Helmholtz-Manakov Vector Solitons

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Abstract

The novel Helmholtz-Manakov equation and its vector soliton solutions are presented for the first time. These solutions exhibit the expected non-trivial Helmholtz-type corrections to their paraxial counterparts. The stability of the Helmholtz solutions is addressed by using a well-tested numerical perturbative approach.

Introduction

Vector solitons are well-known in paraxial wave optics (multi-component self-trapped non-linear structures of low-dimensional systems). A simple extension of the well-known *paraxial* non-linear Schrödinger (NLS) equation was first proposed by Manakov [1]. In the context of spatial soliton propagation, the Manakov equation governs the behaviour of an electric field with *two* transverse field components confined to quasi-2D waveguide geometry. The initial work was extended some years ago [2], with more exotic solutions being found for both Kerr focusing and defocusing media. The work presented here extends the Manakov equation to the Helmholtz-type regime by retaining the $\partial_{\zeta\zeta}$ operator:

$$\kappa\frac{\partial^2\mathbf{U}}{\partial\zeta^2}+i\frac{\partial\mathbf{U}}{\partial\zeta}+\frac{1}{2}\frac{\partial^2\mathbf{U}}{\partial\xi^2}\pm\left(\mathbf{U}^\dagger\mathbf{U}\right)\mathbf{U}=\mathbf{0}\qquad \mathbf{U}=\begin{bmatrix}A\\B\end{bmatrix}$$

Retain Helmholtz term

Non-Trivial Helmholtz-Type Corrections…

- Well-defined angles, e.g. $\tan^2\theta=2\kappa V^2$,
- Angular beam broadening,
- Corrections to phase shifts,
- Modifications to soliton *intrinsic velocity*.

The Helmholtz-Manakov (H-M) equation is a vector extension of the scalar non-paraxial non-linear Schrödinger (NNLS) equation [3]. This framework provides a well-defined connection between angles in the unscaled (x,z) and scaled (ξ,ζ) coordinate systems. In contrast, paraxial models based on the NLS equation support waves propagating only at vanishingly small angles, which themselves are poorly defined quantities. The H-M equation can be solved analytically using Ansatz techniques and also Hirota's method [4].

Bright-Bright Helmholtz Soliton

- Exist for the ***focusing non-linearity***,
- Are topologically trivial,
- Can be found using *Ansatz approach*.

$$\mathbf{U}(\xi,\zeta)=\mathbf{C}\eta\,\mathrm{sech}\left[\frac{\eta(\xi+V\zeta)}{\sqrt{1+2\kappa V^2}}\right]\times\exp\left[i\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-\frac{i\zeta}{2\kappa}\right],$$

$$\mathbf{C}=\begin{bmatrix}e^{i\delta_1}\cos(\alpha)\\e^{i\delta_2}\sin(\alpha)\end{bmatrix}.$$

δ_1 and δ_2 are the phases of the components. α is the *polarization angle*.

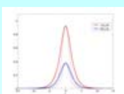

Figure 1. ***Bright-Bright Soliton Profiles.*** Dotted lines correspond to paraxial solutions (no broadening). Parameters: $\kappa=10^{-3}$ and $V=25$, so $\theta\approx48.2°$. The soliton amplitude here is $\eta=1$ with polarization angle $\alpha=\pi/8$.

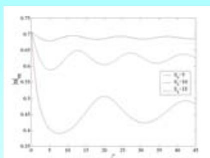

Figure 2. ***Reshaping Bright-Bright Soliton Beam.*** Equal-amplitude components, $\alpha=\pi/4$.

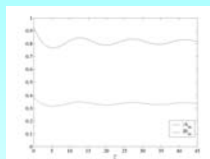

Figure 3. ***Reshaping Bright-Bright Soliton Beam.*** Unequal-amplitude components, $\alpha=\pi/8$.

Initial conditions correspond to exact paraxial bright-bright solitons. The non-paraxial parameter is $\kappa=10^{-3}$ with $\eta=O(1)$, but $\kappa V^2>>1$. This defines the ***Helmholtz-type non-paraxial regime,*** where the propagation angles are non-trivial.

$$\mathbf{U}(\xi,0)=\begin{bmatrix}A(\xi,0)\\B(\xi,0)\end{bmatrix}=\begin{bmatrix}\cos\alpha\\\sin\alpha\end{bmatrix}\mathrm{sech}(\xi)\exp(-iS_0\xi)$$

$S_0=5,10,15$ corresponds to the non-trivial propagation angles of $\theta=12.9°$, $26.6°$ and $42.1°$, respectively, for the value $\kappa=10^{-3}$.

Bright-Dark Helmholtz Soliton

- Exist for the ***focusing non-linearity***,
- Are topologically *non-trivial*,
- Can be found using *Ansatz approach*,
- The solution is constrained by $\eta^2\geq a^2$.

$$A(\xi,\zeta)=\eta\,\mathrm{sech}\left[\frac{a(\xi+V\zeta)}{\sqrt{1+2\kappa V^2}}\right]\times\exp\left[i\sqrt{\frac{1+2\kappa\eta^2}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-\frac{i\zeta}{2\kappa}\right],$$

$$B(\xi,\zeta)=\sqrt{\eta^2-a^2}\tanh\left[\frac{a(\xi+V\zeta)}{\sqrt{1+2\kappa V^2}}\right]\times\exp\left[i\sqrt{\frac{1+4\kappa(\eta^2-a^2)}{1+2\kappa V^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-\frac{i\zeta}{2\kappa}\right].$$

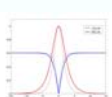

Figure 4. ***Bright-Dark Soliton Profiles.*** Dotted lines correspond to paraxial solutions (no broadening). Parameters: κ and V as in figure 1. Here, $\eta=1$ and $a=0.8$.

