SUMMARY / OVERVIEW OF ...

Mathematical Methods and Applications

HANDOUT 7

MATRICES

- Introduction
  - consistency and number of solutions of 2x2 systems
  - definition of a matrix
  - matrix arithmetic
    * addition and subtraction
    * multiplication by a scalar
    * multiplying matrices

- Solution of equations
  - Cramer's rule
  - Laplace expansion of determinants
  - Classification of systems I.

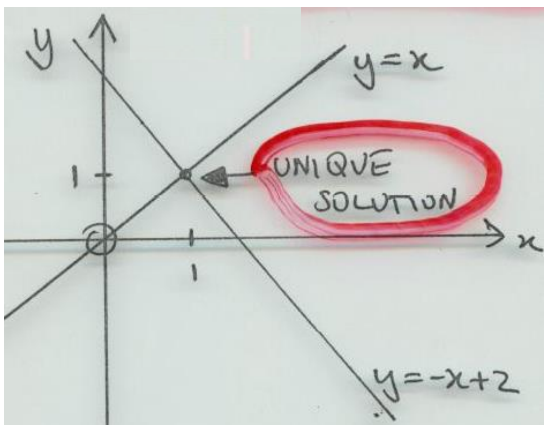
MATRICES

H7 p**191** to p**192**

Commonly used in the solution of simultaneous linear equations. Considering just 2 equations and 2 unknowns (x and y) permits both algebraic and graphical interpretation of **the general patterns** that emerge for other more complicated systems.

Ex. 1

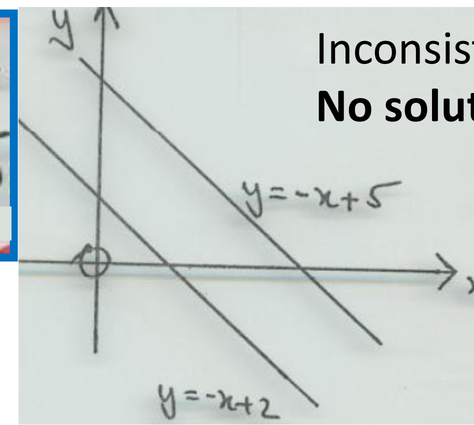
$$x+y=2$$
$$x-y=0$$



Unique (non-zero) solution

Ex. 2

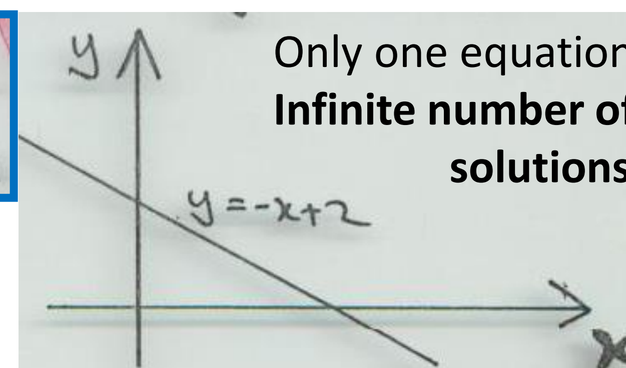
$$x+y=2$$
$$x+y=5$$



Inconsistent.
No solution

Ex. 3

$$x+y=2$$
$$2x+2y=4$$



Only one equation.
Infinite number of solutions

<u>3 INHOMOGENEOUS SYSTEMS</u>

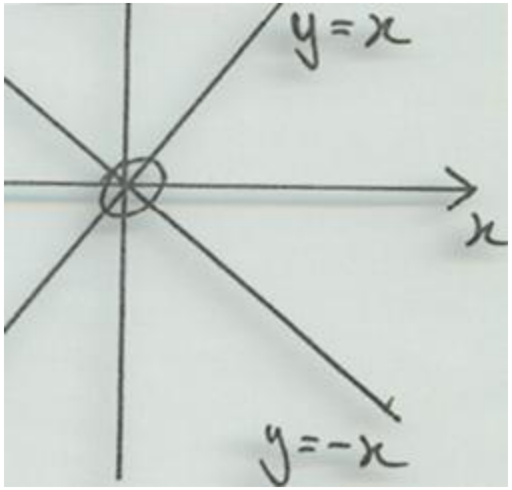
- have non-zero constant(s)

