# Wave envelopes with second-order spatiotemporal dispersion. I. Bright Kerr solitons and cnoidal waves

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We propose a simple scalar model for describing pulse phenomena beyond the conventional slowly varying envelope approximation. The generic governing equation has a cubic nonlinearity, and we focus here mainly on contexts involving anomalous group-velocity dispersion. Pulse propagation turns out to be a problem firmly rooted in frames-of-reference considerations. The transformation properties of the model and its space-time structure are explored in detail. Two distinct representations of exact analytical solitons and their associated conservation laws (in both integral and algebraic forms) are presented, and a range of predictions is made. We also report cnoidal waves of the governing nonlinear equation. Crucially, conventional pulse theory is shown to emerge as a limit of the more general formulation. Extensive simulations examine the role of the new solitons as robust attractors.

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## I. INTRODUCTION

The slowly varying envelope approximation (SVEA) is so widespread in the literature of wave phenomena that it is almost trivially familiar [1–3]. Across a diverse range of contexts—from electromagnetics and quantum mechanics to hydrodynamics and plasma physics—it is routinely deployed to simplify complicated (e.g., elliptic or hyperbolic) governing equations, typically reducing them to the parabolic class. One is often concerned with the evolution of a physical quantity $Q(t,z)$ that is represented by an envelope $q(t,z)$ modulating a rapidly oscillating component according to $Q(t,z) = q(t,z)\exp[i(k_0 z - \omega_0 t)] + q^*(t,z)\exp[-i(k_0 z - \omega_0 t)]$. Here, $t$ and $z$ denote time and space coordinates, respectively, whereas, $k_0$ and $\omega_0$ are the propagation constant and angular frequency of the underlying carrier wave. The quantity $Q$ may correspond to, e.g., electric field, polarization grating, fluid velocity, or ion density. The SVEA (which assumes the longitudinal variation of $q$ is slow on the $\sim 1/k_0$ scale length) tends to go hand in hand with a Galilean coordinate transformation to a frame of reference $(t_{\mathrm{loc}},z_{\mathrm{loc}})$ moving at some characteristic (system-dependent) speed, typically the group velocity $v_g$. Such a transformation has the standard form $t_{\mathrm{loc}} = t - z/v_g$ and $z_{\mathrm{loc}} = z$ (this local time frame is denoted throughout by the "loc" subscript). Together, the SVEA and subsequent coordinate boost form a universal mathematical device that is the cornerstone of conventional pulse modeling.

In recent papers [4], we considered the consequences of neglecting the "SVEA + Galilean boost" combination in a simple nonlinear pulse propagation model that comprises one space dimension plus time. The motivation was to understand more thoroughly the precise role played by the SVEA in generating simplified governing equations and to quantify directly its effect upon the various classes of wave-packet solutions. Such a task may be undertaken most effectively, providing maximum physical insight, when the system being approximated can itself be handled in an exact mathematical way. This seems to be an elementary problem but, to the best of our knowledge, such an analysis has not yet been reported in the literature. For instance, while it is certainly true that spatial Helmholtz solitons for a cubic nonlinearity have been known for many years [5–7], no equivalent time-domain solutions appear to have been published to date.

To facilitate the analysis, we start out with a fully second-order spatiotemporal generalization [4] of the universal nonlinear Schrödinger (NLS) equation [1–3]. The early stages of our approach are somewhat traditional, e.g., the introduction of wave envelopes and the Fourier decomposition of the temporal dispersion operator. However, we dispense with the SVEA + Galilean boost device and instead remain in the laboratory frame (i.e., the frame where the source of pulses is at rest with respect to the observer). Such a choice is clearly allowed physically; it is, after all, the frame in which experiments are usually performed and measurements are made. Although the effects uncovered here are generic in nature (i.e., a consequence of ellipticity or hyperbolicity in the model), one arena where this work may find particular application is in waveguide optics with spatial dispersion that can be of *either sign* (see the Appendix). It is the frame-of-reference feature that distinguishes the following investigation from the historic works of Hasegawa and Tappert [8,9], Zakharov and Shabat [10,11], Manakov [12], Gordon [13], and many others [14–16].

This paper constitutes the first part of our analysis of Kerr-type nonlinear systems with spatiotemporal dispersion. Attention is focused predominantly on regimes involving anomalous group-velocity dispersion (GVD); regimes with normal GVD are discussed in a companion paper [4]. The layout of the current paper is as follows. A generic nonlinear wave equation is proposed in Sec. II, and the incompatibility of the traditional Galilean boost with systems involving spatiotemporal dispersion is detailed. The space-time transformation laws of the model are then discussed, and the velocity combination rule for pulses is obtained. Families of exact analytical forward- and backward-propagating bright solitons are derived in Sec. III along with physical predictions for pulse characteristics and three conservation laws (in both integral and algebraic forms). The subtle notion of rest frames is also addressed, and two new classes of cnoidal waves are reported. Asymptotic emergence of conventional pulse theory (in an appropriate physical limit) is examined in Sec. IV, then, in Sec. V, computer simulations investigate the robustness of the new solitons against perturbations to the pulse shape. We conclude, in Sec. VI, with some remarks about the applicability of our work and its potential for describing novel nonlinear wave phenomena that are directly observable in experiments.

## II. NONLINEAR WAVES WITH SPATIOTEMPORAL DISPERSION

### A. Governing equation and Galilean boosts

We consider the following governing equation for the dimensionless envelope $u$:

$$\kappa\frac{\partial^2 u}{\partial\zeta^2}+i\left(\frac{\partial u}{\partial\zeta}+\alpha\frac{\partial u}{\partial\tau}\right)+\frac{s}{2}\frac{\partial^2 u}{\partial\tau^2}+|u|^2u=0. \tag{1}$$

Here, $\zeta$ denotes the (longitudinal) space coordinate along which the wave is traveling, and $\tau$ denotes time (it is crucial to note $\tau$ does *not* represent a local time variable). There are three parameters appearing in Eq. (1): $\alpha$ is proportional to a ratio of group speeds [e.g., for optical waves, one may have $\alpha \equiv k_1 t_p/|k_2|$, where $t_p$ is a reference pulse duration, and $k_{1,2}$ are coefficients in the Taylor expansion of the mode propagation constant $k(\omega)$ around $\omega_0$—see the Appendix] and $s = \pm 1$ flags the GVD regime ($+1$ for anomalous, $-1$ for normal) [17–19]. The third parameter $\kappa$ can be interpreted as a positive *or negative* spatial dispersion parameter [spatial and temporal dispersion phenomena are identified with the $\kappa\partial^2/\partial\zeta^2$ and $(s/2)\partial^2/\partial\tau^2$ operators, respectively]. Hence, the fundamental structure of the governing equation depends upon the product $s\kappa$: For $\mathrm{sgn}(s\kappa) = +1\ (-1)$, Eq. (1) is elliptic (hyperbolic) in character.

The first term in Eq. (1), $\kappa\partial^2u/\partial\zeta^2$, is routinely neglected by invoking the SVEA; one assumes that $|\kappa\partial^2u/\partial\zeta^2| \ll |\partial u/\partial\zeta|$, and defines a set of local coordinates $\tau_{\mathrm{loc}} = \tau - \alpha\zeta$, $\zeta_{\mathrm{loc}} = \zeta$. This transformation describes a Galilean boost to a reference frame moving at speed $1/\alpha$ in the $+\zeta$ direction. The differential operators transform as $\partial/\partial\zeta = \partial/\partial\zeta_{\mathrm{loc}} - \alpha\partial/\partial\tau_{\mathrm{loc}}$ and $\partial/\partial\tau = \partial/\partial\tau_{\mathrm{loc}}$, combining to leave the familiar parabolic wave operator in the local time frame: $i(\partial/\partial\zeta + \alpha\,\partial/\partial\tau) + (s/2)\partial^2/\partial\tau^2 = i\partial/\partial\zeta_{\mathrm{loc}} + (s/2)\partial^2/\partial\tau_{\mathrm{loc}}^2$ [8–19]. Since the nonlinearity is unchanged by the coordinate boost, the envelope $u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$ satisfies

$$i\frac{\partial u}{\partial\zeta_{\mathrm{loc}}}+\frac{s}{2}\frac{\partial^2 u}{\partial\tau_{\mathrm{loc}}^2}+|u|^2u\simeq 0, \tag{2}$$

which is the canonical form of the universal NLS equation [10,11]. One can now ask what happens when the same boost transformation is applied to Eq. (1). This procedure leads to a governing equation with a cross derivative $\partial^2u/\partial\zeta_{\mathrm{loc}}\partial\tau_{\mathrm{loc}}$ that does not rigorously vanish:

$$\kappa\frac{\partial^2 u}{\partial\zeta_{\mathrm{loc}}^2}+i\frac{\partial u}{\partial\zeta_{\mathrm{loc}}}+\frac{1}{2}\left(s+2\kappa\alpha^2\right)\frac{\partial^2 u}{\partial\tau_{\mathrm{loc}}^2} - 2\kappa\alpha\frac{\partial^2 u}{\partial\zeta_{\mathrm{loc}}\partial\tau_{\mathrm{loc}}}+|u|^2u=0. \tag{3}$$

