SUMMARY / OVERVIEW OF ...

Mathematical Methods and Applications

HANDOUT 6

Handout 6
p156

- VECTOR CALCULUS (concluded)

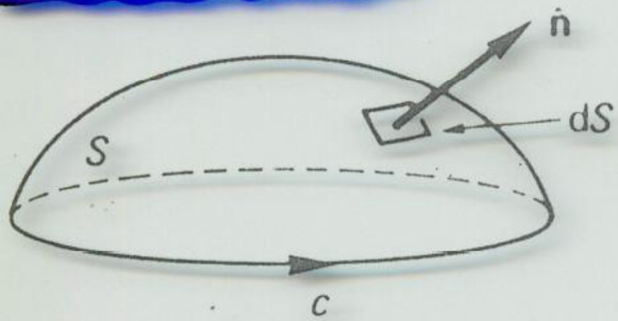
- Stoke's Theorem
  - proof
  - applications
- Conservative Fields – Revisited
  - the <u>five</u> equivalent conditions
  - examples of conservative fields

then ...

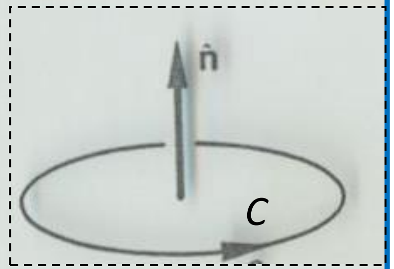
- Examples of solenoidal fields (zero divergence everywhere)
- Alternative space coordinate systems (reference material)

HEADLINE TOPIC OF HANDOUT 6

Stokes Theorem



$\hat{\boldsymbol{n}}$ and C directions



convention

If **F** is a vector field existing over an open surface S and around its boundary closed curve c, then

$$\boxed{\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}}$$

where $\oint \underset{\sim}{F} \cdot d\underset{\sim}{r}$ is called THE CIRCULATION OF $\underset{\sim}{F}$ AROUND THE CURVE C.

Key Features

- Relationship between (flux of) **curl F** and "circulation" / rotation
- Relates open surface integral to closed line integral (bounding curve)
- Can transform between integral and differential forms of physical laws
- Applies to **any** surface *S* with the same bounding curve *C*

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Maxwell's Equations (simple form)

Differential Form	Integral Form
$\boldsymbol{\nabla}.\boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	$\displaystyle\oint_S \boldsymbol{E}.\boldsymbol{dS} = \dfrac{Q}{\varepsilon_0}$
$\boldsymbol{\nabla}.\boldsymbol{B} = 0$	$\displaystyle\oint_S \boldsymbol{B}.\boldsymbol{dS} = 0$

Now …

$\boldsymbol{\nabla} \times \boldsymbol{E} = ?$	$?$
$\boldsymbol{\nabla} \times \boldsymbol{B} = ?$	$?$

where ***E*** and ***B*** are electric and magnetic fields, respectively,

ρ is charge (volume) density, and *Q* is total charge.

The circuital law,

$$\oint_C \underset{\sim}{E} \cdot d\underset{\sim}{l} = -\frac{\partial \Phi}{\partial t} = -\frac{\partial}{\partial t}\int_S \underset{\sim}{B} \cdot d\underset{\sim}{S}$$

Integral Form

ϕ is total flux of ***B***-field through surface *S* with bounding curve *C*

$$= \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{E}) \cdot d\underset{\sim}{S} = -\frac{\partial}{\partial t}\int_S \underset{\sim}{B} \cdot d\underset{\sim}{S}$$

(using Stokes Theorem)

$$\Rightarrow \quad \boxed{\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}}$$

Differential Form

Maxwell-Faraday equation

Maxwell's Equations (simple form)

