A SUMMARY / OVERVIEW OF

HANDOUT 4

Handout 4
P92

— VECTOR CALCULUS (continued)

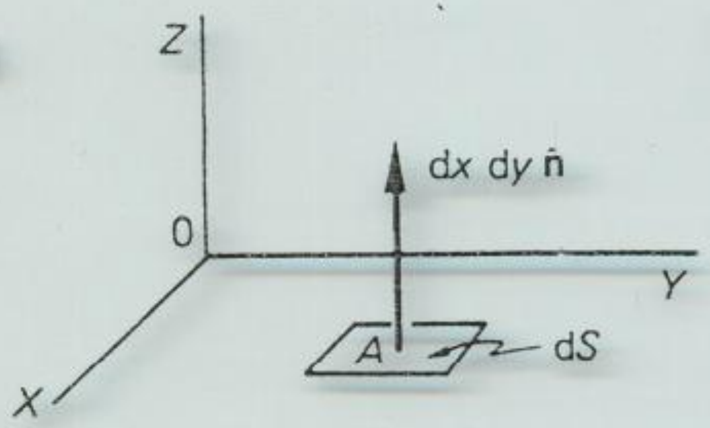
- Flux calculations (surface integrals like $\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$)
- Divergence
- Curl
  - definition
  - physical examples
- Multiple operations
  - grad div, div grad, curl curl
  - curl grad, div curl, div grad

(revisited, Laplacian, physical examples)

- Revision summary (so far)

VECTOR PRODUCT → VECTOR AREA → SURFACE INTEGRALS

H4
p93
bot



If $P(x, y)$ is a point in the xy-plane, the element of area $d\underset{\sim}{S}$ can be written

$$d\underset{\sim}{S} = (\mathbf{i}\, dx) \times (\mathbf{j}\, dy)$$
$$= dx\, dy\, \hat{\mathbf{n}}$$

i.e. a vector of magnitude $dx\, dy$ acting in the direction of $\hat{\mathbf{n}}$ and referred to as the *vector area*.

For a general surface S in space, each element of surface dS has a *vector area* $d\mathbf{S}$ such that $\boxed{d\mathbf{S} = dS\, \hat{\mathbf{n}}}$.

where the surface equation is $\boxed{S(x,y,z) = 0}$
whose unit normals are given by $\hat{\boldsymbol{n}}$ where ...

$$\boxed{\underset{\sim}{\hat{n}} = \frac{\underset{\sim}{\nabla} S}{|\underset{\sim}{\nabla} S|}}$$

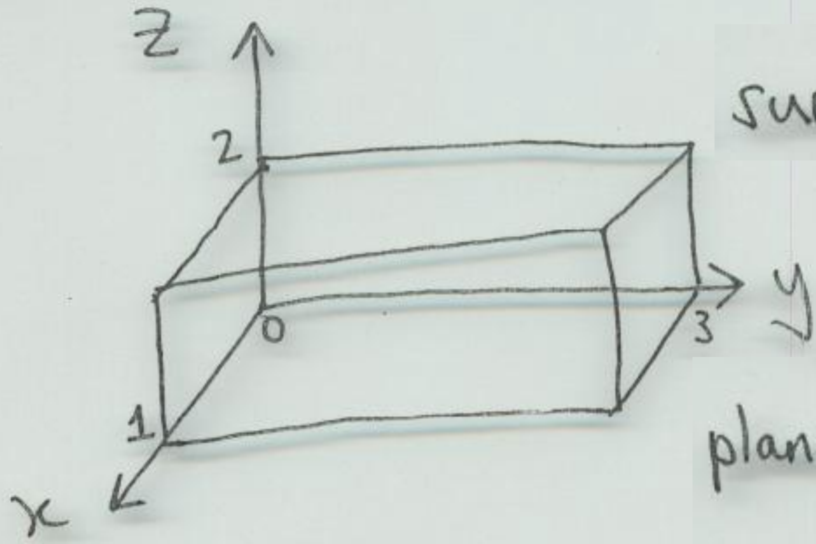
A **flux calculation** (very long example) ...

for a given vector field: $\underset{\sim}{F}(x,y,z) = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$.

What is the total flux of $\underset{\sim}{F}$ over S ?

In other words, what is $\oint_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$?

Ans The surface S is a "box" in x, y, z ...