The cross derivative can hinder a straightforward physical interpretation of Eq. (3) [20–22], and it introduces extra computational complexity. To proceed, one might, for instance, consider only those families of solutions where $|\kappa|\alpha^2 \ll O(1)$, which enables the coefficient of the GVD term to be restored to $s/2$. One could also argue, based on order-of-magnitude considerations, that the cross-derivative term may be dropped (see, for instance, the seminal spatial dispersion analysis of Biancalana and Creatore [23]). In so doing, one arrives at

$$\kappa\frac{\partial^2 u}{\partial\zeta_{\mathrm{loc}}^2}+i\frac{\partial u}{\partial\zeta_{\mathrm{loc}}}+\frac{s}{2}\frac{\partial^2 u}{\partial\tau_{\mathrm{loc}}^2}+|u|^2u\simeq 0, \tag{4}$$

which is a temporal analog of the well-known spatial nonlinear Helmholtz equation [5,7]. Since we wish to minimize approximations, we instead dispense with the Galilean boost and deal with the full generality of Eq. (1) directly.

A leading-order estimate of the contribution made by the $\kappa\partial^2/\partial\zeta^2$ operator in Eq. (1) can be obtained by assuming that $|\kappa\partial^2u/\partial\zeta^2|$ is negligible compared to other terms so that the longitudinal part of the wave operator is approximated by $\partial/\partial\zeta \simeq -\alpha\,\partial/\partial\tau + i(s/2)\partial^2/\partial\tau^2 + i|u|^2$. One can then consider the product $\partial^2u/\partial\zeta^2 = (\partial/\partial\zeta)(\partial u/\partial\zeta)$, which leads to

$$\begin{aligned} &i\left(\frac{\partial u}{\partial\zeta}+\alpha\frac{\partial u}{\partial\tau}\right)+\frac{1}{2}(s+2\kappa\alpha^2)\frac{\partial^2 u}{\partial\tau^2}-\kappa\left(is\alpha\frac{\partial^3 u}{\partial\tau^3}+\frac{1}{4}\frac{\partial^4 u}{\partial\tau^4}\right)\\ &\quad+(1-\kappa|u|^2)|u|^2u \simeq \kappa|u|^2\left(i\alpha\frac{\partial}{\partial\tau}+\frac{s}{2}\frac{\partial^2}{\partial\tau^2}\right)u\\ &\quad+\kappa\left(i\alpha\frac{\partial}{\partial\tau}+\frac{s}{2}\frac{\partial^2}{\partial\tau^2}\right)(|u|^2u). \end{aligned} \tag{5}$$

By deploying this perturbative technique [24,25], the $\partial^2/\partial\zeta^2$ operator has been replaced by an $O(\kappa)$ combination of higher-order (linear and nonlinear) derivatives with respect to the time variable $\tau$.

### B. Coordinate transformation laws and velocity combination rule

When investigating the properties of Eq. (1) and its solutions under transformations in the space-time plane, it is convenient to adopt the notation routinely deployed for spatial solitons. Under the change of coordinates [26],

$$\tau = \frac{\tau' - V\zeta'}{\sqrt{1+2s\kappa V^2}}, \tag{6a}$$

$$\zeta = \frac{2s\kappa V\tau' + \zeta'}{\sqrt{1+2s\kappa V^2}}, \tag{6b}$$

the covariance of Eq. (1) is guaranteed so long as $u$ transforms according to

$$u(\tau,\zeta)=\exp\left[-i\frac{sV\tau'}{\sqrt{1+2s\kappa V^2}}+\frac{i}{2\kappa}\left(1-\frac{1}{\sqrt{1+2s\kappa V^2}}\right)\zeta'\right] \times\exp\left[-is\alpha\frac{\tau'-V\zeta'}{\sqrt{1+2s\kappa V^2}}+is\alpha\tau'\right]u'(\tau',\zeta'). \tag{6c}$$

Here, $V$ is a temporal analog of the transverse velocity parameter of an optical beam [4,26]. Under two successive applications of transformation (6), which are characterized by velocities $V_0$ and $V$, respectively, it can be shown that the net velocity $W$ with respect to $(\tau,\zeta)$ is given by [27]

$$W=\frac{V_0+V}{1-2s\kappa VV_0}. \tag{7}$$

Parameters $V$ and $V_0$ are linked in a way that is strongly reminiscent of the Lorentz velocity combination rule in relativistic kinematics [28]. While this correspondence is exact when $\mathrm{sgn}(s\kappa)=-1$, one should be mindful that $V_0$, $V$, and $W$ are physically related to *inverse* velocities.

It is interesting to note that one can define an invariant interval associated with Eq. (1) according to whether $s=\pm1$. For two points in the $(\tau,\zeta)$ plane separated by coordinate differences $\Delta\zeta$ and $\Delta\tau$, the invariant interval between these points is $\Delta\zeta^2/2s\kappa+\Delta\tau^2$. This numerical quantity is unchanged under transformation (6) with arbitrary $V$.

## III. BRIGHT SOLITON PULSES

### A. Symmetry reduction and quadrature equations

In the anomalous GVD regime (where $s=+1$), it is reasonable to expect families of exact analytical bright solitons to exist [5]. Furthermore, the cubic nonlinearity strongly suggests that one should be able to find sech-shaped pulses [8]. We begin our analysis by looking for solutions that have the quite general form $u(\tau,\zeta)=\rho^{1/2}(\tau,\zeta)\exp[i\Phi(\tau,\zeta)]\exp(-i\zeta/2\kappa)$, where $\rho(\tau,\zeta)\equiv|u(\tau,\zeta)|^2$ is the wave intensity, $\Phi(\tau,\zeta)$ is the phase distribution, and the complex-exponential $\exp(-i\zeta/2\kappa)$ is a manifestation of the underlying carrier wave [this contribution appears explicitly in the envelope solutions of Eq. (1)]. In the following analysis, we focus predominantly on regimes where the total spatial dispersion parameter is positive (i.e., where $\kappa>0$).

Substitution of $u$ into Eq. (1) and isolating the real and imaginary parts yields

$$\frac{2}{\rho}\left(\frac{\partial^2\rho}{\partial\tau^2}+2\kappa\frac{\partial^2\rho}{\partial\zeta^2}\right)-\frac{1}{\rho^2}\left[\left(\frac{\partial\rho}{\partial\tau}\right)^2+2\kappa\left(\frac{\partial\rho}{\partial\zeta}\right)^2\right] -4\left[\left(\frac{\partial\Phi}{\partial\tau}\right)^2+2\kappa\left(\frac{\partial\Phi}{\partial\zeta}\right)^2\right]-8\left(\alpha\frac{\partial\Phi}{\partial\tau}-\frac{1}{4\kappa}-\rho\right)=0, \tag{8a}$$

and

$$\rho\left(\frac{\partial^2\Phi}{\partial\tau^2}+2\kappa\frac{\partial^2\Phi}{\partial\zeta^2}\right)+\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\Phi}{\partial\tau}+2\kappa\frac{\partial\rho}{\partial\zeta}\frac{\partial\Phi}{\partial\zeta}\right)+\alpha\frac{\partial\rho}{\partial\tau}=0. \tag{8b}$$