Differential Form	Integral Form
$\boldsymbol{\nabla}.\boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \boldsymbol{E}.\boldsymbol{dS} = \dfrac{Q}{\varepsilon_0}$
$\boldsymbol{\nabla}.\boldsymbol{B} = 0$	$\oint_S \boldsymbol{B}.\boldsymbol{dS} = 0$
$\boldsymbol{\nabla} \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	$\oint_C \boldsymbol{E}.\boldsymbol{dl} = -\dfrac{\partial \emptyset}{\partial t}$

Now …

$\boldsymbol{\nabla} \times \boldsymbol{B} = ?$	?

where ***E*** / ***B*** are electric / magnetic fields, ρ is charge (volume) density, *Q* is total charge, *t* is time, and ϕ is ***B***-flux in curve *C* .

Ampère's Law

$$\boxed{\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}}$$

Differential Form

J is electric current density (see page 92), whereby total current I is

$$I = \int_S \underset{\sim}{J} \cdot d\underset{\sim}{S}$$

Calculate flux through surface S of both sides of the above equation. Recall total current I in terms of ***J*** (for RHS), and use Stokes Theorem (for LHS):

$$\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{B} \cdot d\underset{\sim}{r}$$

Integral Form

$$\boxed{\oint_C \underset{\sim}{B} \cdot d\underset{\sim}{r} = \mu_0 I}$$

Maxwell's Equations (simple form)

Differential Form	Integral Form
$\boldsymbol{\nabla} . \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \boldsymbol{E} . \boldsymbol{dS} = \dfrac{Q}{\varepsilon_0}$
$\boldsymbol{\nabla} . \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B} . \boldsymbol{dS} = 0$
$\boldsymbol{\nabla} \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	$\oint_C \boldsymbol{E} . \boldsymbol{dr} = -\dfrac{\partial \emptyset}{\partial t}$
$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$	$\oint_C \boldsymbol{B} . \boldsymbol{dr} = \mu_0 I$

where ***E*** / ***B*** are electric / magnetic fields, ρ is charge (volume) density, *Q* is total charge, *t* is time, ϕ is ***B***-flux within curve *C* , ***J*** is electric current density, and I is total current passing through *C* .

[*More details are in the full presentations and handouts!*]

Conservative Fields – Revisited

Earlier, we obtained <u>three</u> equivalent conditions for a vector field $\underset{\sim}{V}$ to be conservative.

These were ...

(I) • the existence of a scalar potential $\phi(x,y,z)$ such that

$$\int_A^B \underset{\sim}{V} \cdot d\underset{\sim}{r} = \int_A^B d\phi = \phi_B - \phi_A$$

[path independence]

(II) • for $\underset{\sim}{V} \cdot d\underset{\sim}{r} = d\phi = V_x\,dx + V_y\,dy + V_z\,dz$

[$\underset{\sim}{V} \cdot d\underset{\sim}{r} = d\phi$, an exact differential]

(III) • the reciprocity relations : $\dfrac{\partial V_x}{\partial y} = \dfrac{\partial V_y}{\partial x}$; $\dfrac{\partial V_x}{\partial z} = \dfrac{\partial V_z}{\partial x}$

and $\dfrac{\partial V_y}{\partial z} = \dfrac{\partial V_z}{\partial y}$

Two more conditions

Firstly, note that if $\underset{\sim}{V} = (V_x, V_y, V_z)$ then

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

i.e. $$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \hat{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \hat{j}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \hat{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

If the following is true (the reciprocity relations):

$$\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x} \; ; \; \frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}$$

III) and $$\frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}$$

then we simply have that:

$$\boxed{\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0} \quad \text{if } \underset{\sim}{V} \text{ is conservative}}$$

i.e. a concise test for whether vector field ***V*** is conservative

Last condition

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Secondly, note condition (II) requiring $d\phi$ to be an exact differential implies that