2 HOMOGENEOUS SYSTEMS - all constants are zero

H7 p**193** to p**194**

Ex. 4

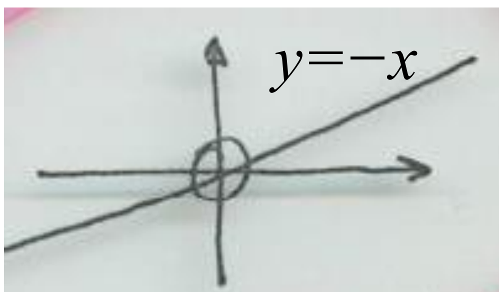
$$x+y=0$$
$$x-y=0$$



Unique solution. But is *always* the "**trivial**"/"null" solution: x=y=0

Ex. 5

$$x+y=0$$
$$2x+2y=0$$



Only one equation. **Infinite number of solutions** (including the **trivial** solution)

These 5 main cases are clear for just 2 equations and 2 unknowns. But, what if we have more equations - even just 3 ?!?

We need systematic ways of determining
- whether solutions exist
- how many exist
- finding them.

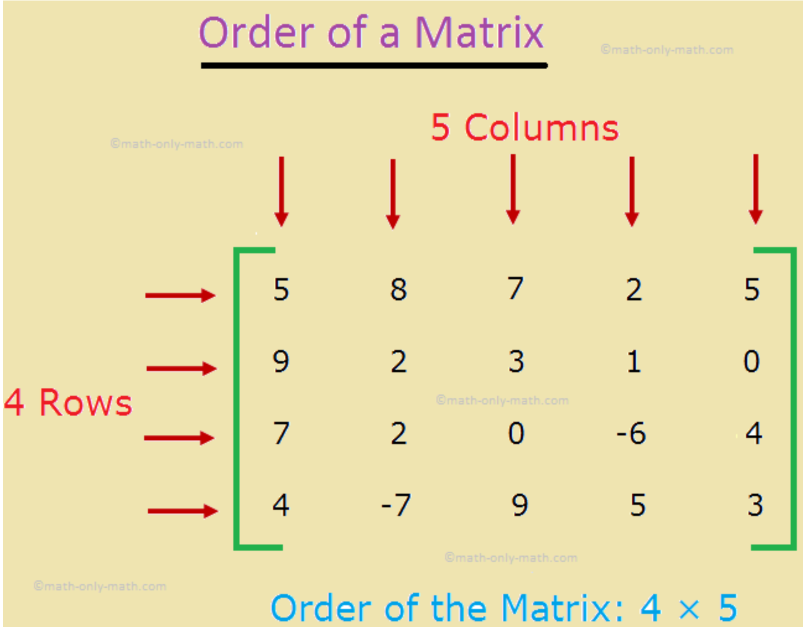
we need to
- write the equations in a way that they can be analysed (MATRICES)
- work out how to answer the above questions (MATRIX THEORY)

“**Matrix**”: rectangular array of elements
(surrounded by round or square brackets)

H7
p**195**
to
p**197**

The “**order**“ of a matrix ...

<u>Always</u> quote rows then columns. Here is one with order 4x5.



If a matrix only has a single column, I tend to denote it with vector notation

e.g. $\underset{\sim}{b} \equiv \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

“MATRIX ARITHMETIC”

(a) Addition. Add the corresponding elements ...

$A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

$A+B = \begin{pmatrix} 2+3 & 1-5 & 4+1 \\ -3+2 & 0+1 & 2+3 \end{pmatrix}$

$= \begin{pmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{pmatrix}$

If they are not the *same order*, then we simply cannot add them.

(b) Subtraction.
Subtract the corresponding elements ...

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \qquad B = \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix}$$

If they are not the *same order*, then we simply cannot subtract them.

$$A - B = \begin{pmatrix} a-g & b-h & c-i \\ d-j & e-k & f-l \end{pmatrix}$$

(c) Multiplication by a scalar.
Multiple every single element by that scalar ...

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$$

$$\lambda A = \begin{pmatrix} 2\lambda & \lambda & 4\lambda \\ -3\lambda & 0 & 2\lambda \end{pmatrix}$$

H7 p**197** to p**197**

So far, these operations are just the same as "vector arithmetic".

Maybe not surprising, since vectors can be written as column matrices (i.e. matrices with only a single column). But, we'll have to define something new to facilitate multiplication of matrices ...

H7
p198
to
p198

(d) Multiplying one matrix by another (more involved !)