Equations (8a) and (8b) can be simplified by seeking solitary solutions that have linear phase profiles; this amounts to setting $\Phi(\tau,\zeta)\equiv\Omega\tau+K\zeta$, where $K$ is the soliton propagation constant and $\Omega$ is the frequency shift (strictly, $\Omega$ is a normalized measure of the deviation of the pulse's center frequency from the carrier frequency). With such a choice, the second partial derivatives of $\Phi$ are zero, and Eqs. (8a) and (8b) may be recast as

$$\frac{2}{\rho}\left(\frac{\partial^2\rho}{\partial\tau^2}+2\kappa\frac{\partial^2\rho}{\partial\zeta^2}\right)-\frac{1}{\rho^2}\left[\left(\frac{\partial\rho}{\partial\tau}\right)^2+2\kappa\left(\frac{\partial\rho}{\partial\zeta}\right)^2\right]=8\beta-8\rho, \tag{9a}$$

and

$$(\alpha+\Omega)\frac{\partial\rho}{\partial\tau}+2\kappa K\frac{\partial\rho}{\partial\zeta}=0. \tag{9b}$$

Parameter $\beta$ in Eq. (9a) has been identified as

$$\beta\equiv\kappa K^2-\frac{1}{4\kappa}+\Omega\left(\alpha+\frac{\Omega}{2}\right), \tag{9c}$$

so that Eq. (9c) defines the soliton dispersion relation whose quadratic character is tightly connected with the presence of the $\kappa\partial^2u/\partial\zeta^2$ term in Eq. (1). Solving for $K$, one naturally obtains two roots,

$$K=\pm\frac{1}{2\kappa}\sqrt{1+4\kappa\beta-4\kappa\Omega\left(\alpha+\frac{\Omega}{2}\right)}, \tag{10}$$

where the $\pm$ sign flags propagation along $\pm\zeta$ [5,26].

By introducing the new space-time coordinate $\xi\equiv(\tau-W\zeta)/(1+2\kappa W^2)^{1/2}$, Eqs. (9a) and (9b) assume the canonical form

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\left(\frac{d\rho}{d\xi}\right)^2\right]=8\beta-8\rho, \tag{11a}$$

and

$$[(\alpha+\Omega)-2\kappa WK]\left(\frac{d\rho}{d\xi}\right)=0. \tag{11b}$$

For Eq. (11b) to hold for arbitrary gradients $d\rho/d\xi$, it must follow that $W=(\alpha+\Omega)/2\kappa K$. The symmetry reduction technique has, thus, transformed Eq. (1) into a simple quadrature equation [that is, Eq. (11a)]. Crucially, this ordinary differential equation has exactly the same formal structure as that obtained in conventional pulse theory. Bright ("bell-shaped") solitons are subject to vanishing-asymptotic boundary conditions of the form

$$\lim_{\xi\to0}\rho(\xi)=\rho_0, \tag{12a}$$

$$\lim_{\xi\to0}\frac{d}{d\xi}\rho(\xi)=0, \tag{12b}$$

$$\lim_{\xi\to\pm\infty}\rho(\xi)=0, \tag{12c}$$

$$\lim_{\xi\to\pm\infty}\frac{d}{d\xi}\rho(\xi)=0, \tag{12d}$$

which, when applied to Eq. (11a), cement the relationship $\beta\equiv\rho_0/2$ between parameter $\beta$ [defined in Eq. (9c)] and the pulse peak intensity $\rho_0$.

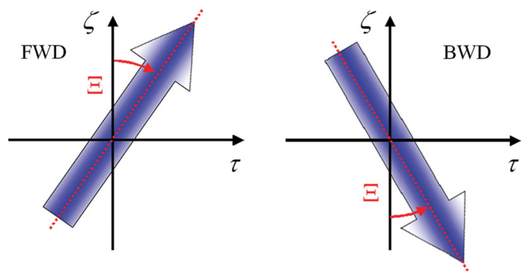


FIG. 1. (Color online) Schematic illustrating forward (FWD) and backward (BWD) pulses in the laboratory frame. For physically meaningful solutions, the FWD (BWD) soliton [upper (lower) signs in Eq. (13a)] must always span the first and third (second and fourth) quadrants of the $(\tau,\zeta)$ plane. This condition, which is captured by $W > 0$, ensures that the peak of the pulse is always moving forward in time. Dotted lines denote the trajectories $\tau \mp W\zeta = 0$.

Two integrations of Eq. (11a) yield $\rho(\xi) = \rho_0 \mathrm{sech}^2(\rho_0^{1/2}\xi)$, and hence, the two families of exact analytical bright solitons can be written as

$$u(\tau,\zeta) = \rho_0^{1/2}\mathrm{sech}\left(\rho_0^{1/2}\frac{\tau \mp W\zeta}{\sqrt{1+2\kappa W^2}}\right) \times \exp\left[i\Omega\tau \pm i\sqrt{1+4\kappa\beta - 4\kappa\Omega\left(\alpha + \frac{\Omega}{2}\right)}\frac{\zeta}{2\kappa}\right] \times \exp\left(-i\frac{\zeta}{2\kappa}\right), \tag{13a}$$

where $W$ is given by

$$W = \frac{\alpha + \Omega}{\sqrt{1+4\kappa\beta - 4\kappa\Omega\left(\alpha + \frac{1}{2}\Omega\right)}}. \tag{13b}$$

The upper (lower) signs correspond to pulses that are traveling in the forward (backward) longitudinal direction (see Fig. 1) [25].

For propagating solutions (i.e., real $K$ and no growth or evanescence in $\zeta$), $\Omega$ must lie within the band $\Omega_- < \Omega < \Omega_+$, where $\Omega_\pm = -\alpha \pm (\alpha^2 + 2\beta + 1/2\kappa)^{1/2}$. In addition, one must also have $W > 0$ for solutions traveling both forward and backward in space (thus, ensuring that the pulse is always moving forward in *time*). This latter condition amounts to $\Omega > -\alpha$. By combining these two simultaneous inequalities, it can be seen that physically meaningful solutions require $-\alpha < \Omega < \Omega_+$. However, for large frequency deviations (e.g., where $\Omega \to \Omega_+$), it should be borne in mind that the parabolic approximation invoked to arrive at the GVD-dominated temporal dispersion operator can become invalid.

### B. Space-time geometry of solitons

The forward-propagating soliton must form a stripe in the $(\tau,\zeta)$ plane that spans the first and third quadrants. The pulse, thus, travels through space-time along its world line $\tau - W\zeta = 0$ at speed $1/W$. This linear trajectory is inclined at angle $\Xi$ to the longitudinal $\zeta$ axis, where $\tan\Xi = -\partial K/\partial\Omega = W$ (see Fig. 1). When $\Omega = 0$, it follows that $W = V_0$, where

$$V_0 \equiv \frac{\alpha}{\sqrt{1+4\kappa\beta}}. \tag{14}$$

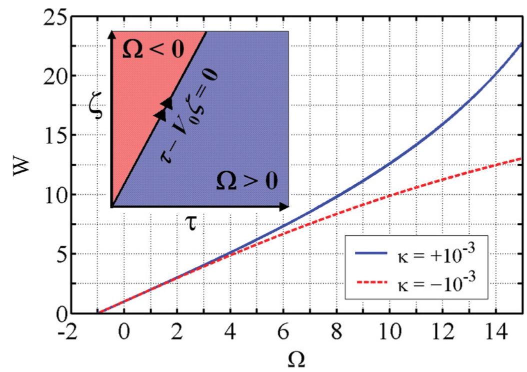


FIG. 2. (Color online) Plot of parameter $W$ [given by Eq. (12b) with $\alpha = 1.0$] as a function of frequency $\Omega$ for solitons with $\rho_0 = 1.0$ and for two (opposite-sign) values of the spatial dispersion parameter $\kappa$. Inset: space-time plane $(\tau,\zeta)$. Forward-propagating solitons with $\Omega = 0$ move along the line $\tau - V_0\zeta = 0$ at uniform speed $1/V_0$. Solutions with $\Omega < 0$ $(\Omega > 0)$ travel at faster (slower) speeds.

One therefore has a natural yardstick, namely $1/V_0 = (1 + 4\kappa\beta)^{1/2}\alpha^{-1}$, against which all other speeds may be measured. For fixed $\kappa\beta$, solitons with $\Omega < 0$ $(\Omega > 0)$ travel faster (slower) than $1/V_0$ because their world lines are associated with smaller (larger) gradients (see Fig. 2).