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz .$$

But this is just the dot product of $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial\phi}{\partial x} + \underset{\sim}{j}\frac{\partial\phi}{\partial y} + \underset{\sim}{k}\frac{\partial\phi}{\partial z}$

and $d\underset{\sim}{r} = \underset{\sim}{i}\,dx + \underset{\sim}{j}\,dy + \underset{\sim}{k}\,dz$

i.e. $d\phi = \underset{\sim}{\nabla}\phi \cdot d\underset{\sim}{r}$ but condition (I) is: $\int_A^B \underset{\sim}{V}\cdot d\underset{\sim}{r} = \int_A^B d\phi$

so ... If we can write a vector field $\underset{\sim}{V}$ as

$\boxed{\underset{\sim}{V} = \underset{\sim}{\nabla}\phi}$, where ϕ is a scalar field,

then $\underset{\sim}{V}$ is a conservative field.

That gives us **FIVE conditions** in total.

Note that ...

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if $\underset{\sim}{V}$ is a conservative vector field

then $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ everywhere

⇒ the field is "IRROTATIONAL"

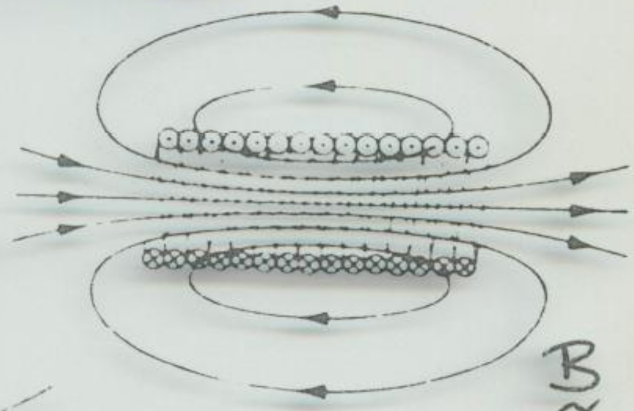
i.e. it has no vortices / circulation / swirl / etc.

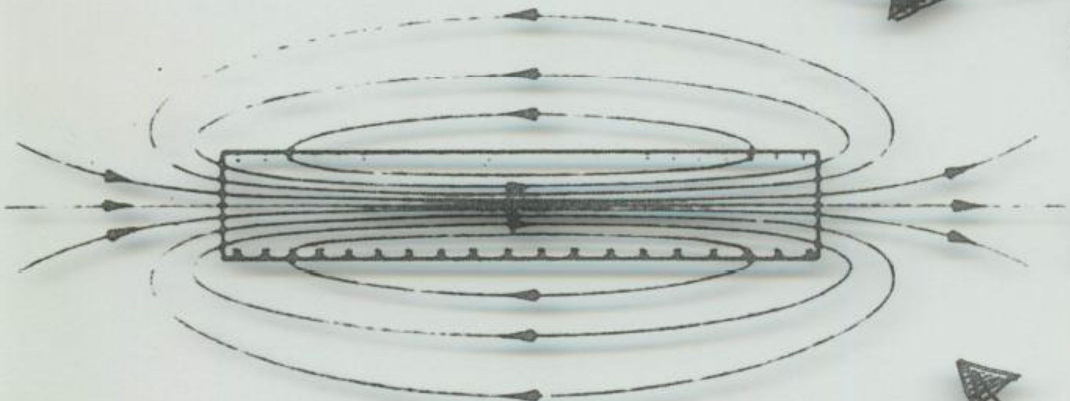
And that ...

Another special type of vector field :

ZERO DIVERGENCE $\Rightarrow$ "SOLENOIDAL"

examples (a) Field of a solenoid!



