SUMMARY / OVERVIEW OF ...

Mathematical Methods and Applications

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HANDOUT 8

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- Linear Independence of
  - equations
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Classification of systems of linear equations

II. The RANK of a matrix

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We need a means to determine parameter m (**the number of independent equations**). For this, we introduce a measure called the **RANK OF A MATRIX**. As seen earlier, the existence and number of solutions depends not only on the coefficient matrix A. We will also need to use the information given by the right-hand-side (constants) vector $\boldsymbol{b}$.

Definition. The **RANK** r of a matrix is the size of the largest non-zero determinant that can be found from its elements (in the order that they appear in the matrix). In other words, rank r implies at least one square sub-matrix with r rows and non-zero determinant, while any square sub-matrix with r+1 rows would have zero determinant.

Some examples ...

- A square sub-matrix of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ could be $\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ or $\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}$

  or even $\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}$ or $\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}$. There are a lot to choose from

- If A is a square matrix of order n then if the rank of A is n it follows that $|A| \neq 0$ and A is non-singular.

- If all 2x2 sub-matrices have zero determinant, then we need at least one non-zero element for the matrix to have rank $r = 1$, otherwise it will have $r = 0$. Note that a 1x1 matrix has determinant equal to the element value (which can be positive, negative, or zero).

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Application of rank to the solution of simultaneous equations

System

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

can be written as:

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

then, one defines an **AUGMENTED COEFFICIENT MATRIX**:

$$A_b = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

combining A and $\mathbf{b}$, and thus capturing all the system information.

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In the notes, we compare rank(A), rank(A_b) and n (the number of unknowns) for our five main 2x2 systems, and find only three distinct cases.

We then conject that this is a general result when A is an nxn matrix ...

- rank(A) = rank(A_b) = n $\Rightarrow$ UNIQUE SOLUTION
- rank(A) = rank(A_b) < n $\Rightarrow$ INFINITE NUMBER OF SOLUTIONS
- rank(A) < rank(A_b) $\Rightarrow$ NO SOLUTION

We also know that ...

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- Homogeneous systems are always consistent because they always have at least the trivial solution.
- det(A) = 0 is required for non-trivial solutions of homogeneous systems.
- Inhomogeneous systems only have non-trivial solutions.

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What does $\det(A) \equiv |A|$ tell us?

(unique solution, but ...)

Homogeneous systems : $|A| \neq 0 \Rightarrow$ only trivial solution

$|A| = 0 \Rightarrow$ infinite number of solutions

Inhomogeneous systems : $|A| \neq 0 \Rightarrow$ unique non-trivial solution

$|A| = 0 \Rightarrow$ infinite number of solutions when $r(A) = r(A_b) = m < n$

OR

no solution when $r(A) < r(A_b)$

additional possibility:

Note that in Cramer's rule we have

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$$x = \frac{|\cdots|}{|A|}, \quad y = \frac{|\cdots|}{|A|}, \text{ etc.}$$

and we cannot find a solution when $|A| = 0$ (using Cramer's rule).

– we cannot divide by zero (when A is singular).

... these are the cases when we <u>don't have</u> either a trivial or non-trivial unique solution ....

$|A| = 0$

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Linear Independence of
- equations
- vectors
- functions

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p**244**

We considered solving our five main 2x2 systems in A_b format and using the following **ELEMENTARY ROW OPERATIONS**:

(i) interchange two rows

(ii) multiply a row by a number

(iii) add a multiple of one row to another row

The scheme we used to solve the equations involved a first stage of reducing A_b to **ECHELON** ("staircase") **FORM**.

We used elementary row operations to reduce A_b to a matrix with only zeroes below the main diagonal, to see what patterns emerged for the different systems.

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For inhomogeneous with a unique solution

no zero rows in A

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right)$$

"echelon form"

For inhomogeneous with an infinite number of solutions

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ \hline 0 & 0 & 0 \end{array}\right)$$

eliminated equation

For inhomogeneous with no solution

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ \hline 0 & 0 & 3 \end{array}\right)$$

an inconsistency

Echelon form shows us the number of independent rows in A.

We can now consider these results in relation to rank(A), rank(A_b) and n

When $\boxed{\text{rank}(A) = \text{rank}(A_b) = n}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \end{pmatrix}$$

we have **m = n linearly independent equations and a unique solution**.

When $\boxed{\text{rank}(A) = \text{rank}(A_b)}$

also true for this case ...

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

we have **consistency.** Note that the number of non-zero rows of A_b in echelon form equals m (the number of linearly independent equations).

But when $\boxed{\text{rank}(A) = \text{rank}(A_b) < n}$

we have **m < n linearly independent equations and an infinite number of solutions** (as in the second example above).

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Finally ...

When $\boxed{\mathrm{rank}(A) < \mathrm{rank}(A_b)}$

$$\left(\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 0 & 0 & 3 \end{array}\right)$$

we have **inconsistency**. The number of non-zero rows of A is less than the number of non-zero rows of A_b (when the system is in echelon form).

Linear Independence of
equations
vectors
functions

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$\mathbf{v}_1 , \mathbf{v}_2 , \ldots , \mathbf{v}_n$ are **linearly dependent** if their linear combination $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots\ a_n\mathbf{v}_n = \mathbf{0}$ for some numbers a_i not all zero.

$\mathbf{v}_1 , \mathbf{v}_2 , \ldots , \mathbf{v}_n$ are **linearly independent** if their linear combination cannot be set to **0** without assuming all a_i are zero.

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to
p**250**

Considering linear combinations of independent vectors in space, such as: $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

the dimension of the space
= the required number of basis vectors
= the number of linearly independent vectors needed to "span" the space

For a given set of vectors, one can find how many are linearly independent by putting them as matrix rows and performing a reduction of the matrix to echelon form. There, note that:

the number of non-zero rows of the matrix in echelon form
= the number of linearly independent rows / vectors
= the rank of that matrix

Linear Independence of
- → equations
- → vectors
- → functions

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bot

Functions

Linear (in)dependence of functions is defined in the same way... Functions $f_1(x), f_2(x), \dots, f_n(x)$ are linearly **dependent** if some linear combination of them is zero

i.e. $a_1 f_1(x) + a_2 f_2(x) + \dots + a_n f_n(x) = 0$

for some numbers $a_1, a_2, \dots, a_n$ not all zero.