In the laboratory frame, the duration of the soliton is $\Delta\tau = (1 + 2\kappa W^2)^{1/2}\Delta\tau_0$, where $\Delta\tau_0 \equiv 2\rho_0^{-1/2}$ is the duration in the rest frame [whose coordinates $(\tau_0,\zeta_0)$ can be obtained from Eqs. (6a) and (6b)]. By substituting for $W$ from Eq. (13b), it can be shown that

$$\frac{\Delta\tau}{\Delta\tau_0} = \left[\frac{1+4\kappa\beta+2\kappa\alpha^2}{1+4\kappa\beta-4\kappa\Omega\left(\alpha+\frac{1}{2}\Omega\right)}\right]^{1/2}. \tag{15}$$

This key result predicts how the pulse duration depends on the system and solution parameters. When $\kappa < 0$, one finds that $\Delta\tau_0 > \Delta\tau$, whereas, when $\kappa > 0$, it follows that $\Delta\tau_0 < \Delta\tau$. Neither effect appears in conventional theory (see Fig. 3), which constitutes the Galilean limit.

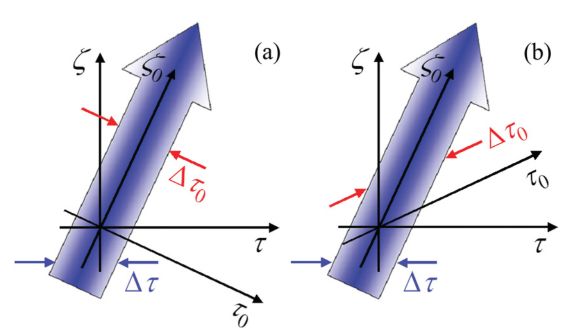


FIG. 3. (Color online) Schematic for a forward-propagating pulse illustrating the cases of (a) $\mathrm{sgn}(s\kappa) = +1$ and (b) $\mathrm{sgn}(s\kappa) = -1$. The "0" subscripted coordinates $(\tau_0,\zeta_0)$ define the rest frame (i.e., the frame in which the pulse is stationary and, hence, travels along the $\zeta_0$ axis); these coordinates can be obtained from Eqs. (6a) and (6b) by selecting $W$ as the velocity parameter. Note that the pulse width must be measured at a fixed longitudinal position (so that $\Delta\zeta = \Delta\zeta_0 = 0$).

### C. Velocity representation

Equation (7) is a geometric relation that has been derived solely on the basis of coordinate transformations (i.e., it is independent of system nonlinearity, and has been obtained without recourse to any particular solution). By applying transformation (6) to the forward and backward $\Omega = 0$ solutions in Eqs. (13a) and (13b), one generates an alternative pair of pulse solutions that may be combined into

$$u(\tau,\zeta) = \rho_0^{1/2}\mathrm{sech}\left(\rho_0^{1/2}\frac{\tau \mp W\zeta}{\sqrt{1+2\kappa W^2}}\right) \times \exp\left[i\sqrt{\frac{1+4\kappa\beta}{1+2\kappa V^2}}\left(V\tau \pm \frac{\zeta}{2\kappa}\right)\right] \times \exp\left(i\alpha\frac{\tau \mp V\zeta}{\sqrt{1+2\kappa V^2}} - i\alpha\tau\right)\exp\left(-i\frac{\zeta}{2\kappa}\right), \tag{16}$$

where $W$ is given by Eq. (7) (with $s = +1$). We refer to this as the velocity representation.

Solitons (13a) and (16) represent the same physical pulse when the frequency shift $\Omega$ and velocity $V$ are connected by

$$\Omega(V) = V\sqrt{\frac{1+4\kappa\beta}{1+2\kappa V^2}} + \alpha\left(\frac{1}{\sqrt{1+2\kappa V^2}} - 1\right). \tag{17a}$$

This mapping is most easily obtained by equating the phase slopes of the two representations. It then follows that the expressions for $W$ given in Eqs. (7) and (13b) are equivalent. After some algebra, it can be shown that

$$V(\Omega) = \frac{(\Omega+\alpha)\sqrt{1+4\kappa\beta-4\kappa\Omega\left(\alpha+\frac{1}{2}\Omega\right)} - \alpha\sqrt{1+4\kappa\beta}}{1+4\kappa\beta-2\kappa(\Omega+\alpha)^2}. \tag{17b}$$

Equation (17b) is entirely consistent with Eqs. (7) and (13b). For example, when $\Omega = 0$, one recovers $V = 0$ and, hence, $W = V_0$.

### D. Rest frames vs laboratory frames

We now consider what happens when performing a linear boost to $(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}}) = (\tau - \alpha\zeta,\zeta)$ coordinates in the forward soliton (13a) with $\Omega = 0$. This procedure generates a solution $u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$ that satisfies Eq. (3),

$$u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}}) = \rho_0^{1/2}\mathrm{sech}\left(\rho_0^{1/2}\frac{\tau_{\mathrm{loc}} - W_{\mathrm{loc}}\zeta_{\mathrm{loc}}}{\sqrt{1+2\kappa V_0^2}}\right) \times \exp\left[i\sqrt{1+4\kappa\beta}\frac{\zeta_{\mathrm{loc}}}{2\kappa}\right]\exp\left(-i\frac{\zeta_{\mathrm{loc}}}{2\kappa}\right), \tag{18}$$

and whose net velocity is $W_{\mathrm{loc}} \equiv V_0 - \alpha = \alpha[(1+4\kappa\beta)^{-1/2} - 1]$. Thus, boosting to the Galilean local time frame used in conventional theory (see Sec. II B) cannot result in a stationary pulse (i.e., one where $W_{\mathrm{loc}} = 0$) of finite amplitude unless $\kappa\beta \to 0$.

One can always describe a soliton in its rest frame (see Fig. 3), whose coordinates are denoted by $(\tau_0,\zeta_0)$. As a simple illustrative example, consider the $\Omega = 0$ forward soliton. Transforming to the rest frame using Eqs. (6a) and (6b) [where the velocity parameter is given by Eq. (14)], the partial-differential operators become $\partial/\partial\tau = (1+2s\kappa V_0^2)^{-1/2}(2s\kappa V_0\partial/\partial\zeta_0 + \partial/\partial\tau_0)$ and $\partial/\partial\zeta = (1+2s\kappa V_0^2)^{-1/2}(\partial/\partial\zeta_0 - V_0\partial/\partial\tau_0)$. The equation that $u(\tau_0,\zeta_0)$ satisfies is then free of mixed derivatives [unlike Eq. (3)], but it is still more complicated than the original model [Eq. (1)]. More precisely, the combination of second-order derivatives is covariant, i.e.,

$$\kappa\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\zeta} + \frac{s}{2}\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau} = \kappa\frac{\partial}{\partial\zeta_0}\frac{\partial}{\partial\zeta_0} + \frac{s}{2}\frac{\partial}{\partial\tau_0}\frac{\partial}{\partial\tau_0}, \tag{19a}$$

but the combination of first-order derivatives transforms according to

$$\frac{\partial}{\partial\zeta} + \alpha\frac{\partial}{\partial\tau} = \frac{1}{\sqrt{1+2s\kappa V_0^2}} \times \left[(1+2s\kappa\alpha V_0)\frac{\partial}{\partial\zeta_0} + (\alpha - V_0)\frac{\partial}{\partial\tau_0}\right]. \tag{19b}$$

In systems with spatiotemporal dispersion, the notion of rest frames also involves an additional subtlety. It follows that $(\tau_0,\zeta_0)$ strictly defines the rest frame of only a subset of $\Omega = 0$ solitons—those solutions parametrized by fixed $\kappa\beta$. In other words, the rest frame of the $\Omega = 0$ solution depends explicitly on the pulse peak amplitude (through the dependence of $V_0$ on $\beta$). To avoid these sorts of complexities, it is more straightforward to consider soliton pulses in the laboratory frame.