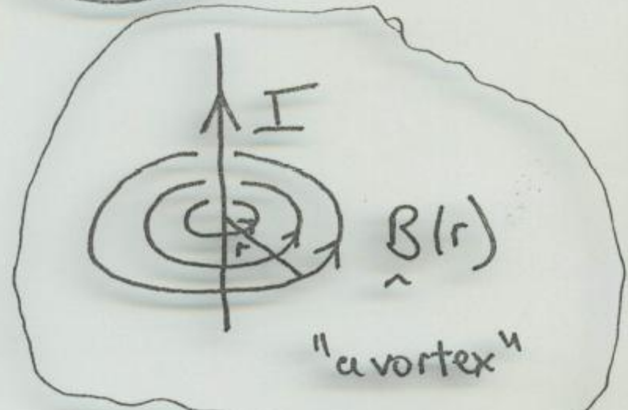


(b) Field of a bar magnet

(c) Field from a current-carrying wire :

: $\nabla \times B = \mu_0 J$

can be non-zero, can have vortices



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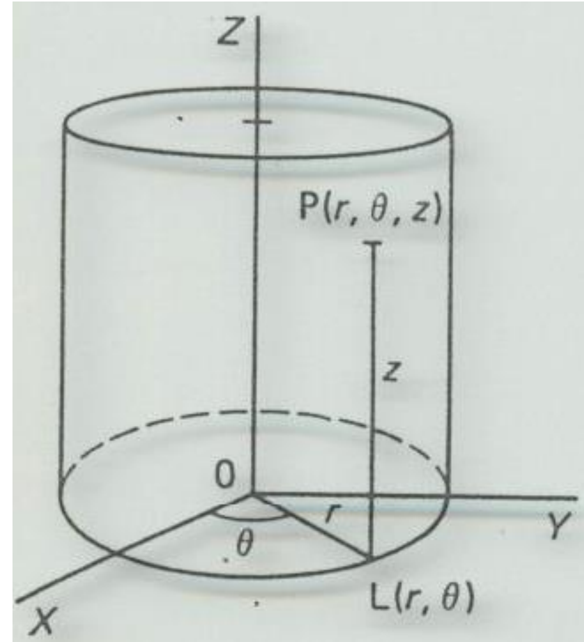
Space coordinate systems

Cylindrical coordinates (r, θ, z)

$$x = r\cos\theta; \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta; \qquad \theta = \arctan(y/x)$$

$$z = z; \qquad z = z$$



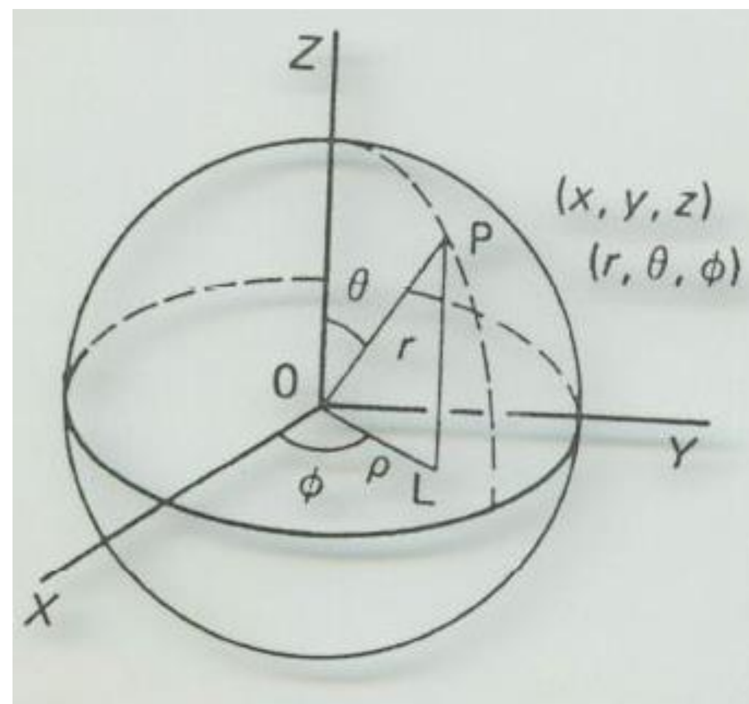
Spherical coordinates (r, θ, ϕ)

$$x = \boxed{r\sin\theta}\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = \boxed{r\sin\theta}\sin\phi \qquad \theta = \arccos(z/r)$$

$$z = r\cos\theta \qquad \phi = \arctan(y/x)$$

(<u>NB</u> $\rho = \boxed{r\sin\theta}$)



