

SUMMARY / OVERVIEW OF ...

Mathematical Methods and Applications

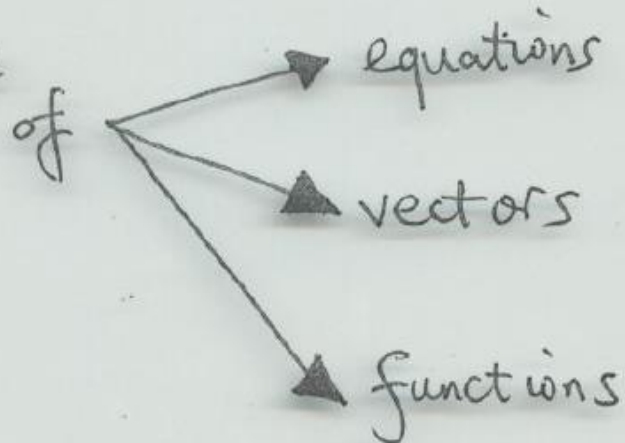
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II. The Rank of a matrix

- Linear Independence



Classification of systems of linear equations

II. The RANK of a matrix

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We need a means to determine parameter m (**the number of independent equations**). For this, we introduce a measure called the **RANK OF A MATRIX**. As seen earlier, the existence and number of solutions depends not only on the coefficient matrix A . We will also need to use the information given by the right-hand-side (constants) vector \mathbf{b} .

Definition. The **RANK** r of a matrix is the size of the largest non-zero determinant that can be found from its elements (in the order that they appear in the matrix). In other words, rank r implies at least one square sub-matrix with r rows and non-zero determinant, while any square sub-matrix with $r+1$ rows would have zero determinant.

Some examples ...

• A square sub-matrix of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ could be $\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ or $\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}$

or even $\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}$ or $\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}$. There are a lot to choose from

• If A is a square matrix of order n then if the rank of A is n it follows that $|A| \neq 0$ and A is non-singular.

- If all 2×2 sub-matrices have zero determinant, then we need at least one non-zero element for the matrix to have rank $r = 1$, otherwise it will have $r = 0$. Note that a 1×1 matrix has determinant equal to the element value (which can be positive, negative, or zero).

Application of rank to the solution of simultaneous equations

System

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

can be written as:

$$A \underset{\sim}{x} = \underset{\sim}{b}$$

then, one defines an **AUGMENTED COEFFICIENT MATRIX**:

$$A_b = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

combining A and \mathbf{b} ,
and thus capturing all
the system information.

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In the notes, we compare $\text{rank}(A)$, $\text{rank}(A_b)$ and n (the number of unknowns) for our five main 2×2 systems, and find only three distinct cases.

We then conjecture that this is a general result when A is an $n \times n$ matrix ...

- $\text{rank}(A) = \text{rank}(A_b) = n \Rightarrow$ UNIQUE SOLUTION
- $\text{rank}(A) = \text{rank}(A_b) < n \Rightarrow$ INFINITE NUMBER OF SOLUTIONS
- $\text{rank}(A) < \text{rank}(A_b) \Rightarrow$ NO SOLUTION

We also know that ...

• Homogeneous systems are always consistent because they always have at least the trivial solution.

• $\det(A) = 0$ is required for non-trivial solutions of homogeneous systems.

• Inhomogeneous systems only have non-trivial solutions.

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What does $\det(A) \equiv |A|$ tell us?

(unique solution,
but ...)