### E. Conservation laws

Model (1) and its complex conjugate can be regarded as the Euler-Lagrange equations for a Lagrangian density $\mathscr{L}$, where

$$\mathscr{L} = \frac{i}{2}\left[u^*\left(\frac{\partial u}{\partial\zeta} + \alpha\frac{\partial u}{\partial\tau}\right) - u\left(\frac{\partial u^*}{\partial\zeta} + \alpha\frac{\partial u^*}{\partial\tau}\right)\right] - \frac{s}{2}\frac{\partial u^*}{\partial\tau}\frac{\partial u}{\partial\tau} - \kappa\frac{\partial u^*}{\partial\zeta}\frac{\partial u}{\partial\zeta} + \frac{1}{2}|u|^4. \tag{20a}$$

By identifying the canonically conjugate momenta as

$$\pi \equiv \frac{\partial\mathscr{L}}{\partial(\partial_\zeta u)} = \left(\frac{i}{2} - \kappa\frac{\partial}{\partial\zeta}\right)u^*, \tag{20b}$$

$$\tilde{\pi} \equiv \frac{\partial\mathscr{L}}{\partial(\partial_\zeta u^*)} = -\left(\frac{i}{2} + \kappa\frac{\partial}{\partial\zeta}\right)u, \tag{20c}$$

and deploying standard field-theoretic techniques [28], the following three conserved quantities can be derived on the basis of Noether's theorem:

$$J = \int_{-\infty}^{+\infty} d\tau\left[|u|^2 - i\kappa\left(u^*\frac{\partial u}{\partial\zeta} - u\frac{\partial u^*}{\partial\zeta}\right)\right], \tag{21a}$$

$$M = \int_{-\infty}^{+\infty} d\tau \left[ \frac{i}{2}\left( u^* \frac{\partial u}{\partial \tau} - u \frac{\partial u^*}{\partial \tau} \right) - \kappa \left( \frac{\partial u^*}{\partial \zeta}\frac{\partial u}{\partial \tau} + \frac{\partial u^*}{\partial \tau}\frac{\partial u}{\partial \zeta} \right) \right], \tag{21b}$$

$$H = \int_{-\infty}^{+\infty} d\tau \left[ \frac{s}{2}\frac{\partial u^*}{\partial \tau}\frac{\partial u}{\partial \tau} - \kappa \frac{\partial u^*}{\partial \zeta}\frac{\partial u}{\partial \zeta} - \frac{i\alpha}{2}\left( u^* \frac{\partial u}{\partial \tau} - u \frac{\partial u^*}{\partial \tau} \right) - \frac{1}{2}|u|^4 \right]. \tag{21c}$$

The integrals in Eqs. (21a)–(21c) represent the pulse energy flow, momentum, and Hamiltonian, respectively; they are conserved in the sense that $dJ/d\zeta = 0$, $dM/d\zeta = 0$, and $dH/d\zeta = 0$. By substituting $u(\tau,\zeta) = \rho^{1/2}(\tau,\zeta)\exp[i\Phi(\tau,\zeta)]\exp(-i\zeta/2\kappa)$ into Eqs. (21a)–(21c) and applying boundary conditions (12a)–(12d), it can be shown that when $s = +1$,

$$J = \pm\sqrt{1 + 4\kappa\beta + 2\kappa\alpha^2}\,P, \tag{22a}$$

$$M = -\Omega J \pm \frac{2\kappa\,(\alpha+\Omega)\,Q}{\sqrt{1+4\kappa\beta+2\kappa\alpha^2}}, \tag{22b}$$

$$H = \frac{J}{2\kappa} - \left(\frac{1}{2\kappa}\right)\sqrt{1+4\kappa\beta-4\kappa\Omega\left(\alpha+\frac{\Omega}{2}\right)} \times\left[\sqrt{1+4\kappa\beta+2\kappa\alpha^2}\,P - \frac{2\kappa Q}{\sqrt{1+4\kappa\beta+2\kappa\alpha^2}}\right], \tag{22c}$$

where the integrals $P$ and $Q$ (which have positive-definite integrands) are defined by

$$P \equiv \int_{-\infty}^{+\infty} d\xi\ \rho(\xi), \tag{22d}$$

$$Q \equiv \frac{1}{4}\int_{-\infty}^{+\infty} d\xi \frac{1}{\rho(\xi)}\left[\frac{d}{d\xi}\rho(\xi)\right]^2. \tag{22e}$$

For Kerr bright solitons, it can be shown that $P = 2\rho_0^{1/2}$ and $Q = (2/3)\rho_0^{3/2}$.

### F. Cnoidal waves

So far, we have derived and have analyzed the exact bright solitons of Eq. (1), which describe isolated structures in the time domain. For completeness, we now report the existence of two classes of cnoidal waves when $s = +1$. These solutions are given by Jacobi elliptic functions of the first kind and describe periodic trains of similar nondispersive pulses. The first solution class is

$$u(\tau,\zeta) = m\rho_0^{1/2}\,\mathrm{cn}\left(\rho_0^{1/2}\frac{\tau \mp W_{\mathrm{cn}}\zeta}{\sqrt{1+2\kappa W_{\mathrm{cn}}^2}};m\right) \times \exp\left[i\Omega\tau \pm i\sqrt{1+4\kappa\beta_{\mathrm{cn}}-4\kappa\Omega\left(\alpha+\frac{\Omega}{2}\right)}\frac{\zeta}{2\kappa}\right] \times\exp\left(-i\frac{\zeta}{2\kappa}\right), \tag{23a}$$

where $\beta_{\mathrm{cn}} \equiv \rho_0(m^2 - 1/2)$, whereas, the second class is

$$u(\tau,\zeta) = \rho_0^{1/2}\mathrm{dn}\left(\rho_0^{1/2}\frac{\tau \mp W_{\mathrm{dn}}\zeta}{\sqrt{1+2\kappa W_{\mathrm{dn}}^2}};m\right) \times \exp\left[i\Omega\tau \pm i\sqrt{1+4\kappa\beta_{\mathrm{dn}}-4\kappa\Omega\left(\alpha+\frac{\Omega}{2}\right)}\frac{\zeta}{2\kappa}\right] \times\exp\left(-i\frac{\zeta}{2\kappa}\right), \tag{23b}$$

where now $\beta_{\mathrm{dn}} \equiv \rho_0(1 - m^2/2)$. For both cn and dn solutions (which could also be written in the velocity representation of Sec. III C), $W \equiv W_{\{\mathrm{cn,dn}\}}$ is given by Eq. (13b) when $\beta = \beta_{\{\mathrm{cn,dn}\}}$. It is noteworthy that, when $\alpha = 0$, the two cnoidal-wave solutions satisfy the spatial nonlinear Helmholtz equation [5], where $\tau$ [in Eq. (1)] represents the transverse spatial coordinate.

Wave trains (23a) and (23b) are parametrized by the modulus $m$, where $0 \leqslant m \leqslant 1$ controls the temporal period between successive pulses. In the limit that $m \to 1$, the period of the cn and dn functions diverges, and one recovers the sech soliton [see solution (13a)]. As $m \to 0$, the cn wave train tends toward a cosine modulation (of vanishingly small amplitude), whereas, the dn wave train asymptotes to a flat (continuous-wave) solution.

## IV. CONVENTIONAL BRIGHT SOLITONS AND CNOIDAL WAVES

A systematic analysis of pulses of a generic nonlinear wave equation with spatiotemporal dispersion has been developed throughout this section. We now show that, in a simultaneous multiple limit, the predictions of conventional pulse theory can be recovered. Intuitively, one should expect to find this type of asymptotic convergence—conventional pulses must emerge in regimes where the SVEA is valid.

When all contributions from the $\kappa\partial^2/\partial\zeta^2$ operator can be neglected simultaneously [4,5,26,27], Eq. (1) may be replaced by the (approximate) parabolic model [8,9,17],

$$i\left(\frac{\partial u}{\partial \zeta} + \alpha\frac{\partial u}{\partial \tau}\right) + \frac{s}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u \simeq 0. \tag{24}$$

In the limit that $\kappa V^2 \to 0$, one recovers the transformation laws for Eq. (24) from Eqs. (6a)–(6c),

$$\tau \simeq \tau' - V\zeta', \tag{25a}$$

$$\zeta \simeq \zeta', \tag{25b}$$

$$u(\tau,\zeta) \simeq \exp\left[-isV\tau' + is\left(\frac{V^2}{2} + \alpha V\right)\zeta'\right]u'(\tau',\zeta'). \tag{25c}$$

One also finds the classic Galilean velocity combination rule $W \simeq V_0 + V$ emerging from Eq. (7).