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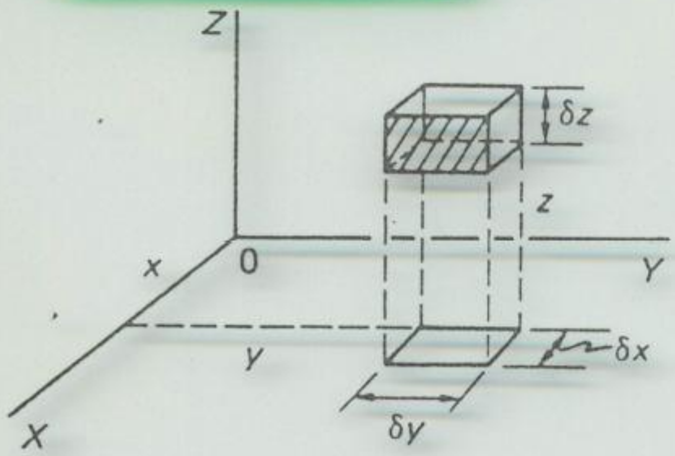
Element of volume in space in the three coordinate systems

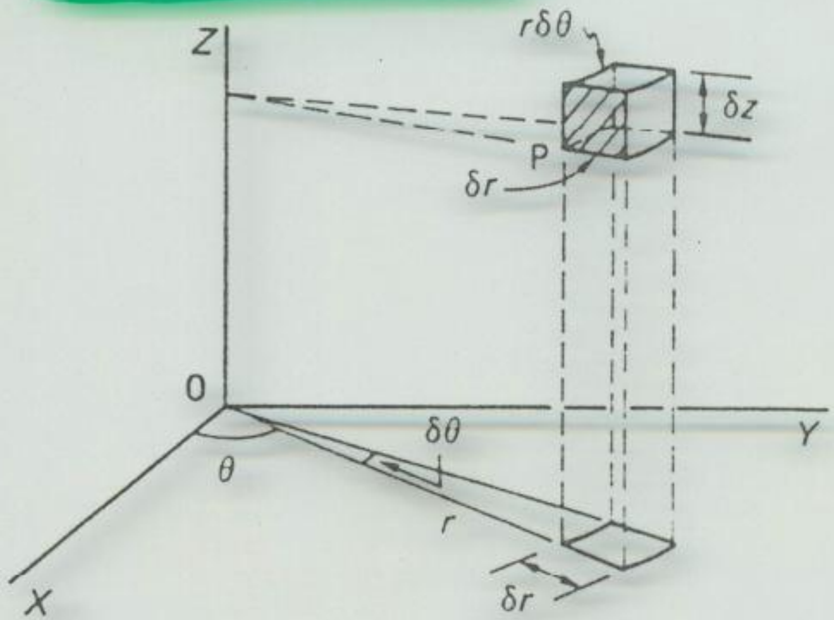
1. *Cartesian coordinates*

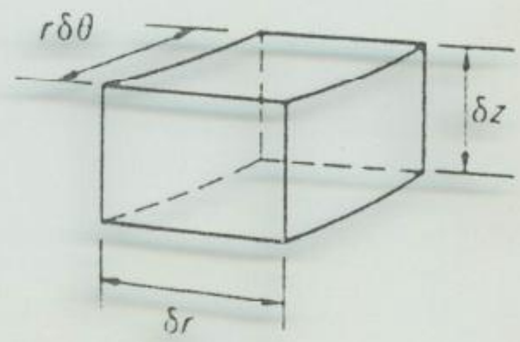


We have already used this many times.

