Basic Engineering

Binary Numbers 2

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of the multiplication and division of binary numbers.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Binary Numbers (Introduction)

In **Binary Numbers 1**, binary numbers were introduced, as well as the techniques of their addition and subtraction. This package covers the methods of multiplication and division but to begin, here is a reminder of the rules of binary addition and subtraction.

Basic Rules for Binary Addition

0+0	=	0	0 plus 0 equals 0
0+1	=	1	0 plus 1 equals 1
1+0	=	1	1 plus 0 equals 1
1+1	=	10	1 plus 1 equals 0
			with a carry of 1 (binary 2)

$0-0$	=	0	0 minus 0 equals 0
$1-1$	=	0	1 minus 1 equals 0
$1-0$	=	1	1 minus 0 equals 1
10_2-1	=	1	10_2 minus 1 equals 1

Basic Rules for Binary Subtraction

As with decimal numbers, multiplication of binary numbers requires the technique of addition, whilst division of binary numbers requires the technique of subtraction. As a refresher to these ideas, here some questions for you to do.

Exercise 1. (Click on the green letters for solutions.)

(a) Convert the binary number 1011 into decimal form.

(b) Convert the binary number 1.011 into decimal form.

(c) Convert the numbers 15 and 12 into binary form, add the two binary numbers together and convert the answer to decimal form to check that the sum is correct.

(d) Convert the numbers 9 and 6 into binary form. Use this to find $9 - 6$ in binary form. Check that the answer is correct by converting the binary answer into decimal form.

2. Binary Multiplication

Table of Basic Rules for Binary Multiplication

$$\begin{array}{ccc} 0 \times 0 & = & 0 \\ 0 \times 1 & = & 0 \\ 1 \times 0 & = & 0 \\ 1 \times 1 & = & 1 \end{array}$$

The multiplication process for binary numbers is similar to that for decimal numbers. Partial products are formed, with each product shifted one place to the left. This is illustrated below.

Example 7 Multiply $7_{10} = 111$ and $5_{10} = 101$ in binary form.

Solution

$$\begin{array}{rrrrrr} & & & 1 & 1 & 1 \\ & & \times & 1 & 0 & 1 \\ \hline & & & 1 & 1 & 1 \\ & & 0 & 0 & 0 & 0 \\ + & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \end{array}$$

The third row is the multiplication of 111 by 1. In the fourth row, the 0 is the shift left before 111 is multiplied by 0. In the fifth row, the 00 is the shift left before 111 is multiplied by 1. The final row is the binary sum of the preceding three rows.

Exercise 2.
In each of the questions below, a product is written in *decimal* form. In each case, convert both numbers to binary form, multiply them in binary form and check that the solution is correct by converting the answer to decimal form. (Click on the green letters for solutions.)

(a) 3×2, (b) 4×4, (c) 5×10,

(d) 6×7, (e) 9×6, (f) 11×7.

Quiz Choose the correct answer from below for the result of the *binary* multiplication 1101×110.

(a) 1001111, (b) 1010110, (c) 1001110, (d) 1011111.

3. Binary Division

Binary division follows a similar process to that of decimal division.

Example 8 Divide **(a)** 15_{10} by 5_{10} in binary form, and **(b)** 15_{10} by 6_{10} in binary form.

Solution In binary form $15_{10} = 1111$, $5_{10} = 101$ and $6_{10} = 110$. The process for each of these is shown below.

(a)
$$\begin{array}{ccc|cccc}
 & & & & & 1 & 1 \\ \cline{4-7}
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
 & & & 1 & 0 & 1 & \\ \cline{4-6}
 & & & & 1 & 0 & 1 \\
 & & & & 1 & 0 & 1 \\ \cline{5-7}
 & & & & 0 & 0 & 0
\end{array}$$

(b)
$$\begin{array}{ccc|ccccc}
 & & & & & 1 & 0 & .1 \\ \cline{4-8}
1 & 1 & 0 & 1 & 1 & 1 & 1 & .0 \\
 & & & 1 & 1 & 0 & & \\ \cline{4-6}
 & & & & & 1 & 1 & 0 \\
 & & & & & 1 & 1 & 0 \\ \cline{6-8}
 & & & & & 0 & 0 & 0
\end{array}$$

In decimal form, $15_{10} \div 5_{10} = 3_{10}$, and 3_{10} is 11 in binary, which is the answer in the left hand array.