Homogeneous systems : $|A| \neq 0 \Rightarrow$ only trivial solution
 $|A| = 0 \Rightarrow$ infinite number of solutions

Inhomogeneous systems : $|A| \neq 0 \Rightarrow$ unique non-trivial solution
 $|A| = 0 \Rightarrow$ $\left\{ \begin{array}{l} \text{infinite number of solutions} \\ \text{when } r(A) = r(A_b) = m < n \end{array} \right.$
additional possibility: \rightarrow $\left\{ \begin{array}{l} \text{no solution} \\ \text{when } r(A) < r(A_b) \end{array} \right.$

Note that in Cramer's rule we have

$$x = \frac{| \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} |}{|A|}, \quad y = \frac{| \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} |}{|A|}, \quad \text{etc.}$$

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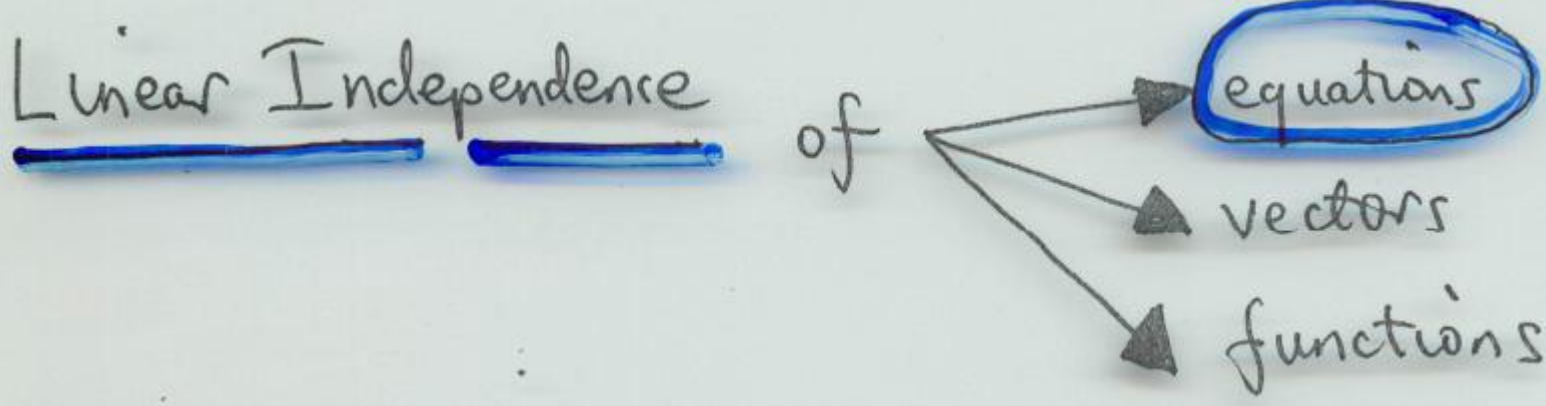
and we cannot find a solution when $|A| = 0$ (using Cramer's rule).

- we cannot divide by zero (when A is singular).

... these are the cases
when we don't have
either a trivial or non-trivial
unique solution

$$|A| = 0$$

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We considered solving our five main 2×2 systems in A_b format and using the following **ELEMENTARY ROW OPERATIONS**:

- (i) interchange two rows
- (ii) multiply a row by a number
- (iii) add a multiple of one row to another row

The scheme we used to solve the equations involved a first stage of reducing A_b to **ECHELON** ("staircase") **FORM**.

We used elementary row operations to reduce A_b to a matrix with only zeroes below the main diagonal, to see what patterns emerged for the different systems.

For inhomogeneous with a unique solution

no zero rows in A



$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & \\ \hline 0 & -2 & -2 & \end{array} \right)$$

"echelon form"

For inhomogeneous with an infinite number of solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & \\ \hline 0 & 0 & 0 & \end{array} \right)$$



eliminated equation

For inhomogeneous with no solution

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & \\ \hline 0 & 0 & 3 & \end{array} \right)$$



an inconsistency

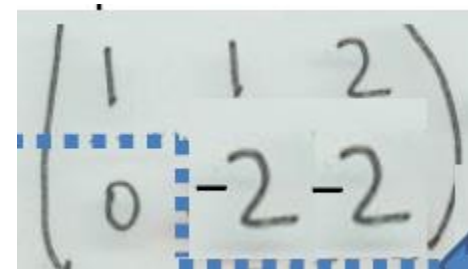
Echelon form shows us the number of independent rows in A.