$$\delta v = \delta x \delta y \delta z$$

2. *Cylindrical coordinates*





$$\delta v = r\,\delta\theta\,\delta r\,\delta z$$

$$\therefore\ \delta v = r\,\delta r\,\delta\theta\,\delta z$$

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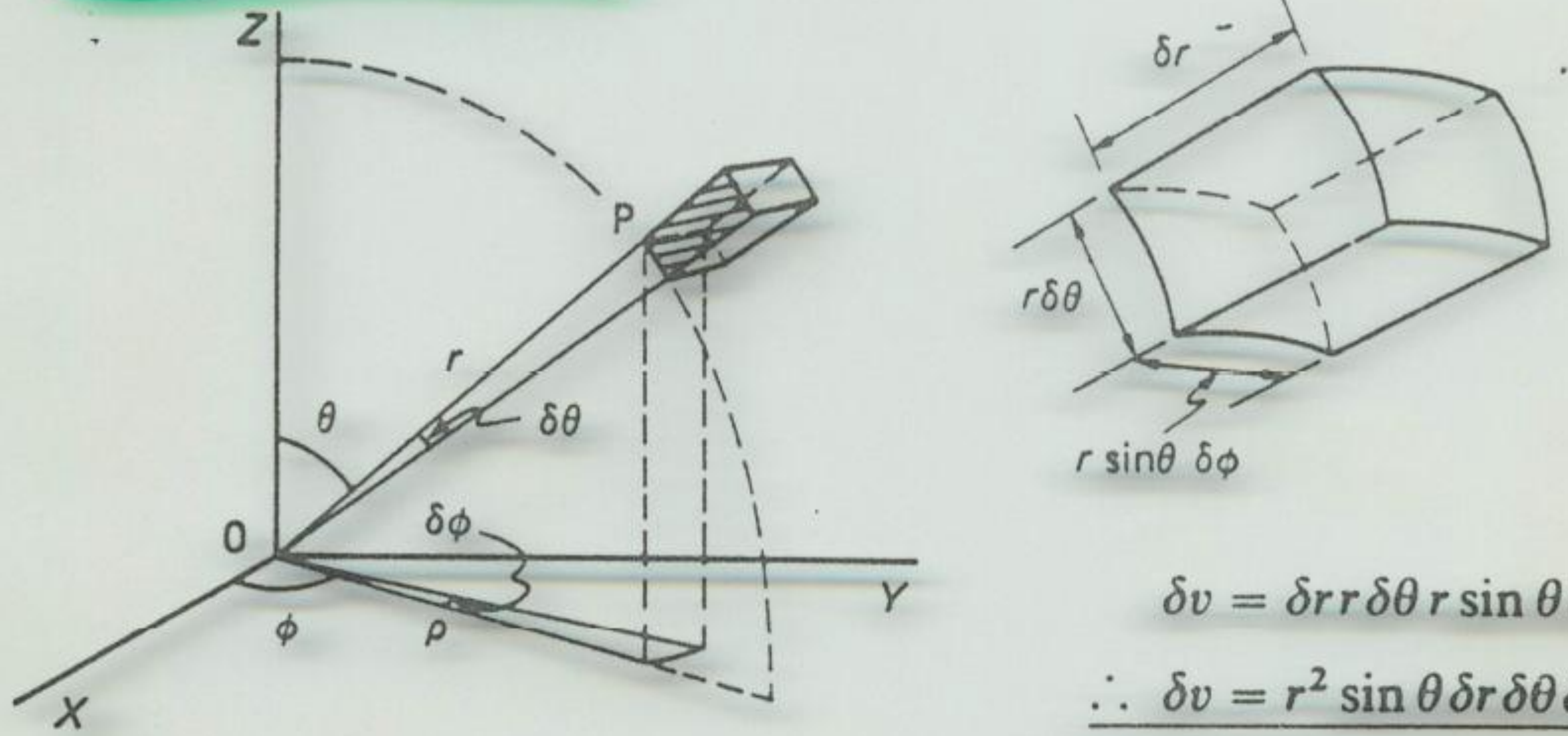
Element of volume in space in the three coordinate systems