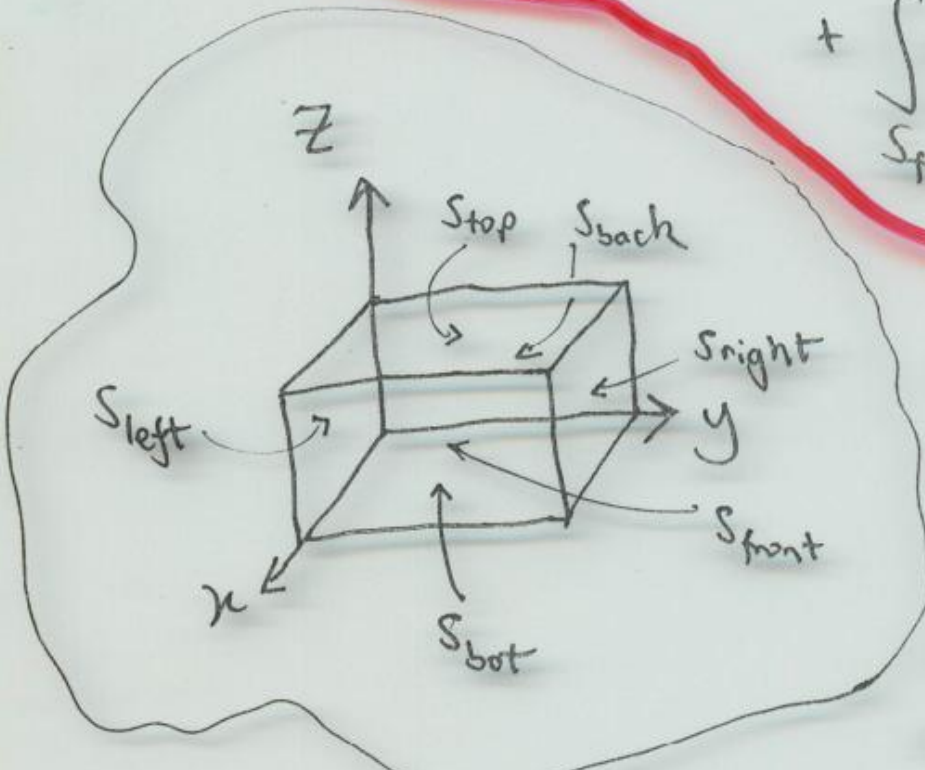
surface S with flat sides

planes $x=0$, $y=0$, $z=0$,
$x=1$, $y=3$, $z=2$

To work out the flux of $\underset{\sim}{F}$ over the whole surface, consider each side of the box in turn.

$$\oint_S \hat{F}.\underset{\sim}{dS} = \int_{S_{bot}} \hat{F}.\hat{dS} + \int_{S_{top}} \hat{F}.\underset{\sim}{dS} + \int_{S_{right}} \hat{F}.\hat{dS} + \int_{S_{left}} \hat{F}.\underset{\sim}{dS}$$

$$+ \int_{S_{front}} \hat{F}.\hat{dS} + \int_{S_{back}} \hat{F}.\underset{\sim}{dS}$$



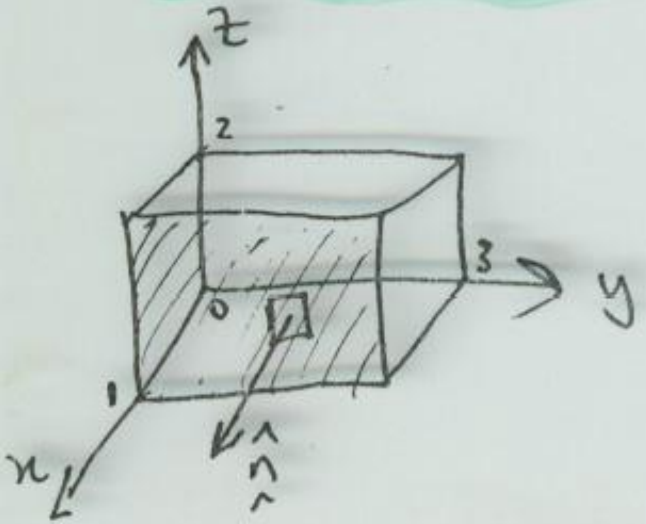
And recall that, for this closed surface, each $\underset{\sim}{dS}$ will point OUTWARDS from the enclosed volume.