### A. Bright solitons

Enforcing the fourfold *simultaneous* limit $\kappa \to 0$ (negligible spatial dispersion), $\kappa\beta \to 0$ (negligible nonlinear phase shift), $\kappa\Omega(\alpha + \Omega/2) \to 0$ (negligible frequency shift), and $\kappa W^2 \to 0$ in solution (13a) leads to the approximate result,

$$\begin{aligned} u(\tau,\zeta) \simeq \rho_0^{1/2}\mathrm{sech}\left[\rho_0^{1/2}(\tau \mp W\zeta)\right] \\ \times \exp\left[i\Omega(\tau \mp \alpha\zeta) \pm i\left(\beta - \frac{\Omega^2}{2}\right)\zeta\right] \\ \times \exp\left[-i(1\mp 1)\frac{\zeta}{2\kappa}\right], \end{aligned} \tag{26}$$

where $W \simeq \alpha + \Omega$. It is notable that a $\kappa$-dependent rapid phase factor $\exp[-i2(\zeta/2\kappa)]$ survives the limit process for the backward waves. Hence, the backward solutions in Eq. (26) cannot satisfy Eq. (24). This situation points to the intrinsic unidirectionality of the conventional model, which describes evolution along a single longitudinal direction only. Under a Galilean boost to local coordinates $(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$, Eq. (24) transforms into Eq. (2), as discussed in Sec. I, and the familiar Kerr soliton emerges from the forward solution [9],

$$\begin{aligned} u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}}) \simeq \rho_0^{1/2}\mathrm{sech}\left[\rho_0^{1/2}(\tau_{\mathrm{loc}} - \Omega\zeta_{\mathrm{loc}})\right] \\ \times \exp\left[i\Omega\tau_{\mathrm{loc}} + i\left(\beta - \frac{\Omega^2}{2}\right)\zeta_{\mathrm{loc}}\right]. \end{aligned} \tag{27}$$

When $\Omega = 0$, soliton (27) is stationary in the local frame, *irrespective of* $\beta$. That is, $(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$ defines the rest frame of *all* $\Omega = 0$ solutions, whose peak always sits at $\tau_{\mathrm{loc}} = 0$ [recall from Sec. III D, that there is no such universality for solitons of Eq. (1)]. For the forward pulse, the conserved quantities given in Eqs. (22a)–(22c) asymptote to $J \simeq P$, $M \simeq -\Omega P$, and $H \simeq \Omega(\alpha + \Omega/2)P - \beta P + Q$; for the backward pulse, one obtains $J \simeq -P$, $M \simeq +\Omega P$, and $H \simeq \Omega(\alpha + \Omega/2)P - 3\beta P + Q - (\alpha^2 + 1/\kappa)P$. We note that, in the latter case, $H$ contains a term at $O(|\kappa|^{-1})$ and so is formally singular.

### B. Cnoidal waves

The same formal limit procedure that we applied to soliton (13a) can now be used to asymptote cnoidal waves (23a) and (23b), in which case, one obtains

$$\begin{aligned} u(\tau,\zeta) \simeq \rho_0^{1/2}\begin{Bmatrix} m\ \mathrm{cn} \\ \mathrm{dn}\end{Bmatrix}\left[\rho_0^{1/2}(\tau \mp W\zeta);m\right] \\ \times \exp\left[i\Omega(\tau \mp \alpha\zeta) \pm i\left(\beta_{\{\mathrm{cn,dn}\}} - \frac{\Omega^2}{2}\right)\zeta\right] \\ \times \exp\left[-i(1\mp 1)\frac{\zeta}{2\kappa}\right]. \end{aligned} \tag{28a}$$

The forward solutions converge to their conventional counterparts, as they should, and in the $(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$ frame, one finds that [29,30]

$$\begin{aligned} u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}}) \simeq \rho_0^{1/2}\begin{Bmatrix} m\ \mathrm{cn} \\ \mathrm{dn}\end{Bmatrix}\left[\rho_0^{1/2}(\tau_{\mathrm{loc}} - \Omega\zeta_{\mathrm{loc}})\right] \\ \times \exp\left[i\Omega\tau_{\mathrm{loc}} + i\left(\beta_{\{\mathrm{cn,dn}\}} - \frac{\Omega^2}{2}\right)\zeta_{\mathrm{loc}}\right]. \end{aligned} \tag{28b}$$

As expected, the backward cnoidal waves have no analog in this frame.

A similar handling of the velocity representation of solutions is also possible, whereupon one finds exactly the same algebraic results as in Eqs. (26)–(28b) but with $\Omega$ replaced by $V$. This follows directly from Eqs. (17a) and (17b), which show that $\Omega \simeq V$. That is, frequency shifts and velocities are completely interchangeable in conventional theory since they have the same mathematical status under the SVEA. However, the same is clearly not true in the (more general) spatiotemporal formalism where the connection between $\Omega$ and $V$ is more intricate. It can also be shown, from Eq. (14), that the intrinsic velocity is $V_0 \simeq \alpha$.

### C. A note on soliton momentum

Finally, we note some interesting observations about the structure and interpretation of conserved quantities in both spatiotemporal and conventional pulse models. Equation (22c) shows that spatiotemporal bright solitons possess a nonvanishing Hamiltonian when $\Omega = 0$; this quantity may be interpreted as a zero-point energy. Equation (22b) reveals a more intuitive symmetry between forward and backward solutions, namely, that they have opposite momenta. Like $H$, the invariant $M$ is nonzero when $\Omega = 0$, and it is tempting to interpret this zero-point momentum physically through frame-of-reference considerations: Solutions (13a) and (16) describe pulses that are moving with respect to the laboratory frame, and such relative motion may be associated with a nonvanishing momentum.

To highlight the limitations of the standard analogy with particle mechanics, it is instructive to first consider conventional solutions with $\Omega = 0$ [see Eq. (26)] and that are, hence, moving in the forward direction with $W = V_0 = \alpha$ in the laboratory frame. Although one naturally expects such solitons to have zero momentum in their rest frame [which, recall, is identical to the $(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$ coordinates for the subset of $\Omega = 0$ solutions], analysis shows that their momentum is zero *even in the laboratory frame* (i.e., *before* one transforms to the rest frame). Hence, one may conclude that the zero-point momentum is eliminated by the SVEA, rather than by the Galilean boost.

One can extend these considerations to the corresponding spatiotemporal soliton, which has momentum $M = +2\kappa\alpha Q/(1 + 4\kappa\beta + 2\kappa\alpha^2)^{1/2}$ [see Eq. (22b)]. Coordinate transformation Eqs. (6a) and (6b) can be used to show that, in its rest frame, this soliton is represented by

$$\begin{aligned} u(\tau_0,\zeta_0) = \rho_0^{1/2}\mathrm{sech}\left(\rho_0^{1/2}\tau_0\right) \\ \times \exp\left[i(\Omega_0\tau_0 + K_0\zeta_0)\right]\exp\left(-i\frac{\zeta_0}{2\kappa}\right), \end{aligned} \tag{29a}$$

where the frequency shift $\Omega_0$ and propagation constant $K_0$ are given by

$$\Omega_0 \equiv -\frac{V_0}{\sqrt{1+2\kappa V_0^2}}(\sqrt{1+4\kappa\beta}-1), \tag{29b}$$

and

$$K_0 \equiv \frac{1}{2\kappa}\left[\frac{1}{\sqrt{1+2\kappa V_0^2}}(\sqrt{1+4\kappa\beta}-1)+1\right]. \tag{29c}$$

Solution (29a) satisfies a governing equation that can be obtained as described in Sec. III D; that equation and its complex conjugate correspond to the Euler-Lagrange equations for a Lagrangian density,

$$\begin{aligned}\mathscr{L}_0 = \frac{i}{2}\frac{1}{\sqrt{1+2s\kappa V_0^2}}&\left[(1+2s\kappa\alpha V_0)\left(u^*\frac{\partial u}{\partial\zeta_0}-u\frac{\partial u^*}{\partial\zeta_0}\right)\right.\\ &\left.+(\alpha-V_0)\left(u^*\frac{\partial u}{\partial\tau_0}-u\frac{\partial u^*}{\partial\tau_0}\right)\right]\\ &-\frac{s}{2}\frac{\partial u^*}{\partial\tau_0}\frac{\partial u}{\partial\tau_0}-\kappa\frac{\partial u^*}{\partial\zeta_0}\frac{\partial u}{\partial\zeta_0}+\frac{1}{2}|u|^4.\end{aligned} \tag{30}$$

This expression for $\mathscr{L}_0$ can be obtained by transforming Eq. (20a) with Eqs. (19a) and (19b). Subsequent analysis following the methods in Sec. III E reveals that the momentum of soliton (29a) is $M_0 = \alpha[(1+4\kappa\beta)^{1/2}-1]P$, where $P = 2\rho_0^{1/2}$. Thus, in its rest frame, the forward spatiotemporal soliton is associated with a momentum $M_0$ that is nonvanishing unless $\kappa\beta \to 0$ (which, recall, is one contribution to the SVEA).