3. *Spherical coordinates*



$$\delta v = \delta r \, r\,\delta\theta \, r\sin\theta\,\delta\phi$$

$$\therefore\ \delta v = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

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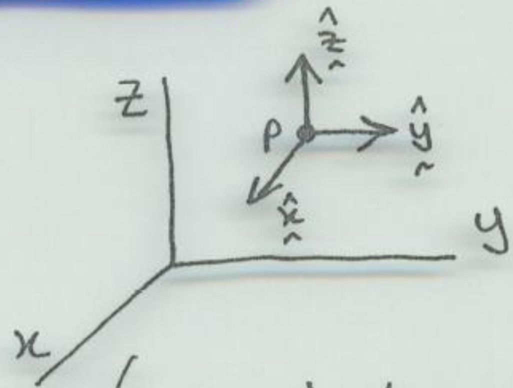
UNIT BASIS VECTORS

Cartesian

$\underset{\sim}{e}_x \equiv \underset{\sim}{i} \equiv \underset{\sim}{\hat{x}}$

$\underset{\sim}{e}_y \equiv \underset{\sim}{j} \equiv \underset{\sim}{\hat{y}}$

$\underset{\sim}{e}_z \equiv \underset{\sim}{k} \equiv \underset{\sim}{\hat{z}}$



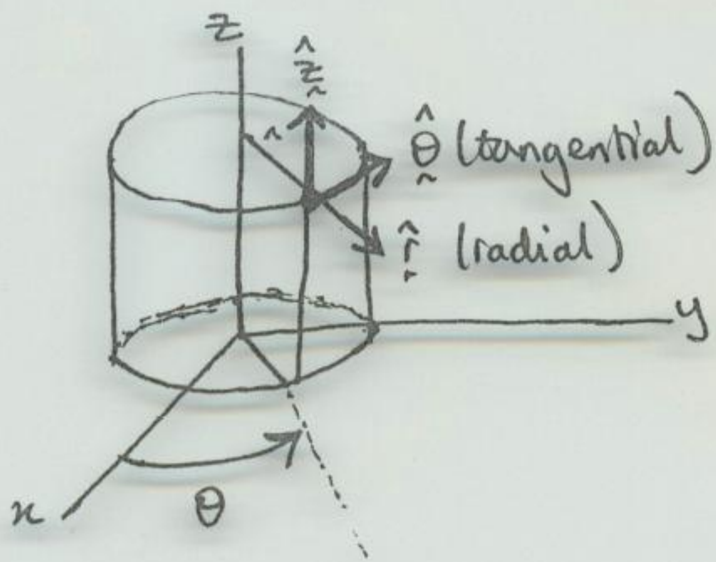
(in direction of increasing x, y, z)

Cylindrical

$\underset{\sim}{e}_r \equiv \underset{\sim}{\hat{r}}$

$\underset{\sim}{e}_\theta \equiv \underset{\sim}{\hat{\theta}}$

$\underset{\sim}{e}_z \equiv \underset{\sim}{\hat{z}}$



(in direction of increasing r, θ, z)

