y)$ = height,

i.e. in direction towards maximum lower gravitational potential

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Ex Electrostatics . . .

V = scalar potential field

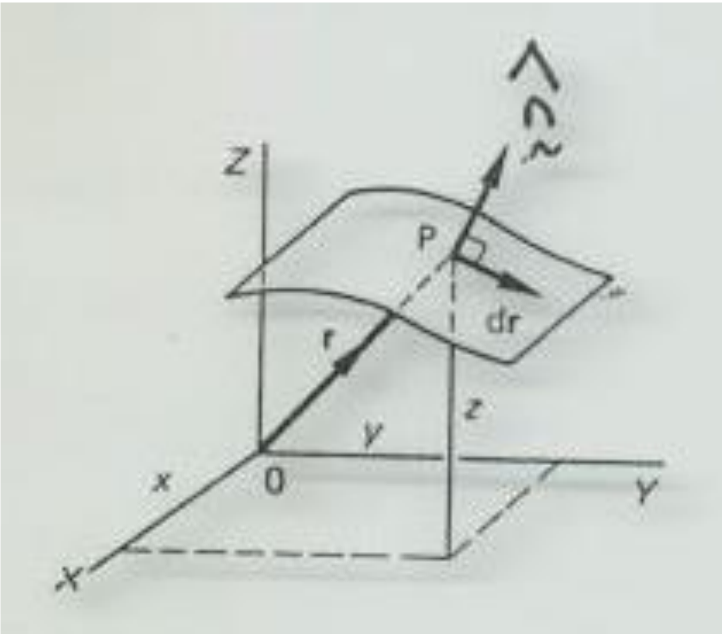
$\underset{\sim}{E}$ = vector electric field

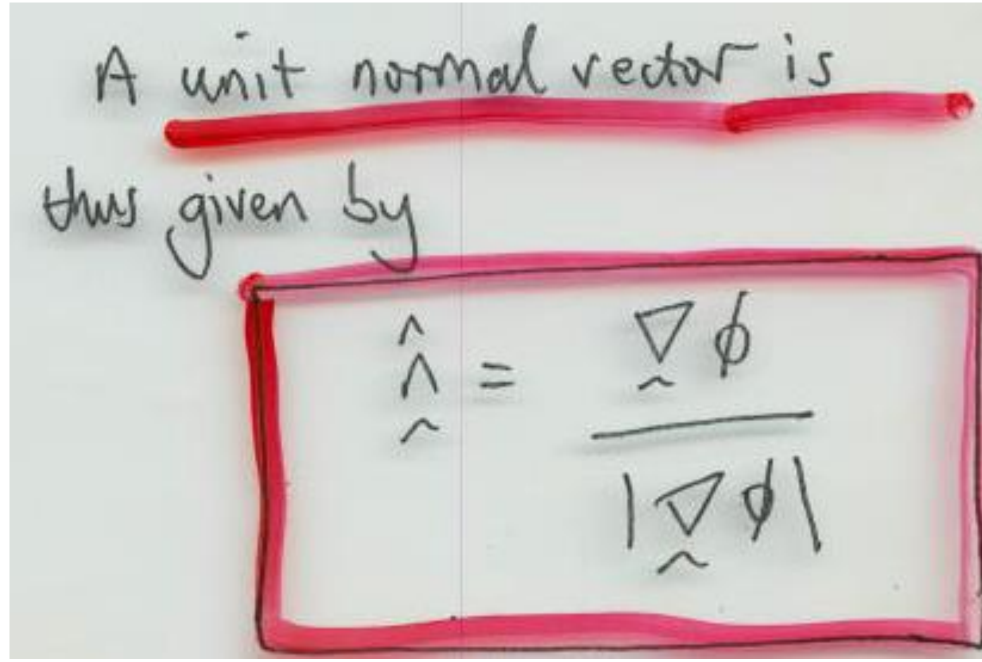
In one dimension x, $E = -\frac{dV}{dx}$

In three dimensions x, y, z, $\underset{\sim}{E} = -\underset{\sim}{\nabla} V$

Force acts to give maximum <u>decrease</u> in electrostatic potential

Geometry. Unit Normal Vectors to a Surface with equation $\emptyset(x, y, z) = \mathbf{const}$.