A natural conclusion to draw from these frame-of-reference considerations is that the canonical field momentum [e.g., the $M$ integral defined in Eq. (21b)] for continuously distributed objects in space-time cannot generally be considered analogous to the kinematic momentum of a point particle in classical mechanics.

## V. STABILITY OF BRIGHT SOLITON PULSES

### A. Stability criterion

The stability properties of pulses are usually analyzed in the local time frame where the conventional governing equation has the NLS form [see Eq. (2)]. It is well known that a localized solution of this equation is robust against small perturbations if its power $P(\beta)$ satisfies the inequality,

$$\frac{dP}{d\beta} \equiv \frac{d}{d\beta}\int_{-\infty}^{+\infty} d\tau_{\mathrm{loc}}\,|u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})|^2 > 0, \tag{31}$$

where $P$ is the integral quantity on the right-hand side of Eq. (31). This inequality, which is the Vakhitov-Kolokolov (VK) integral criterion [31], tends to be a necessary but not sufficient condition for predicting stability [32]. For exact soliton (27), it follows that $P(\beta) = 2(2\beta)^{1/2}$, and so the VK condition is always satisfied.

Physical symmetry principles demand that instability cannot arise as a consequence of changing one's coordinate system. For example, if a solution $u(\tau_{\mathrm{loc}},\zeta_{\mathrm{loc}})$ of Eq. (2) is found to be robust against perturbations, it follows that the corresponding solution $u(\tau-\alpha\zeta,\zeta)$ of Eq. (24) must be equally robust. In other words, one cannot have a solution that is stable in the local time frame and, simultaneously, unstable in the laboratory frame. By extension, if a solution of the conventional model [Eq. (24)] is stable, one might expect the spatiotemporal generalization of that solution to also be stable (at least, if the $\kappa\partial^2u/\partial\zeta^2$ contribution is small). As these arguments are not conclusive and, in any case, would only consider stability with respect to small perturbations, we complement these considerations with full numerical investigations.

### B. Initial-value problems

The stability of the new bright solitons is now addressed computationally through an initial-value problem. The input pulse for Eq. (1) is chosen to be

$$u(\tau,0) = \rho_0^{1/2}\mathrm{sech}(\rho_0^{1/2}\tau)\exp(i\Omega\tau), \tag{32}$$

which corresponds to exact solution (26) of the conventional pulse model. The following selection of simulations, thus, addresses the system dynamics when the injected wave form does not take full account of the $\kappa\partial^2/\partial\zeta^2$ operator. By using Eqs. (6a)–(6c) and (17b) to transform to the rest frame of the input pulse, it can be seen that initial condition (32) corresponds to a solution whose width deviates from the value required for an exact soliton by the factor $(1+2\kappa W^2)^{1/2}$ [26]. When $|\kappa| \ll O(1)$ and $|\kappa|\alpha^2 \ll O(1)$, the asymptotic parameters (amplitude, width, and area) of any emergent solitons can be predicted using inverse-scattering techniques [14] (though one must then be mindful to transform back to the laboratory frame).

Equation (1) is integrated numerically using a direct generalization of the difference-differential algorithm [33] to allow for $i\alpha\,\partial/\partial\tau$ in the linear wave operator. This additional operation can be implemented using fast Fourier transforms and results in a negligible increase in computational overheads. In regimes where $\kappa < 0$ and $s = +1$, the governing equation has a hyperbolic structure, but computations can still be performed using essentially the same method.

### C. Bright solitons as robust attractors

Results are first presented from a range of simulations when $\alpha = 1.0$, $\Omega = 5$, 10, and 15 and where $\kappa = +10^{-3}$. Since $\kappa > 0$, the duration of the input pulse (hence, its power $P$) is *reduced* relative to the exact (i.e., the unperturbed) solution. One, therefore, expects to find qualitatively similar self-reshaping characteristics to those uncovered for Helmholtz-Kerr spatial solitons [26] (see Fig. 4). The pulse parameters exhibit monotonically decaying oscillations that tend to vanish as $\zeta \to \infty$, leaving a stationary state (i.e., an exact spatiotemporal pulse). When $\kappa$ is decreased (for instance, by a factor of 10), the normalized reshaping curves shown in Fig. 4 are nearly unchanged. However, the longitudinal scaling can be such that the oscillations take place over a greater propagation length in the unscaled coordinate $z$ [26].

When $\kappa < 0$, the input pulse duration and power are increased relative to the unperturbed solution. One, therefore, expects the nonlinearity to dominate the initial stages of evolution and, accordingly, for the peak amplitude to increase.

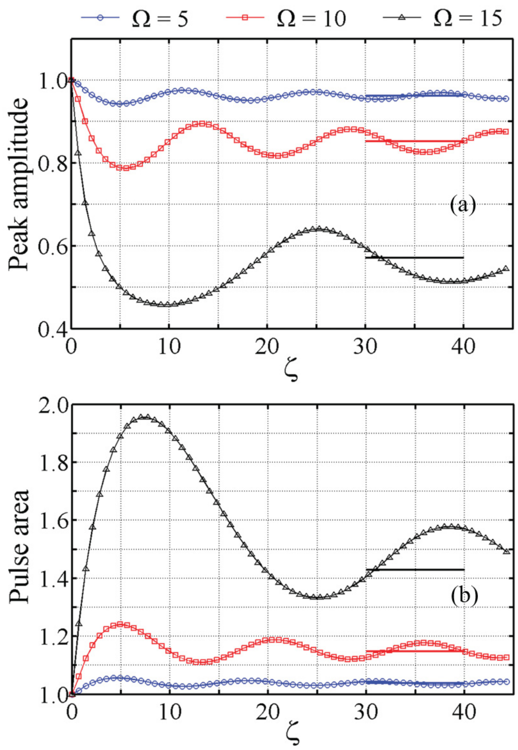


FIG. 4. (Color online) Self-reshaping of a unit-amplitude pulse ($\rho_0 = 1.0$) from initial condition (32) where $\kappa = +10^{-3}$ and $\alpha = 1.0$. The peak amplitude decreases, and the pulse width tends to increase in such a way that the area increases. Horizontal bars denote the asymptotic soliton parameters as predicted by inverse-scattering techniques [14].

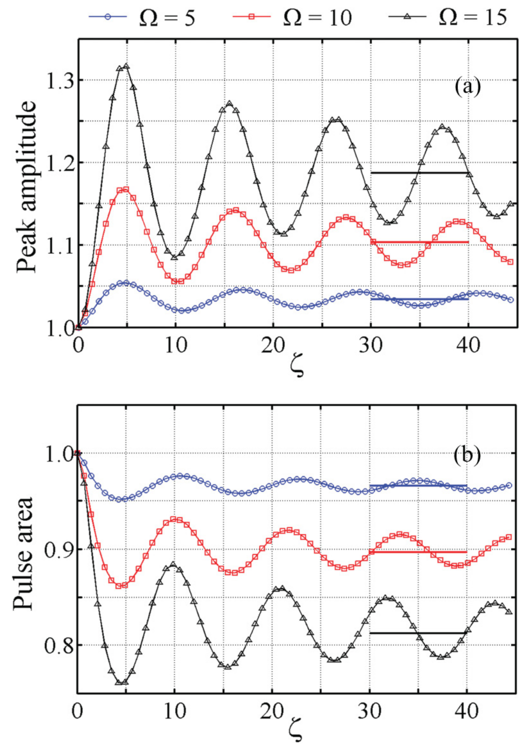


FIG. 5. (Color online) Self-reshaping of a unit-amplitude pulse ($\rho_0 = 1.0$) from initial condition (32) where $\kappa = -10^{-3}$ and $\alpha = 1.0$. The peak amplitude increases, and the pulse width tends to decrease in such a way that the area decreases. Horizontal bars denote the asymptotic soliton parameters as predicted by inverse-scattering techniques [14].

Results are shown in Fig. 5 using the same values of $\Omega = 5$, 10, and 15 but with $\kappa = -10^{-3}$. The reshaping pulses still exhibit their monotonically decaying oscillations as they evolve toward stationary solutions of Eq. (1), but there are qualitative differences. For instance, increasing the strength of the perturbation tends to result in an asymptotic pulse whose area is progressively less than that of the input pulse.