For example (considering just one of the six sides) ...

$$\underset{\sim}{F}(x,y,z) = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$$

p95 to p101

(v) Front side of the box, S_{front}



Here, $\boxed{x=1}$ and $\underset{\sim}{F} = \underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$.

$$\boxed{d\underset{\sim}{S}_{front} = \underset{\sim}{i}\,dy\,dz}$$

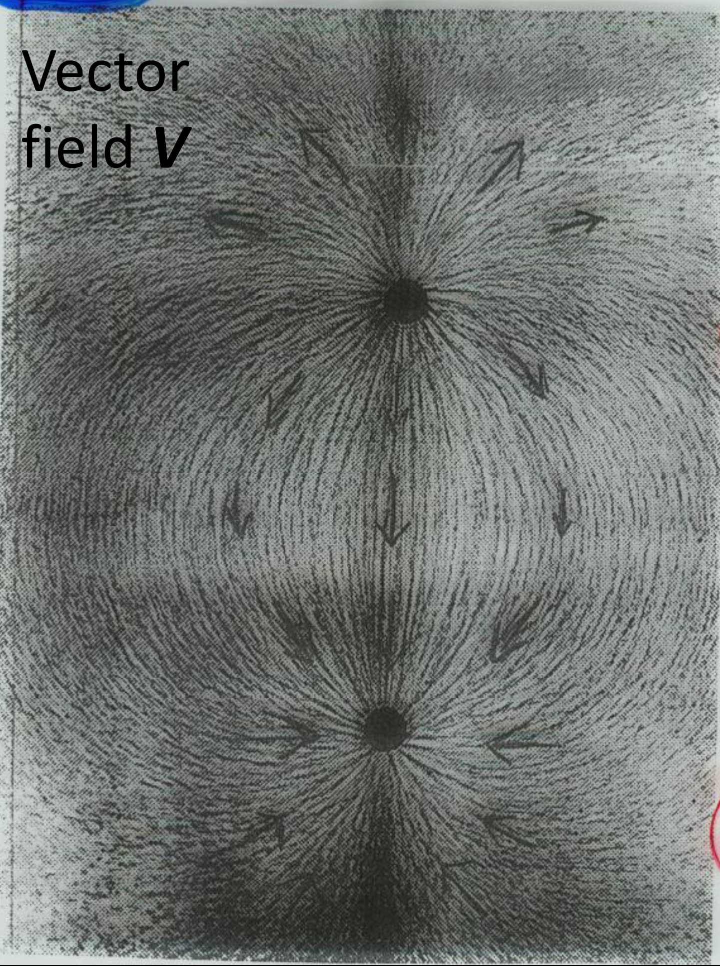
and $$\int_{S_{front}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \boxed{\int_{y=0}^{y=3}\int_{z=0}^{z=2} (1)\,dy\,dz} = 6.$$

DIV (THE DIVERGENCE OF A VECTOR FUNCTION)

p**101** to p**106**

Physical / geometrical picture:

div vol. density of flux sources



div $\boldsymbol{V} > 0$

SOURCE

(OUTFLOW OF FLUX)

div $\boldsymbol{V} < 0$

SINK

(INFLOW OF FLUX)

Mathematically:

$$\boldsymbol{V}(x,y,z) = \boldsymbol{i}\,V_x + \boldsymbol{j}\,V_y + \boldsymbol{k}\,V_z$$

$$\textbf{grad} = \boldsymbol{\nabla} = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z}$$

div $\boldsymbol{V}$ = $\boldsymbol{\nabla}.\boldsymbol{V}$ (*a scalar field*)

i.e. $$\text{div}\,\boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

NET OUTFLOW OF FLUX PER UNIT VOLUME (AT A POINT)

div $\boldsymbol{V} = 0$ for "*solenoidal fields*"

CURL (THE CURL OF A VECTOR FUNCTION)

p**107** to p**114**

Recall that $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and $\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$

curl *V* (also called **rot *V***) of a vector field $\boldsymbol{V}(x,y,z) = \boldsymbol{i}\,V_x + \boldsymbol{j}\,V_y + \boldsymbol{k}\,V_z$ is given by:

$$\text{curl } \underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Mathematical description

$$= \underset{\sim}{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

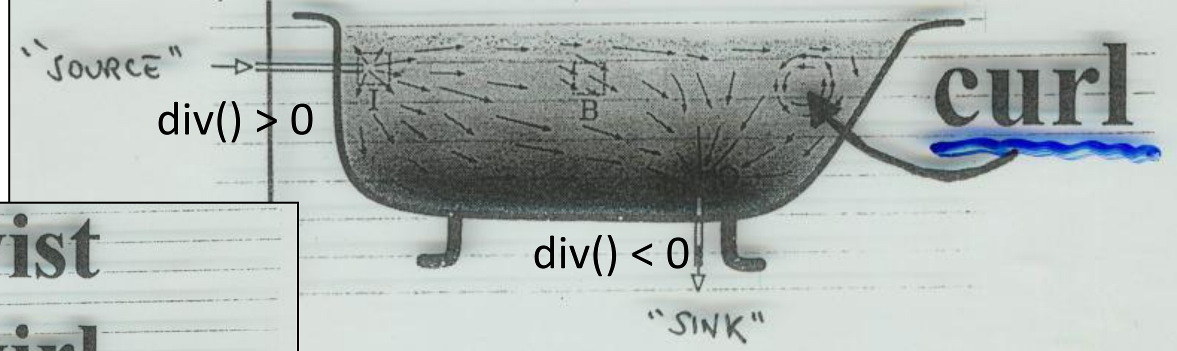
Physical description

curl *V* is a **vector field** describing direction & magnitude of:

