# SUMMARY / OVERVIEW OF ...

## Mathematical Methods and Applications

HANDOUT 5

Handout 5
p131

- VECTOR CALCULUS (continued)

- curlcurl → Laplacian
- Divergence Theorem
  - interpretation
  - electrostatics
  - magnetism
  - hydrodynamics
  - proof
  - worked example
- Summary of integrals

Recall from p123,

$$\text{div grad}\,\phi = \underset{\wedge}{\nabla}\cdot(\underset{\sim}{\nabla\phi}) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

and that we can write this operation on scalar field ϕ using notation for a new operator:

$$\nabla^2 = \underset{\wedge}{\nabla}\cdot\underset{\sim}{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

THE LAPLACIAN

The Laplacian appears in many important model equations such as in Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

and in the wave equation:

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

The manipulation of vector calculus equations can involve forming **curl curl** ( ) and this can be related to using the Laplacian.

$$\text{curl curl}\,\underset{\sim}{A} = \underset{\wedge}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{A})$$

We explore this connection next ...

Referring back to p61, dealing with the vector triple product, the following **vector identities** were used:

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$$

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c} = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a}$$

In the first identity, if we replace **c** with a vector field **V** and both **a** and **b** with ***∇*** we get:

$$\boxed{\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{V}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{V}) - \nabla^2 \underset{\sim}{V}}$$

or, in words:

$$\text{curl curl } \underset{\sim}{V} = \text{grad div } \underset{\sim}{V} - \text{Laplacian } \underset{\sim}{V}$$

This result is proved directly in the main notes.

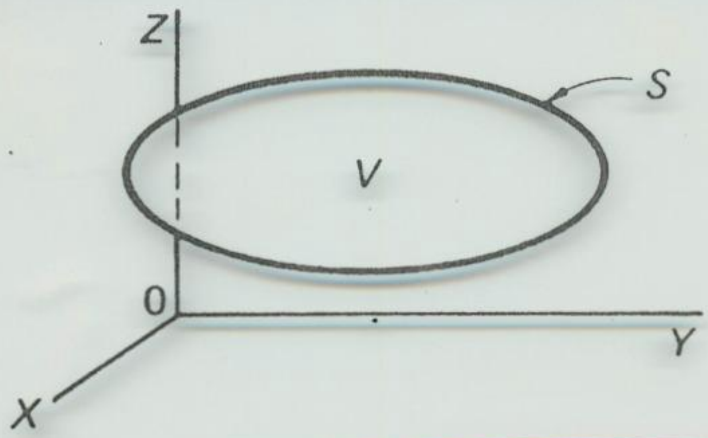
***In cases where the first term on the right-hand-side is zero***, we can make the following useful direct substitution:

$$\boxed{\text{curl curl } \underset{\sim}{V} = - \text{Laplacian } \underset{\sim}{V}}$$

H5 p**134** to p**137**

The **HEADLINE TOPIC** of Handout 5 is below.

**Divergence Theorem (Gauss' theorem)**




For a closed surface $S$, enclosing a region $V$ in a vector field $\mathbf{F}$,

$$\int_V \operatorname{div} \mathbf{F} \, dV = \oint_S \mathbf{F} \cdot d\mathbf{S}$$

H5 p**137** to p**144**

**Key features here are**:

- Relating vol. integral to a closed surface integral
- RHS is total flux of **F** through closed surface *S*
- div **F** is then a volume density of sources and sinks of flux, as RHS can be non-zero
- The theorem can be used to derive Gauss' flux law in differential form from the integral form

Some **overarching themes** of Handouts 5 and 6 are:

- Introducing Maxwell's Equations (*in the simplest notation only*)
- Thereby providing the simplest examples of the roles of div and **curl** in a physical system
- And interpretations/pictures of what div and **curl** can mean.

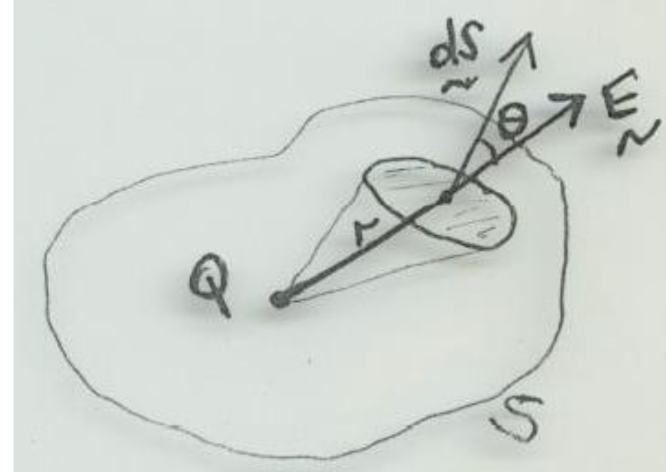
As in other physical laws, Maxwell's equations have **two different general forms**: *differential* (at a point) and *integral* (over region of space)

We want to complete the following "questions"

| Differential Form | Integral Form |
|---|---|
| $\nabla.\boldsymbol{E} = ?$ | ? |
| $\nabla.\boldsymbol{B} = ?$ | ? |
| $\nabla \times \boldsymbol{E} = ?$ | ? |
| $\nabla \times \boldsymbol{B} = ?$ | ? |

H5
p**142**
to
p**144**

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are electric and magnetic fields, respectively.