UNIT BASIS VECTORS

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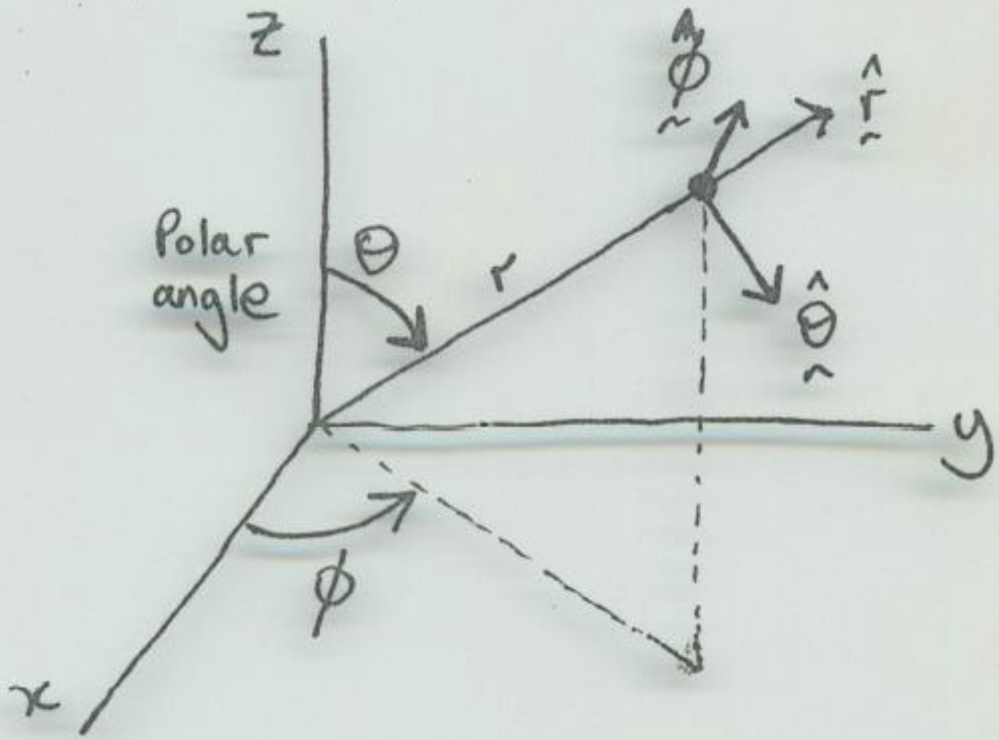
Spherical

$\underset{\sim}{e}_r \equiv \underset{\sim}{\hat{r}}$

$\underset{\sim}{e}_\theta \equiv \underset{\sim}{\hat{\theta}}$

$\underset{\sim}{e}_\phi \equiv \underset{\sim}{\hat{\phi}}$



(in direction of increasing r, θ, ϕ)

SUMMARY – DIFFERENTIAL OPERATORS IN OTHER ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

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Vector components: labelled as 1, 2, & 3

- Cartesian coordinates (x, y, z)

$$\mathbf{e}_x \equiv \mathbf{i}\,, \qquad \mathbf{e}_y \equiv \mathbf{j}\,, \qquad \mathbf{e}_z \equiv \mathbf{k}$$

$$\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$$

$$(\text{grad}\, f)_1 = \frac{\partial f}{\partial x}$$

$$(\text{grad}\, f)_2 = \frac{\partial f}{\partial y}$$

$$(\text{grad}\, f)_3 = \frac{\partial f}{\partial z}$$

$$\text{div}\,\mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$(\text{curl}\,\mathbf{A})_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}$$

$$(\text{curl}\,\mathbf{A})_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}$$

$$(\text{curl}\,\mathbf{A})_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- cylindrical polar coordinates (r, θ, z)

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \qquad z = z$$

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\,, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,, \qquad \mathbf{e}_z = \mathbf{k}$$

$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_z\,\mathbf{e}_z$$

$$(\text{grad}\, f)_r = \frac{\partial f}{\partial r}$$

$$(\text{grad}\, f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$(\text{grad}\, f)_z = \frac{\partial f}{\partial z}$$

$$\text{div}\,\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$(\text{curl}\,\mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$

$$(\text{curl}\,\mathbf{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\text{curl}\,\mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

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- spherical polar coordinates (r, θ, ϕ)

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

$$\mathbf{A} = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_\phi\,\mathbf{e}_\phi$$

$$(\mathrm{grad}\, f)_r = \frac{\partial f}{\partial r}$$
$$(\mathrm{grad}\, f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$
$$(\mathrm{grad}\, f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\mathrm{div}\,\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$(\mathrm{curl}\,\mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}$$
$$(\mathrm{curl}\,\mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$
$$(\mathrm{curl}\,\mathbf{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

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