We can now consider these results in relation to $\text{rank}(A)$, $\text{rank}(A_b)$ and n

When

$$\text{rank}(A) = \text{rank}(A_b) = n$$

we have **$m = n$ linearly independent equations and a unique solution.**

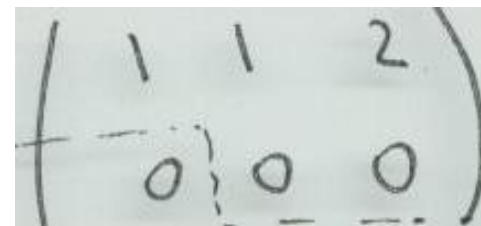

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \end{pmatrix}$$

When

$$\text{rank}(A) = \text{rank}(A_b)$$

we have **consistency**. Note that the number of non-zero rows of A_b in echelon form equals m (the number of linearly independent equations).

also true for this case ...


$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

But when

$$\text{rank}(A) = \text{rank}(A_b) < n$$

we have **$m < n$ linearly independent equations and an infinite number of solutions** (as in the second example above).

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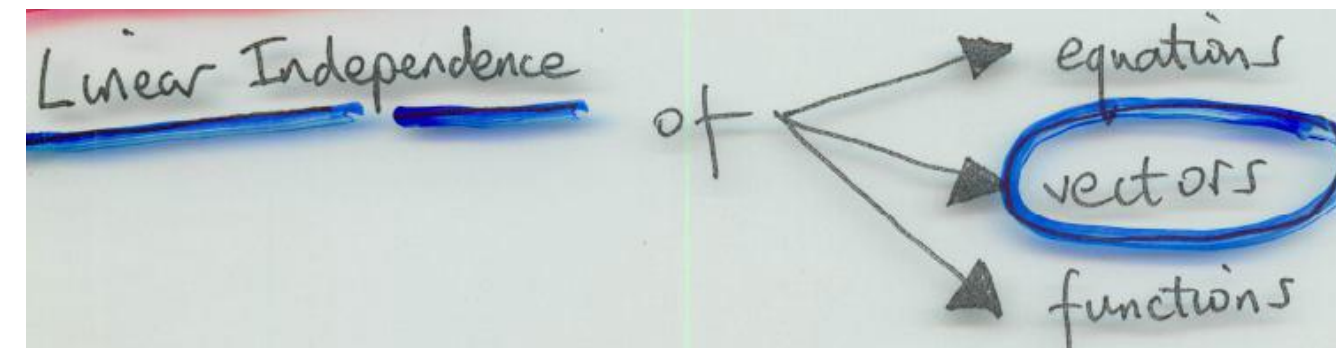
Finally ...

When

$$\text{rank}(A) < \text{rank}(A_b)$$

$$\begin{pmatrix} 1 & 1 & 2 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 3 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

we have **inconsistency**. The number of non-zero rows of A is less than the number of non-zero rows of A_b (when the system is in echelon form).



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$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are **linearly dependent** if their linear combination $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = \mathbf{0}$ for some numbers a_i not all zero.

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are **linearly independent** if their linear combination cannot be set to $\mathbf{0}$ without assuming all a_i are zero.

Considering linear combinations of independent vectors in space, such as:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

the dimension of the space

= the required number of basis vectors

= the number of linearly independent vectors needed to "span" the space

For a given set of vectors, one can find how many are linearly independent by putting them as matrix rows and performing a reduction of the matrix to echelon form. There, note that:

the number of non-zero rows of the matrix in echelon form
= the number of linearly independent rows / vectors
= the rank of that matrix

Linear Independence

of

equations

vectors

functions

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Functions

Linear (in)dependence of functions is defined in the same way^{ooo}. Functions $f_1(x), f_2(x), \dots, f_n(x)$ are linearly dependent if some linear combination of them is zero

$$\text{i.e. } a_1 f_1(x) + a_2 f_2(x) + \dots + a_n f_n(x) = 0$$

for some numbers a_1, a_2, \dots, a_n not all zero.