In decimal form, $15_{10} \div 6_{10} = 2.5_{10}$, and 2.5_{10} is 10.1 in binary, which is the answer in the right hand array.

Exercise 3.

In each of the questions below, a division is written in *decimal* form. In each case, convert both numbers to binary form, perform the division in binary form and check that the solution is correct by converting the answer to decimal form. (Click on the green letters for solutions.)

(a) $6 \div 2$, (b) $8 \div 2$, (c) $9 \div 3$,

(d) $10 \div 4$, (e) $21 \div 7$, (f) $18 \div 8$.

Quiz Choose the correct answer from below for the result of the *binary* division $11011 \div 1001$.

(a) 10, (b) 101, (c) 11, (d) 110.

4. Quiz on Binary Numbers

Begin Quiz

1. Which of the following is the binary product 1001×111?
   (a) 110111 (b) 111111, (c) 111011, (d) 111110.
2. Which of the following is the binary product 1101×1011?
   (a) 10101111, (b) 10001111, (c) 10101011, (d) 10111011.
3. Which of the following is the binary division $10101 \div 11$?
   (a) 100, (b) 110, (c) 101, (d) 111.
4. Which of the following is the binary division $100011 \div 1010$?
   (a) 10.1, (b) 11.11, (c) 11.01, (d) 11.1.

End Quiz

Score: Correct

Solutions to Exercises

Exercise 1(a)

The binary number 1011 is

$$1011 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1 \times 8 + 1 \times 2 + 1 \times 1$$

which in decimal form is 11_{10}.
Click on the green square to return

□

Exercise 1(b)

The binary number 1.011 is

$$\begin{aligned} 1.011 &= 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} \\ &= 1 \times 1 + 1 \times 0.25 + 1 \times 0.125 \end{aligned}$$

which in decimal form is 1.375_{10}.

Click on the green square to return

□

Exercise 1(c)

To add the numbers $15 + 12$, in binary form, note that

$$15_{10} = 8 + 4 + 2 + 1 = 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1111\,,$$

while

$$12_{10} = 8 + 4 = 1 \times 2^3 + 1 \times 2^2 = 1100\,.$$

The sum $15 + 12$ in binary form is shown below.

$$\begin{array}{r} 1111 \\ 1100 \\ \hline 11011 \end{array}$$

Note that in the third column, $1 + 1 = 0$ with a carry of 1 to the next column. In the left-hand column, $1 + 1 + 1 = 1$ with a of carry 1 to the next column.

Thus, in binary, $1111 + 1100 = 11011$, which in decimal form is $11011 = 2^4 + 2^3 + 2^1 + 2^0 = 16 + 8 + 2 + 1 = 27_{10}$.

Click on the green square to return

Exercise 1(d)

To find the difference $9 - 6$ in binary form note that $9_{10} = 1001$ and $6_{10} = 110$. The subtraction, in binary form, is shown below.

$$\begin{array}{r} 1001 \\ -\ 110 \\ \hline 011 \end{array}$$

In the right-hand column $1 - 0 = 1$.
In the next column, borrow a 1 from the third column (at the top) and pay it back at the bottom of the third column. Then $10 - 1 = 1$. The bottom of the third column is now $1+1 = 10$. The final step is thus $10 - 10 = 00$.

Thus, in binary, $1001 - 110 = 11$. In decimal form this is 3_{10}.

Click on the green square to return

Exercise 2(a)

To find the product 3×2 in binary form recall that $3_{10} = 11$ and $2_{10} = 10$. The multiplication is as shown.

$$\begin{array}{rrrr} & & 1 & 1 \\ & \times & 1 & 0 \\ \hline & & 0 & 0 \\ + & 1 & 1 & 0 \\ \hline & 1 & 1 & 0 \end{array}$$

The third row is the multiplication of 11 by 0. In the fourth row, the 0 is the shift left before 11 is multiplied by 1. The final row is the binary sum of the preceding two rows.

Thus $3_{10} \times 2_{10}$ in binary form is 110 which in decimal form is 6_{10}.

Click on the green square to return

□

Exercise 2(b)

To find the product 4×4 in binary form note that $4_{10} = 100$. Therefore 4×4, in binary form, is calculated as shown.