circulation,
rotation,
vorticity,
twist

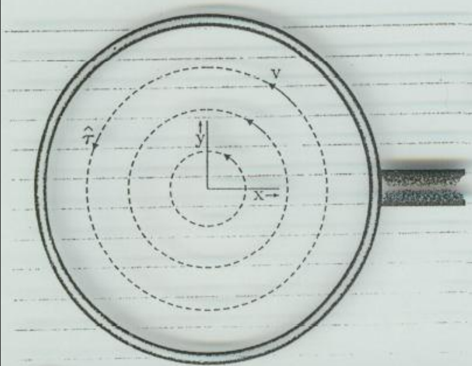
of field $\boldsymbol{V}(x,y,z)$

Physical pictures...



twist
swirl
rotation *(rot)*
circulation
curl

From the small ...



Storm in a teacup

... to the large

Hurricane Mitch

p**108** to p**114**

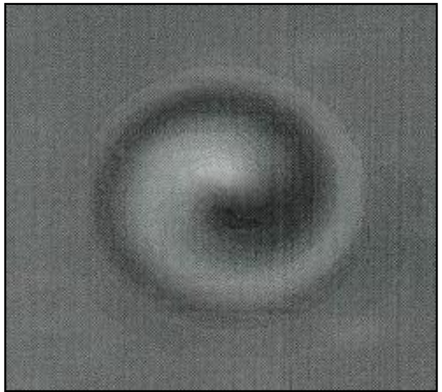
note right-hand rule direction

curl *V*
like angular velocity vector, $\boldsymbol{\omega}$

Velocity field ***V*** of rotating body

see p112 to p114

Vortex (e.g. optical in laser beam X-section)



Examples from fluid dynamics ...

Ex (a)

UNIFORM FLOW OF FLUID

(space independent)

Say, $\underset{\sim}{V} = \rho \underset{\sim}{v}$ where ρ = fluid density

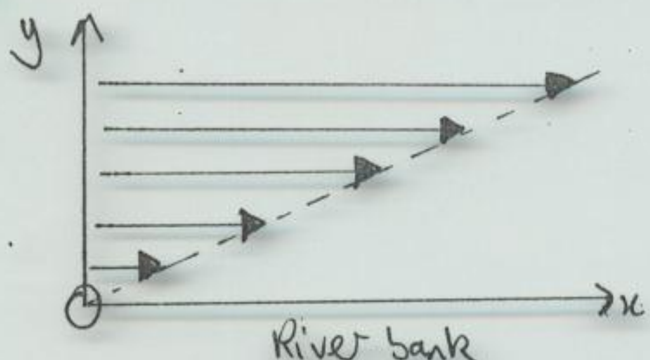
$\underset{\sim}{v}$ = fluid velocity

V is constant (vector) everywhere

curl *V* = *0* (i.e. everywhere)

Field ***V*** is called **"IRROTATIONAL".**

(b) Flow near a river bank (at $y = 0$)



$\underset{\sim}{V} = \alpha y \, \underset{\sim}{i}$, for example

(α = constant)

curl *V* = − α ***k***

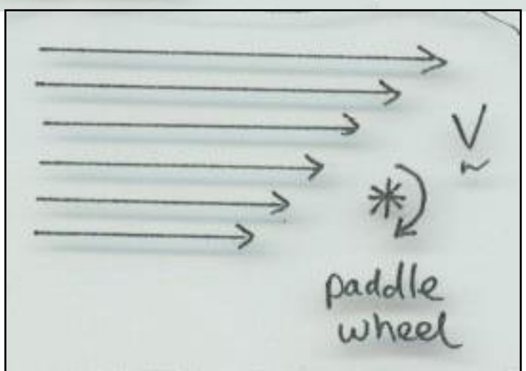
("into the page")

Field ***V*** is called **"ROTATIONAL"**

(has some "circulation").

But, not all non-uniform flows are rotational !