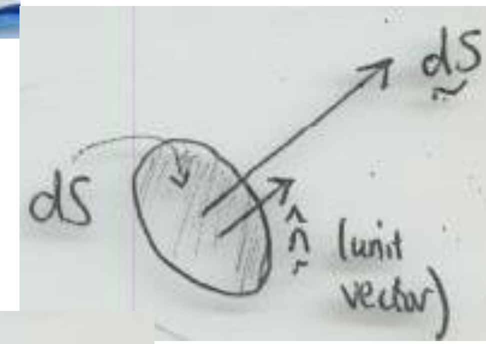


(gives a vector field of all the unit normals)

Consider surface as representing a 2D contour surface.
grad ϕ is perpendicular to every 1D contour on the surface.
And, therefore, is perpendicular to the surface itself.

Flux of a vector over a surface

Surface (vector area) element ***dS*** is $\boxed{\underset{\sim}{dS} = \hat{\underset{\sim}{n}}\, dS}$



Let $\underset{\sim}{V}$ be <u>any</u> vector defined over dS.

If $\underset{\sim}{V}$ constant over dS then

$$\boxed{\text{THE FLUX OF } \underset{\sim}{V} \text{ OVER } dS = V\, dS \cos\theta = \underset{\sim}{V} \cdot \underset{\sim}{dS}}$$

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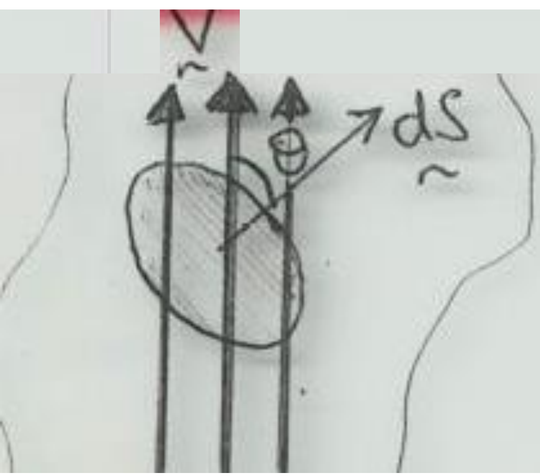


i.e. $\underset{\sim}{V} \parallel^{\text{el}} \underset{\sim}{dS} \rightarrow$ max. flux

$\underset{\sim}{V} \perp^{\text{ar}} \underset{\sim}{dS} \rightarrow$ zero flux

Then ...

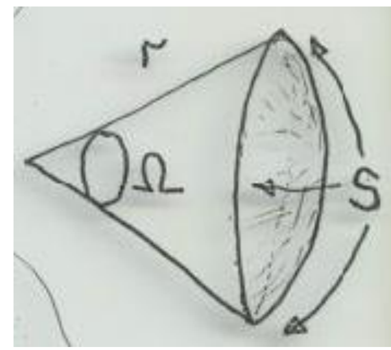
Flux of $\underset{\sim}{V}$ over larger surface S is the sum of the fluxes over all the constituent dS elements of S

i.e. $\boxed{\text{FLUX OF } \underset{\sim}{V} \text{ OVER } S = \int_S \underset{\sim}{V} \cdot d\underset{\sim}{S}}$

Surface integral = double integral

SOLID ANGLE

Solid angle, $\boxed{\Omega = \frac{S}{r^2}}$

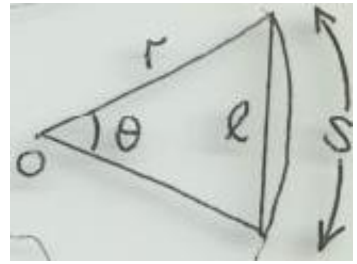


S is surface area subtended at a sphere of radius *r* .

Units of solid angle are **steradians** (it's a generalisation of the 2D angle measure of radians).

Full 2D angle = 2π radians. Full 3D (solid angle) = 4π steradians.

Small angle elements



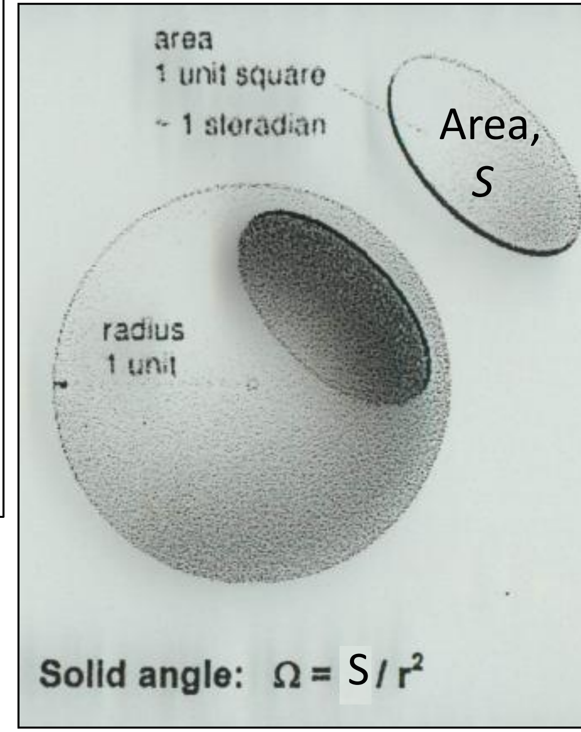
2D angle:

$$\theta = \frac{S}{r}$$

(radians)