			1	0	0
		$\times$	1	0	0
			0	0	0
		0	0	0	0
+	1	0	0	0	0
	1	0	0	0	0

The third row is the multiplication of 100 by 0.
In the fourth row, the 0 is the shift left before 100 is multiplied by 0.
In the fifth row, the 00 is the shift left before 100 is multiplied by 1.
The final row is the binary sum of the preceding three rows.

Thus, in binary, $4 \times 4 = 10000$. In decimal form this is 16_{10}.

Click on the green square to return

Exercise 2(c)

To find the product 5×10 in binary form note that $5_{10} = 101$ and $10_{10} = 1010$. Therefore 5×10, in binary form, is calculated as shown.

$$
\begin{array}{ccccccc}
 & & & & 1 & 0 & 1 \\
 & & \times & 1 & 0 & 1 & 0 \\
\hline
 & & & & 0 & 0 & 0 \\
 & & 0 & 1 & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

In the final row performing the binary sum we use $1 + 1 = 10$ and carry 1 to the next column.

Thus, in binary form, $5_{10} \times 10_{10} = 110010$. In decimal form this is 50_{10}.

Click on the green square to return

□

Exercise 2(d)

To find the product 6×7 in binary form note that $6_{10} = 110$ and $7_{10} = 111$. The product is calculated as shown.

$$
\begin{array}{cccccc}
 & & & 1 & 1 & 0 \\
 & & \times & 1 & 1 & 1 \\
\hline
 & & & 1 & 1 & 0 \\
 & & 1 & 1 & 0 & 0 \\
+ & 1 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 1 & 0
\end{array}
$$

Note that performing the binary sum we use $1 + 1 = 10$ and carry 1 to the next column.

Thus, in binary, $6 \times 7 = 101010$. In decimal form this is 42_{10}.

Click on the green square to return

Exercise 2(e)

To find the product 9×6 in binary form note that $9_{10} = 1001$ and $6_{10} = 110$. The product, in binary form, is as shown.

			1	0	0	1
		×		1	1	0
			0	0	0	0
		1	0	0	1	0
+	1	0	0	1	0	0
	1	1	0	1	1	0

Thus, in binary, $9 \times 6 = 110110$. In decimal form this is 54_{10}.

Click on the green square to return

Exercise 2(f)

To find the product $11_{10} \times 7_{10}$ in binary form note that $11_{10} = 1011$ and $7_{10} = 111$. The product, in binary form, is as shown.

$$\begin{array}{ccccccc} & & & 1 & 0 & 1 & 1 \\ & & \times & & 1 & 1 & 1 \\ \hline & & & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 1 & 1 & 0 \\ + & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}$$

In performing the binary sum use $1 + 1 = 10$ with a carry of 1 to the next column.

Thus, in binary, $11 \times 7 = 1001101$. In decimal form this is 77_{10}.

Click on the green square to return

Exercise 3(a)

To find $6 \div 2$ in binary form first convert the numbers into binary form. Thus $6_{10} = 110$ and $2_{10} = 10$. The division process is shown.

$$
\begin{array}{cc|ccc}
 & & & 1 & 1 \\ \cline{3-5}
1 & 0 & 1 & 1 & 0 \\
 & & 1 & 0 & \\ \cline{3-4}
 & & & 1 & 0 \\
 & & & 1 & 0 \\ \cline{4-5}
 & & & 0 & 0
\end{array}
$$

In binary form, $6 \div 2$ is 11 which, in decimal form, is 3_{10}.

Click on the green square to return

Exercise 3(b)

To find $8 \div 2$ in binary form first convert the numbers into binary form. Thus $8_{10} = 1000$ and $2_{10} = 10$. The division process is shown.

$$
\begin{array}{cc|cccc}
 & & & 1 & 0 & 0 \\ \cline{3-6}
1 & 0 & 1 & 0 & 0 & 0 \\
 & & 1 & 0 & & \\ \cline{3-4}
 & & & 0 & 0 & \\
 & & & 0 & 0 & \\ \cline{4-5}
 & & & & 0 & 0 \\ \cline{5-6}
 & & & & 0 & 0
\end{array}
$$

In binary form, $8 \div 2$ is 100 which in decimal form is 4_{10}.