<u>Physical interpretation</u>:



Differential forces on the paddle wheel make it rotate. Using a right-hand-rule, the **curl *V*** vector points into the page; compare with **ω** of a rotating body.

p**115** to p**117**

Multiple Operations

p**117** to p**121**

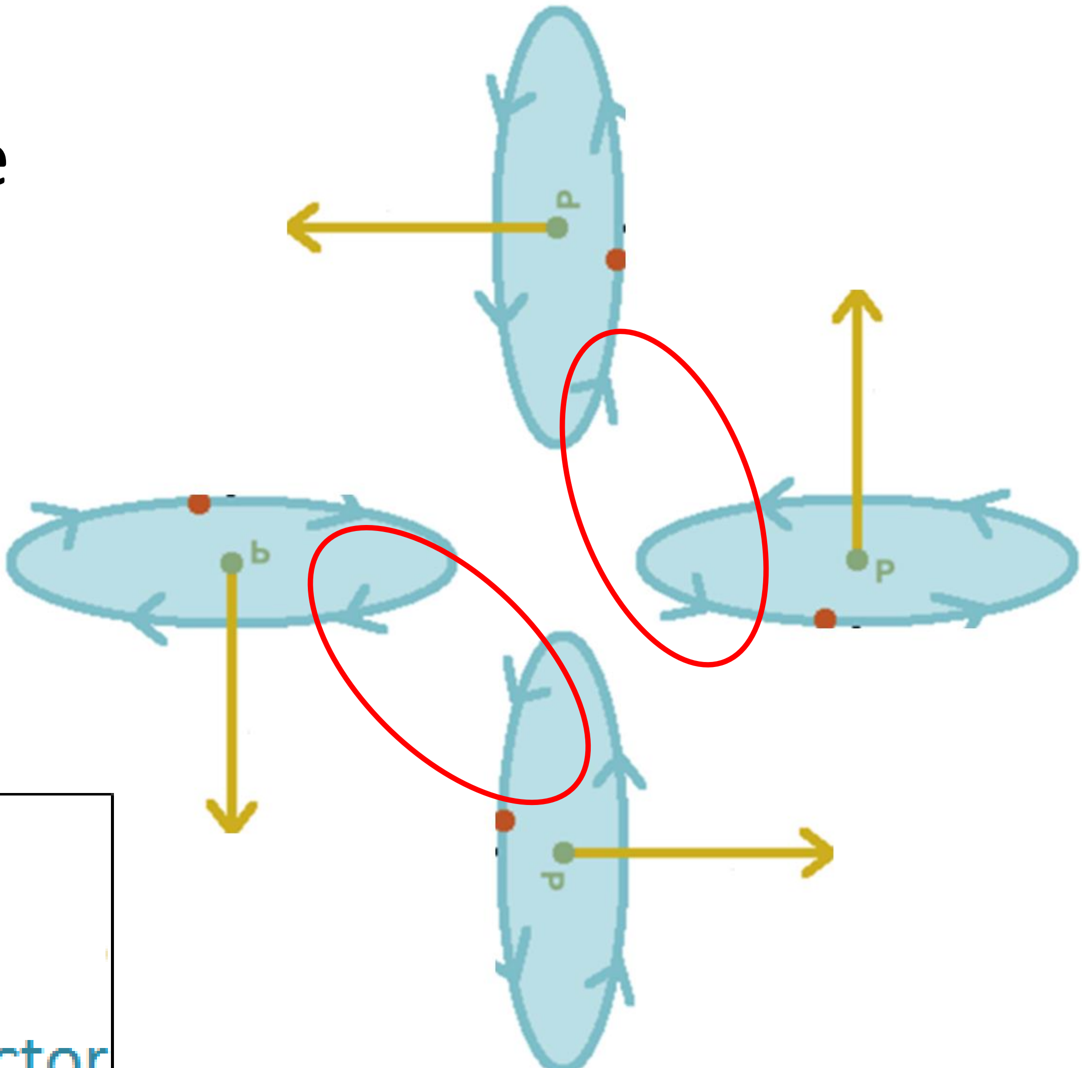
- grad div $\underset{\sim}{A} = \underset{\sim}{\nabla}(\hat{\nabla}.\hat{A})$ — vector field = gradient vectors of the net flux outflow in ***A***
- div grad $\phi = \underset{\sim}{\nabla}.(\hat{\nabla}\phi)$ — scalar field = net flux outflow of the gradient vectors of ϕ
- curl curl $\underset{\sim}{A} = \hat{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{A})$ — vector field = "the circulation of the circulation of ***A***"
- curl grad $\phi = \hat{\nabla}\times(\hat{\nabla}\phi) = \underset{\sim}{0}$