- when can we do it?
- how do we do it?

Ex $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$

a 2x3 matrix   a 3x2 matrix

Then, $AB = \begin{bmatrix} (2)(3)+(1)(2)+(4)(4) & (2)(5)+(1)(-1)+(4)(2) \\ (-3)(3)+(0)(2)+(2)(4) & (-3)(5)+(0)(-1)+(2)(2) \end{bmatrix}$

i.e. $AB = \begin{pmatrix} 24 & 17 \\ -1 & -11 \end{pmatrix}$ ...... a 2x2 matrix !!

H7
p**199** to p**200**

What have we done?

- To get row 1, column 1 element of AB (ie 24)

we multiplied corresponding elements of ...

row 1 of A and column 1 of B

-- it was like a scalar product of vectors

i.e. (2, 1, 4) times $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ gave $(2)(3) + (1)(2) + (4)(4) = 24$

row 1 of A

column 1 of B

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$$

need to be equal

matrix orders: $2 \times \boxed{3 \quad 3} \times 2$

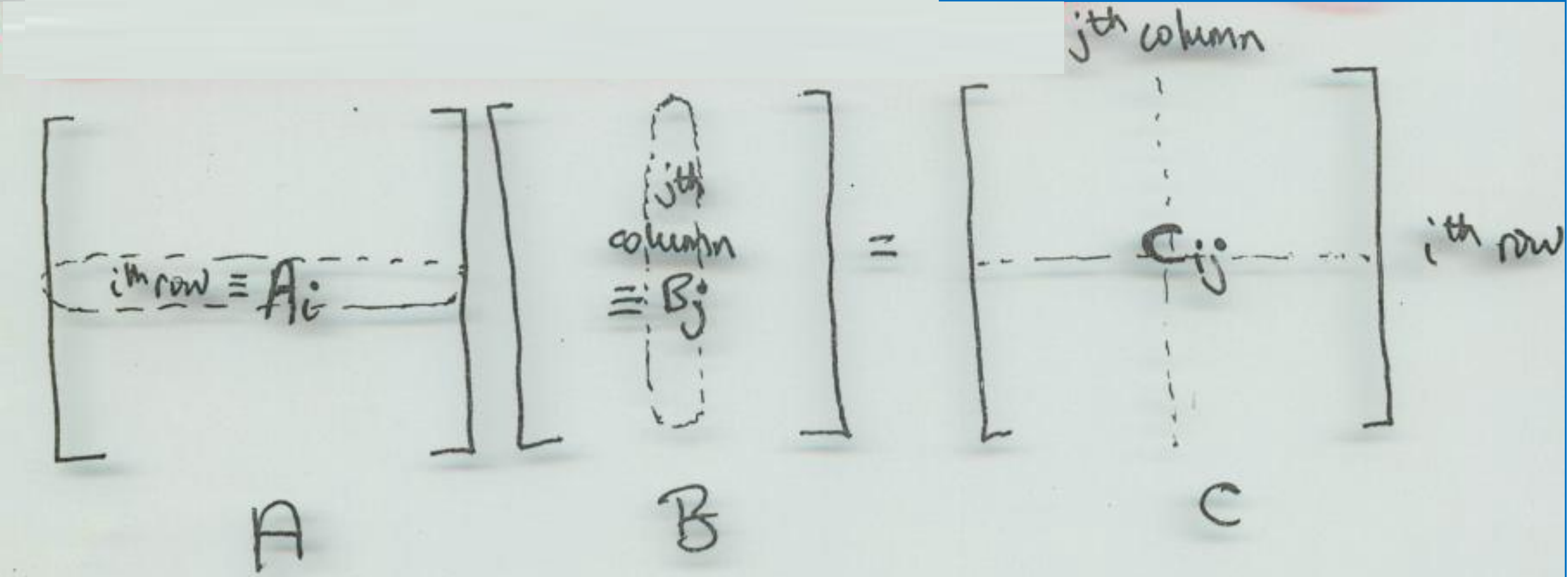
Note, for this rule of forming product AB we need the number of columns of A to equal the number of rows of B.

The matrices are then said to be **CONFORMABLE**.

The outer two numbers then give the order of the product AB. Here, we get a 2x2 result.