## VI. CONCLUSIONS

We have proposed a generalization of the classic nonlinear Schrödinger pulse equation, which is based on a formalism that describes spatiotemporal dispersion. By retaining a more complete description of the linear wave operator, we have been able to apply the mathematical tools and computational techniques of Helmholtz spatial solitons to phenomena in the time domain. While the essence of this approach was suggested some three decades ago [34], it appears to have received little subsequent attention in the literature. One of the surprising outcomes of this research is just how much progress can be made when abandoning the SVEA. A host of exact analytical results has been obtained, including transformation laws, exact analytical solutions (stationary isolated pulses and extended wave trains), and conserved quantities (integral and algebraic forms). We have also uncovered a general velocity combination rule, identified the invariant interval, and analyzed the characteristics of soliton rest frames. Crucially, the predictions of conventional nonlinear pulse theory are recovered in an appropriate limit. One important aspect not discussed here is the stability of the spatiotemporal cnoidal-wave solutions (22a) and (22b) [35]; such considerations are reserved for future research.

Extensive simulations, in parallel with the VK criterion [31] and traditional inverse-scattering techniques [14], have established that the stability properties of spatiotemporal solitons are similar to those of their nonlinear beam counterparts [26]. These new solutions have been shown to behave as robust attractors in the face of significant temporal shape perturbations, and their innate stability appears insensitive to the *sign* of the $\kappa$ parameter.

The analysis presented in this paper has a wide appeal, particularly within the arena of universal soliton-supporting evolution equations. One avenue to explore is the case of more involved nonlinearities, such as cubic-quintic [36] and saturable [37]; another is the systematic generalization of a whole range of pulse models, including the classic equations of Manakov [12], Hirota [38], Kaup and Newell [39], Davey-Stewartson [40], and many more besides (e.g., the real-valued nonlinear Klein-Gordon equation) [41]. Pulse interaction geometries [42] in coupled nonlinear systems with spatiotemporal dispersion are also of intrinsic interest [43]. Although our research may find application in the field of spatially dispersive waveguide optics [23], we believe that the effects uncovered here are generic in nature [4] and that the modeling approach will be applicable to wave-propagation problems in other dispersive nonlinear systems.

## APPENDIX

### 1. Application to waveguide optics

Historically, the theory of optical pulses has been firmly rooted in the nonlinear Schrödinger setting with all its advantages and disadvantages [1–3,8–19]. In the classic scalar-wave optics approach, the transverse spatial profile of the electric field $\mathbf{E}$ is confined by the structure of the waveguide and the polarization scrambling term in Maxwell's equations, namely, $\nabla(\nabla\cdot\mathbf{E})$ can be safely neglected. One proceeds by seeking pulse solutions that have the form

$$E(t,z) = A(t,z)\exp\left[i\left(k_0 z - \omega_0 t\right)\right] + A^*(t,z)\exp\left[-i\left(k_0 z - \omega_0 t\right)\right], \tag{A1}$$

where $z$ is the longitudinal coordinate along the axis of the waveguide and $t$ is the time coordinate. The envelope is $A(t,z)$, $\omega_0$ is the optical carrier frequency, and $k_0 \equiv n_0\omega_0/c$ is the propagation constant. Here, $n_0$ is the linear refractive index of the medium (at frequency $\omega_0$), and $c$ is the vacuum speed of light.

By substituting field ansatz (A1) into the corresponding Maxwell equations and transforming to the frequency domain, it can be shown that [8–19]

$$\frac{\partial^2\tilde{A}}{\partial z^2} + i2k_0\frac{\partial\tilde{A}}{\partial z} + \left(k^2 - k_0^2\right)\tilde{A} = 0, \tag{A2}$$

where $\tilde{A} \equiv \tilde{A}(\omega-\omega_0,z)$ denotes the Fourier transform of the pulse envelope and $k^2$ is the mode eigenvalue (obtained by solving Maxwell's equations for the transverse part of the confined field—see Ref. [18]). The term $(k^2 - k_0^2)$ is routinely approximated by $2k_0(k - k_0)$, and assuming that $\tilde{A}$ remains centered within the vicinity of $\omega_0$, $k(\omega)$ is Taylor expanded around $\omega_0$ so that

$$k(\omega-\omega_0) - k_0 = \sum_{j=1}^{\infty}\frac{k_j}{j!}(\omega-\omega_0)^j + \Delta k_{\mathrm{NL}}, \tag{A3}$$

where $k_0 \equiv k(\omega_0)$ and $k_j \equiv (\partial^j k/\partial\omega^j)_{\omega=\omega_0}$ for $j = 1, 2, 3,\ldots$. The (small) nonlinear correction to $k$ is taken to be $\Delta k_{\mathrm{NL}} = (\omega_0/c)n_2 I$, where $n_2$ is the Kerr coefficient and $I$ is the intensity. By keeping the first two terms in the summation on the right-hand side of Eq. (A3) and Fourier transforming back to the time domain using the prescription $(\omega-\omega_0)^j = (i\partial/\partial t)^j$, it can be shown that $A(t,z)$ satisfies the wave equation,

$$\frac{1}{2k_0}\frac{\partial^2 A}{\partial z^2} + i\left(\frac{\partial A}{\partial z} + k_1\frac{\partial A}{\partial t}\right) - \frac{k_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\omega_0}{c}n_2|A|^2 A = 0. \tag{A4}$$

The coefficients,

$$k_1 \equiv \left(\frac{\partial k}{\partial\omega}\right)_{\omega=\omega_0} = \frac{1}{v_g}, \tag{A5}$$

$$k_2 \equiv \left(\frac{\partial^2 k}{\partial\omega^2}\right)_{\omega=\omega_0} \tag{A6}$$

parametrize the (inverse) group velocity and (inverse) group-velocity dispersion, respectively, and we have identified $I \equiv |A|^2$.

Equation (A4) shows that electromagnetic modes have an intrinsic "propagation" contribution to $\partial^2/\partial z^2$ in the form of the $1/2k_0$ coefficient. Biancalana and Creatore [23] have recently shown that light in some semiconductor waveguides (such as ZnCdSe/ZnSe superlattices) can also exhibit a potentially dominant "material" contribution, whose physical origin lies in the coupling of the confined electric field to diffusing excitons. This spatial dispersion phenomenon appears through a modification to the coefficient of $\partial^2 A/\partial z^2$ in Eq. (A4), whereby the propagation contribution $1/2k_0$ can be augmented by the exciton term to yield a lumped coefficient,

$$\frac{1}{2k_0} + \frac{n_0\Gamma\Delta\tilde{\omega}_0}{2\delta\omega^2 c}. \tag{A7}$$

Here, $\Gamma \equiv \hbar/2M_x^*$, $M_x^*$ is the effective exciton mass, $\tilde{\omega}_0$ is a resonant frequency, $\Delta$ is a dimensionless parameter related to the oscillator strength for the coherent exciton-photon interaction, and $\delta\omega$ is a frequency detuning (for a more complete account, the reader is directed to Ref. [23]). A salient point is that the second term in Eq. (A4) can, in principle, become negative when $M_x^* < 0$.

After rescaling, the dimensionless envelope $u$ is governed by Eq. (1). The normalized space and time coordinates are $\zeta = z/L$ and $\tau = t/t_p$, respectively, where $t_p$ is the duration of a reference pulse with dispersion length $L = t_p^2/|k_2|$. The sign of the group-velocity dispersion is flagged by $s = \pm 1 = -\mathrm{sgn}(k_2)$ ($+1$ for anomalous, $-1$ for normal), and $\alpha \equiv k_1 t_p/|k_2|$. The spatial dispersion parameter is $\kappa = \kappa_0 + D$, where $\kappa_0 \equiv 1/2k_0 L = c|k_2|/2n_0\omega_0 t_p^2$ and $D \equiv n_0\Gamma\Delta\tilde{\omega}_0/2\delta\omega^2 cL = |k_2|n_0\Gamma\Delta\tilde{\omega}_0/2\delta\omega^2 c t_p^2$. Finally, the electric-field amplitude is scaled according to $u = A/A_0$, where $A_0 \equiv (n_0/n_2 k_0 L)^{1/2} = (n_0|k_2|/k_0 n_2 t_p^2)^{1/2}$.


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