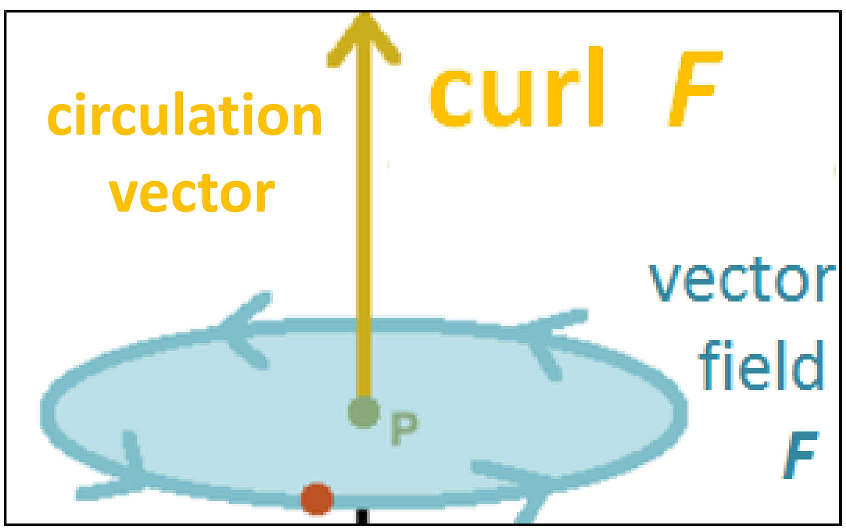
  TRUE FOR ANY SCALAR FIELD ϕ

  Can't have circulation of gradient vectors:
  -> "an impossible staircase"

Circulation of the **circulation vectors** ?

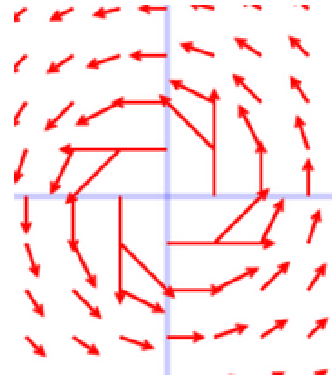
Topologically possible
for a vector field *F* .





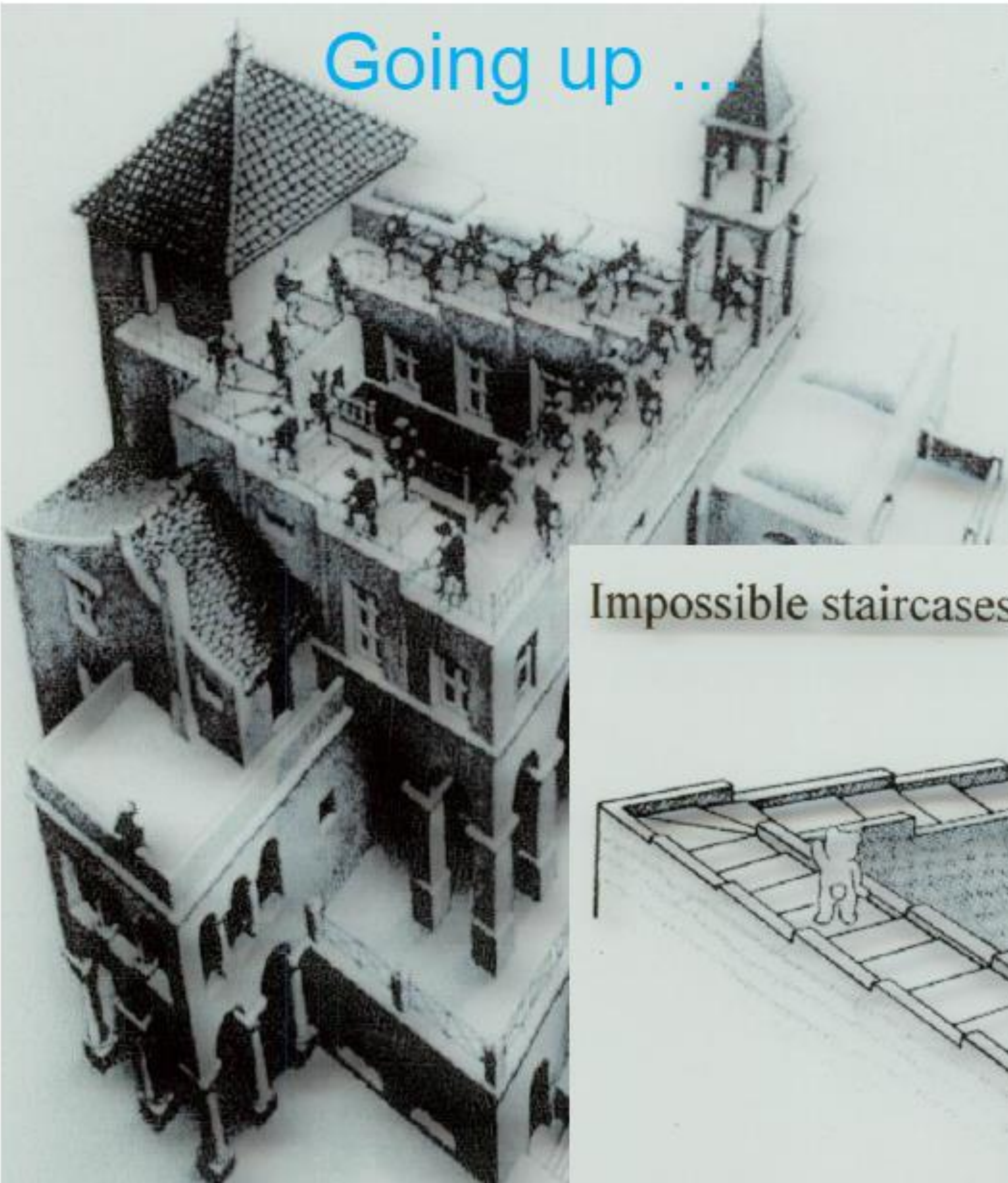
nearby **fields** point in approximately
the same direction