Q > 0 gives a net outflow of ***E***-flux through *S*

Considering a static charge Q inside a surface *S* and the flux of the electric field ***E*** over *S* ,

we derive **Gauss's flux law in integral form**:

$$\oint_S \underset{\sim}{E} \cdot d\underset{\sim}{S} = \frac{Q}{\varepsilon_0}$$

Using the Divergence Theorem, one can then derive **Gauss's flux law in differential form**:

$$\mathrm{div}\, \underset{\sim}{E} = \frac{\rho}{\varepsilon_0}$$

Introducing charge (volume) density ρ , identifies div ***E*** as the volume density of sources and sinks of ***E***-flux

This allows us to complete the first line of unknowns in our Maxwell's equations table:

| Differential Form | Integral Form |
|---|---|
| $\nabla . \boldsymbol{E} = ?$ | ? |
| $\nabla . \boldsymbol{B} = ?$ | ? |
| $\nabla \times \boldsymbol{E} = ?$ | ? |
| $\nabla \times \boldsymbol{B} = ?$ | ? |

H5 p**142** to p**144**

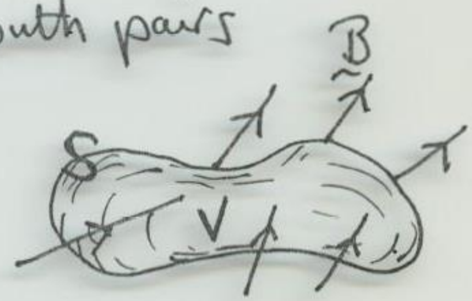
| Differential Form | Integral Form |
|---|---|
| $\nabla . \boldsymbol{E} = ?$ | ? |
| $\nabla . \boldsymbol{B} = ?$ | ? |

$B$ = magnetic induction

$\equiv$ magnetic flux density

- Here, we cannot isolate a magnetic pole inside a surface $S$.

  They always occur as north-south pairs




Therefore, no sources or sinks of magnetic field anywhere !!

i.e. $$\boxed{\oint_S B . dS = 0}$$ Integral Form

and, ***B***-flux into a surface *S* = ***B***-flux out of *S*

But, divergence theorem states $\int_V \text{div}\, B\, dV = \oint_S B . dS$

$\therefore$ $$\boxed{\text{div}\, B = 0}$$ at every point

i.e. it is a "solenoidal" vector field.

Differential Form

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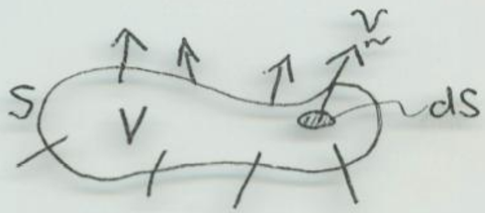
p**148**

Application of the divergence theorem in hydrodynamics...




$\underset{\sim}{v}$ = velocity of fluid particles

$\rho(x,y,z)$ = density of fluid $\left(\frac{\text{mass}}{\text{vol.}}\right)$

$\frac{\partial \rho}{\partial t}$ = rate of increase of density

Considering conservation of mass gives that:

$$\oint_S \rho \underset{\sim}{v} \cdot d\underset{\sim}{S} = -\int_V \frac{\partial \rho}{\partial t}\, dV$$

But divergence theorem tells us that:

$$\int_V \operatorname{div}(\rho \underset{\sim}{v})\, dV = \oint_S \rho\, \underset{\sim}{v} \cdot d\underset{\sim}{S}$$

We then get the **Equation of Continuity** (mass conservation) at each point.

$$\boxed{\operatorname{div}(\rho \underset{\sim}{v}) = -\frac{\partial \rho}{\partial t}}$$

H5 p**149** to p**150**

## INTEGRALS INVOLVING

## SCALAR $\phi(x,y,z)$ AND VECTOR $\underset{\sim}{V}(x,y,z)$ FIELDS

LINE INTEGRALS : $\int_C \hat{V}\cdot d\underset{\sim}{r}$ , $\int_C \underset{\sim}{V}\, dr$ , $\int_C \phi\, d\underset{\sim}{r}$

eg Work Done

(see p130)

SURFACE INTEGRALS : $\int_S \hat{V}\cdot d\hat{S}$ , $\int_S \underset{\sim}{V}\, dS$ , $\int_S \phi\, d\underset{\sim}{S}$

FLUX

similar

VOLUME INTEGRALS : $\int_V \underset{\sim}{V}\, dV$ , $\int_V \phi\, dV$

eg div. Th$^{\underline{m}}$

[where, for example, $\phi = \mathrm{div}(\underset{\sim}{V})$]

**Finally, we looked at two examples of vector calculus integrals that we have not yet dealt with ...**

H5

**For the first type, we can write:**

$$\int_C \phi \, d\underset{\sim}{r} = \int_C \phi \left( \underset{\sim}{i}\, dx + \underset{\sim}{j}\, dy + \underset{\sim}{k}\, dz \right)$$

$$= \underset{\sim}{i} \int_C \phi \, dx + \underset{\sim}{j} \int_C \phi \, dy + \underset{\sim}{k} \int_C \phi \, dz$$

To demonstrate another aspect, we looked at *parametric representation of the curve C* .

Then, we re-write everything: ϕ , dx, dy, dz and integral limits only in terms of that new parameter (we used parameter *u*) .

**For the second type, we can write:**

$$\int_S \phi \, d\underset{\sim}{S} = \int_S \phi \, \hat{\underset{\sim}{n}} \, dS ,$$

where $\hat{\underset{\sim}{n}} = \dfrac{\nabla \underset{\sim}{S}}{|\nabla \underset{\sim}{S}|}$ , i.e. unit normal to the surface

Again, to demonstrate another aspect, we looked at *transforming the integral from Cartesian* ( $x, y, z$) *coordinates to cylindrical coordinates* $(\rho, \theta, z)$.

H5 p**154** to p**155**