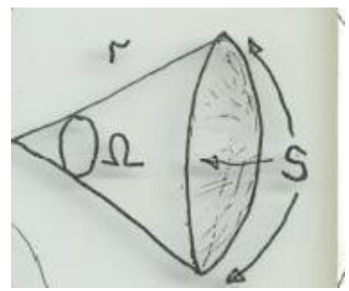
arc length = s
chord length = l
arc element length = ds
chord element length = dl

small angle $\boxed{d\theta = ds/r = dl/r}$



3D (solid) angle:

$$\Omega = \frac{S}{r^2}$$

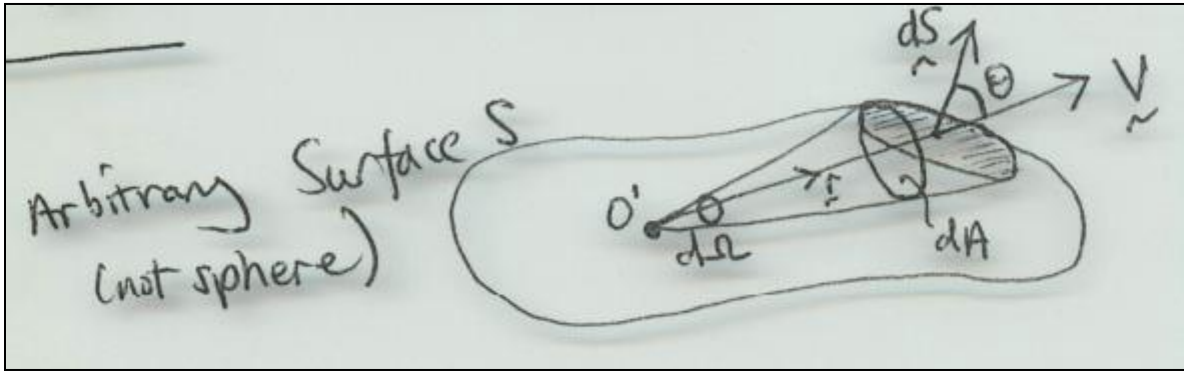


small (solid) angle $$\boxed{d\Omega = \frac{dS}{r^2} = \frac{dA}{r^2}}$$

where dS is area of surface element,
dA is area of flat plane across
the “mouth” of the cone.

Further topics in notes

1. Flux of vector field **V** over an arbitrary closed surface *S* that contains co-ordinate origin O'
(uses solid angle elements)

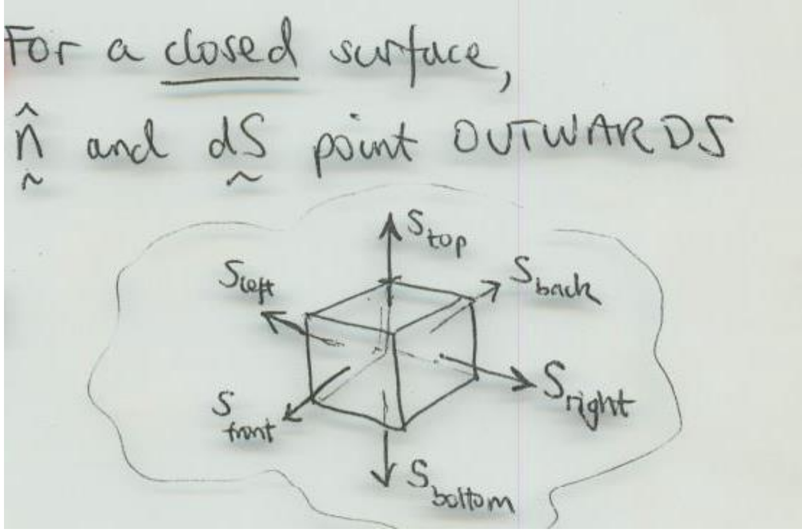


2. Total current I is the flux of the current density *J* :

$$I = \int_S \hat{J} \cdot d\underset{\sim}{S}$$

3. Conventions and Notations

(a)



(b)

For a closed surface: $\int_S \rightarrow \oint_S$

For a closed curve: $\int_C \rightarrow \oint_C$