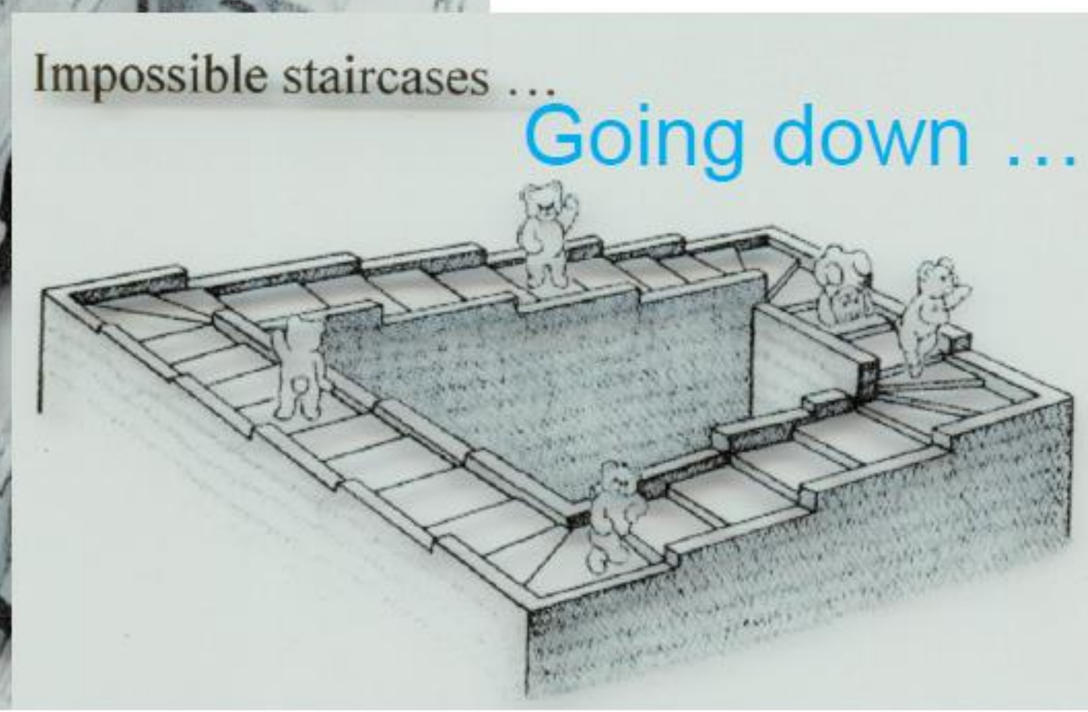
Circulation of gradient vectors ?



gradient vectors ?

curl grad ϕ ?