Click on the green square to return

Exercise 3(c)

To find $9 \div 3$ in binary form convert the numbers 9 and 3 into binary form, i.e. $9_{10} = 1001$ and $3_{10} = 11$. The division process is shown.

$$
\begin{array}{rr|rrrr}
 & & & 0 & 1 & 1 \\ \cline{3-6}
1 & 1 & 1 & 0 & 0 & 1 \\
 & & & 1 & 1 & \\ \cline{3-5}
 & & & & 1 & 1 \\
 & & & & 1 & 1 \\ \cline{5-6}
 & & & & 0 & 0
\end{array}
$$

Here the extra (red) zero has been written because the division process requires 2 shifts to the right (11 does not divide 10 but does divide 100!). Thus, in binary form, $9 \div 3$ is 11 which, in decimal form, is 3_{10}.

Click on the green square to return

Exercise 3(d)

To find $10 \div 4$ in binary form convert the numbers 10 and 4 into a binary form, i.e. $10_{10} = 1010$ and $4_{10} = 100$. The division process is shown.

$$
\begin{array}{ccc|ccccc}
 & & & & 1 & 0. & 1 & \\ \cline{4-7}
1 & 0 & 0 & 1 & 0 & 1 & 0 & \\
 & & & 1 & 0 & 0 & & \\ \cline{4-6}
 & & & & & 1 & 0 & 0 \\
 & & & & & 1 & 0 & 0 \\ \cline{6-8}
 & & & & & 0 & 0 & 0
\end{array}
$$

Thus, in binary form, $10 \div 4 = 10.1$. In decimal form this is 2.5_{10}.

Click on the green square to return

Exercise 3(e)

To find $21 \div 7$ in binary form convert the numbers 21 and 7 into binary form, i.e. $21_{10} = 10101$ and $7_{10} = 111$. The division process is shown.

$$\begin{array}{ccc|ccccc}
 & & & & 0 & 1 & 1 & \\ \cline{4-8}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & & & & 1 & 1 & 1 & \\ \cline{5-7}
 & & & & & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 \\ \cline{6-8}
 & & & & & 0 & 0 & 0
\end{array}$$

Here the extra (red) zero has been written because the division process requires 2 shifts to the right (111 does not divide 101 but does divide 1010!).Thus, in binary form, $21 \div 7 = 11$ which, in decimal form, is 3_{10}.

Click on the green square to return □

Exercise 3(f)

To find $18 \div 8$ in binary form first convert the numbers into binary form, i.e. $18_{10} = 10010$ and $8_{10} = 1000$. The division process is as shown.

$$
\begin{array}{cccc|ccccccc}
 & & & & 1 & 0. & 0 & 1 & & & \\
\cline{5-9}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & & \\
 & & & & 1 & 0 & 0 & 0 & & & \\
\cline{5-8}
 & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & 1 & 0 & 0 & 0 \\
\cline{8-10}
 & & & & & & & 0 & 0 & 0 & 0
\end{array}
$$

Thus, in binary form $18 \div 8 = 10.01$ which in decimal form is 2.25_{10}.

Click on the green square to return

Solutions to Quizzes

Solution to Quiz:

The binary multiplication of numbers 1101×110 is given below

$$\begin{array}{ccccccc} & & & 1 & 1 & 0 & 1 \\ & & \times & & 1 & 1 & 0 \\ \hline & & & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 1 & 0 \\ + & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

The result of multiplication in decimal form is $1001110 = 2^6 + 2^3 + 2^2 + 2 = 64 + 8 + 4 + 2 = 78$. Converting the numbers $1101 = 2^3 + 2^2 + 1 = 13_{10}$ and $110 = 2^2 + 2 = 6$. One can check easily the above given result of multiplication.

End Quiz

Solution to Quiz: The binary division $11011 \div 1001$ is given below.

$$
\begin{array}{cccc|ccccc}
 & & & & & & & 1 & 1 \\ \cline{5-9}
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
 & & & & 1 & 0 & 0 & 1 & \\ \cline{5-9}
 & & & & & 1 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 1 \\ \cline{6-9}
 & & & & & 0 & 0 & 0 & 0
\end{array}
$$

In decimal form $11011 = 27_{10}$ and $1001 = 9_{10}$. The result of the binary division above is $11 = 3_{10}$, which is the correct answer.

End Quiz