More generally … **THE " ROW DOT COLUMN RULE "**

to get the ij^{th} element of $AB = C$ …



where scalar $c_{ij} \equiv \underset{\sim}{A}_i \cdot \underset{\sim}{B}_j = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$

$= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

$$A \quad B = C$$
$$n \times m \quad m \times p \qquad n \times p$$

H7 p**201** to p**202**

H7
p**202**
to
p**203**

Some multiplication properties

- $A(BC) = (AB)C$ associative
- $A(B+C) = AB + AC$ distributive
- $(B+C)A = BA + CA$ distributive
- $AB \neq BA$ non-commutation **! !**

Solutions of equations

e.g. $ax + by = e$

$cx + dy = f$

i.s $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

$(2\times 2) \quad (2\times 1) \to (2\times 1)$

... as a matrix equation:

$$\boxed{A\underset{\sim}{x} = \underset{\sim}{b}}$$

where

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **coefficient matrix**

$\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ **solution vector**

and

$\underset{\sim}{b} = \begin{pmatrix} e \\ f \end{pmatrix}$

CRAMER'S RULE for solving systems

H7
p**203**
to
p**205**

$$ax + by = e$$
$$cx + dy = f$$

(a "2x2 system")

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Determinant, D, of A with **b** swapped in

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Determinant, D, of coefficient matrix A

Recall that D = ad-bc here

CRAMER'S RULE for solving systems

H7
p**206**
to
p**206**

(a "3x3 system")

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{D}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{D} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

"LAPLACE EXPANSION" (working out matrix determinants)

In the vector calculus section, I gave the particular case

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p**207**
to
p**207**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \left(b_2c_3 - c_2b_3\right) - b_1 \left(a_2c_3 - c_2a_3\right) + c_1 \left(a_2b_3 - b_2a_3\right)$$

i.e. go along row $\underline{1}$ and, in each case, cover up the row and column of that element to find the 2×2 determinant.

In fact, one can use <u>any</u> row or <u>any</u> column, as long as you keep the signs of each term right. The pattern that gives the signs is $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ in the 3×3 case, $\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$ in the 4×4 case, So, one can also use the first column to get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

NB.

Sign table

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

equivalent to:

$$(-1)^{i+j}$$

- The above process is called “**LAPLACE EXPANSION**” or “**LAPLACE DEVELOPMENT**”
- The 2x2 determinants that result are called “**MINORS**”
- The “**signed minor**” (including the sign table factor) is called the “**COFACTOR**” and is denoted $\mathbf{A_{ij}}$

H7 p**208** to p**212**

The determinant is then

$$= \sum_{\substack{\text{along row} \\ \text{(or down column)}}} a_{ij} (-1)^{i+j} (\text{minor of } a_{ij}) = \sum_{\substack{\text{row/} \\ \text{column}}} a_{ij} A_{ij}$$

CLASSIFICATION OF SYSTEMS OF LINEAR EQUATIONS

I. *Dependence, Consistency, (In)homogeneous, Singularity*

● Equations are called **DEPENDENT** if one is a multiple of the other. Otherwise, they are called **INDEPENDENT**. Let's introduce the notation that we might have ***m independent equations in n unknowns***.

● If one or more solutions exist, the equations are called **CONSISTENT**. Otherwise, they are called **INCONSISTENT**.

H7 p**213** to p**216**

● If all the constants are zero, i.e. **b** = **0** (a null vector), then the system is said to be **HOMOGENEOUS**. Otherwise, it is **INHOMOGENEOUS**.

● The *four main possibilities are*: **i)** unique non-trivial solution, **ii)** unique but trivial solution, **iii)** no solutions, & **iv)** an infinite number of solutions.

● When the determinant of the coefficient matrix is zero, i.e. $|A| = 0$, *the matrix* A *is said to be* **SINGULAR**.

One can examine more general examples of the five systems that were listed at the start of this summary and classify them in terms of the terminologies listed on the previous page. From the results, one can correctly conject that ...

H7 p**216** to p**223**

For nxn <u>inhomogeneous</u> systems ...

- If $|A| \neq 0$ and $m = n$, have a unique non-trivial solution
- If $|A| = 0$ and $m < n$, have an infinite number of solutions
- If $|A| = 0$ and $m = n$, have no solutions

For nxn <u>homogeneous</u> systems ...

- If $|A| \neq 0$, have a unique (but trivial) solution
- If $|A| = 0$, have an infinite number of solutions (including the trivial)

Note that one cannot have inconsistency (i.e. no solutions) with homogeneous systems as the trivial solution always exists.