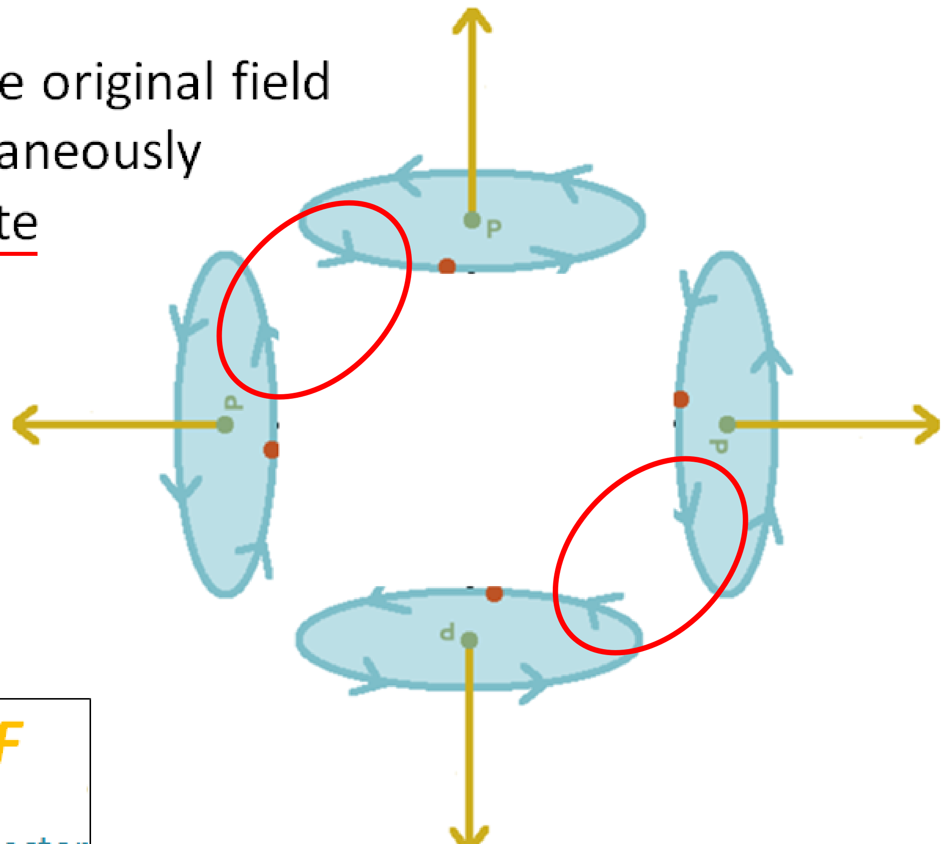
- $\text{div curl } \underset{\sim}{A} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{A}) = 0$.

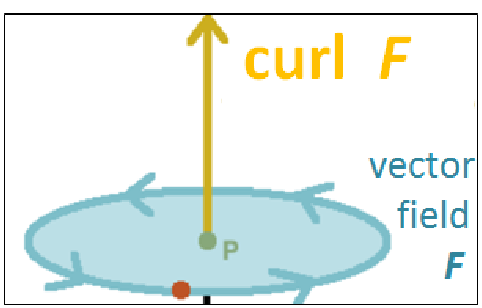
  TRUE FOR ANY VECTOR FIELD $\underset{\sim}{A}$

It is topologically impossible for any vector field ***A*** to have divergence of its circulation vectors.

divergence of the circulation vectors ?

To have this, the original field *F* has to simultaneously point in opposite directions at individual locations





Let's return to the second multiple operation ...

- $\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi)$ scalar field = net flux outflow of the gradient vectors of ϕ

For example, this could be positive at the top of a hill in a 2D contour map. Mathematically ...

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \underset{\sim}{i} + \frac{\partial \phi}{\partial y} \underset{\sim}{j} + \frac{\partial \phi}{\partial z} \underset{\sim}{k}$$

$$\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

So ...

$$\boxed{\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$$

p**123**

$$\text{div grad}\,\phi = \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\phi)$$

This has its own symbol ... ∇^2 and its own name ...

$$\nabla^2 = \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

THE LAPLACIAN

As it's a scalar operator, it can act on either scalar or vector fields:

$\nabla^2\phi$ or $\nabla^2\underset{\sim}{V}$

And, it appears in very many standard model equations (more examples in the notes):

$$\nabla^2\phi = 0$$

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

p**123** to p**127**

Finally, a couple of extra results appearing in the end summary ...

H4
p129
mid

4. *Scalar triple product* $\mathbf{A}.(\mathbf{B}\times\mathbf{C})$

From
Determinant
properties
OR
Parallelepiped
volumes

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \mathbf{B}.(\mathbf{C}\times\mathbf{A}) = \mathbf{C}.(\mathbf{A}\times\mathbf{B})$$

Unchanged by cyclic change of vectors.
Sign reversed by non-cyclic change of vectors.

H4
p130
top

9. *Integration of vectors*

$$\int_a^b \mathbf{A}\,\mathrm{d}u = \mathbf{i}\int_a^b a_x\,\mathrm{d}u + \mathbf{j}\int_a^b a_y\,\mathrm{d}u + \mathbf{k}\int_a^b a_z\,\mathrm{d}u$$