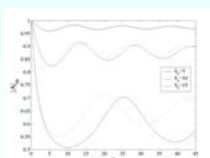

Figure 5. ***Reshaping Bright-Dark Soliton Beam.***

For $a=0.99$ and Helmholtz soliton formation occurs. Dotted curves are for $a=1$ where the dark component B is null.

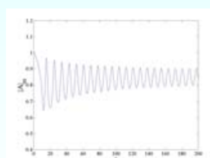

Figure 6. ***Reshaping Bright-Dark Soliton Beam.***

With $a=0.8$ and $S_0=5$, Helmholtz soliton formation does not occur. Instead there are long-lived persistent oscillations.

The initial conditions are taken as:

$$A(\xi,0)=\mathrm{sech}(a\xi)\exp(-iS_0\xi)$$
$$B(\xi,0)=\sqrt{1-a^2}\tanh(a\xi)\exp(-iS_0\xi)$$

Dark-Bright Helmholtz Soliton

- Exist for the ***defocusing non-linearity***,
- Are topologically *non-trivial*,
- Must be found using *Hirota's method*,
- The solution is constrained by $A_0^2\cos^2\phi\geq a^2$.

$$A(\xi,\zeta)=A_0\left[\cos\phi\tanh\left[\frac{a(\xi+W\zeta)}{\sqrt{1+2\kappa W^2}}\right]+i\sin\phi\right]\times\exp\left[i\sqrt{\frac{1-4\kappa A_0^2}{1+2\kappa W^2}}\left(-V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-\frac{i\zeta}{2\kappa}\right],$$

$$B(\xi,\zeta)=\sqrt{A_0^2\cos^2\phi-a^2}\,\mathrm{sech}\left[\frac{a(\xi+W\zeta)}{\sqrt{1+2\kappa W^2}}\right]\times\exp\left[i\sqrt{\frac{1+2\kappa(a^2-2A_0^2)}{1+2\kappa W^2}}\left(-W\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-i\frac{\zeta}{2\kappa}\right].$$

The initial conditions are taken as:

$$A(\xi,0)=\tanh(a\xi)\exp(-iS_0\xi)$$
$$B(\xi,0)=\sqrt{1-a^2}\,\mathrm{sech}(a\xi)\exp(-iS_0\xi)$$

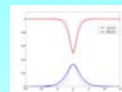

Figure 7. ***Dark-Bright Soliton Profiles.*** Dotted lines correspond to paraxial solutions (no broadening). Parameters: κ and V as in figure 1. Here, $A_0=1$, $a=0.8$ and $\phi=\pi/6$.

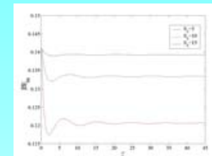

Figure 8. ***Reshaping Dark-Bright Soliton Beam.***

For $a=0.99$ and Helmholtz soliton formation occurs fairly quickly compared to the focusing case, figure 5.

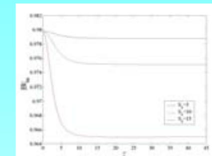

Figure 9. ***Reshaping Dark-Bright Soliton Beam.***

For $a=0.2$ the asymptotic soliton emerges more rapidly, with the oscillatory component almost completely suppressed (c.f. Figure 8 and also Figure 6, with decreasing a).

The dark-bright soliton has a *net velocity W* and an *intrinsic velocity V_0*, in addition to its transverse velocity V:

$$V_0=\frac{a\tan\phi}{\sqrt{1-2\kappa\left(2A_0^2+a^2\tan^2\phi\right)}}$$

$$W=\frac{V-V_0}{1+2\kappa VV_0}$$

NON-TRIVIAL CORRECTIONS TO PARAXIAL THEORY ALSO INCLUDE A MAXIMUM INTRINSIC VELOCITY.

Dark-Dark Helmholtz Soliton

- Exist for the ***defocusing non-linearity***,
- Are topologically *non-trivial*,
- Must be found using *Hirota's method*,
- The solution is constrained by $A_0^2\cos^2\phi_1+B_0^2\cos^2\phi_2=a^2$.

$$A(\xi,\zeta)=A_0\left[\cos\phi_1\tanh\left[\frac{a(\xi+W\zeta)}{\sqrt{1+2\kappa W^2}}\right]+i\sin\phi_1\right]\times\exp\left[i\sqrt{\frac{1-4\kappa\chi}{1+2\kappa V_1^2}}\left(-V_1\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-\frac{i\zeta}{2\kappa}\right],$$

$$B(\xi,\zeta)=B_0\left[\cos\phi_2\tanh\left[\frac{a(\xi+W\zeta)}{\sqrt{1+2\kappa W^2}}\right]+i\sin\phi_2\right]\times\exp\left[i\sqrt{\frac{1-4\kappa\chi}{1+2\kappa V_2^2}}\left(-V_2\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left[-\frac{i\zeta}{2\kappa}\right].$$

Here, $\chi^2=A_0^2+B_0^2$. The intrinsic velocity is,

$$V_{0j}=\frac{a\tan\phi_j}{\sqrt{1-2\kappa\left[2\left(A_0^2+B_0^2\right)+a^2\tan^2\phi_j\right]}}$$

where $j=1,2$ labels polarization component A,B.

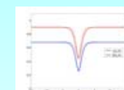

Figure 10. ***Dark-Dark Soliton Profiles.*** Dotted lines correspond to paraxial solutions (no broadening). Parameters: $\kappa=10^{-3}$ and $V_j=25$, so $\theta_j\approx48.1°$. Also, $\phi_1=\pi/6$, $\phi_2=\pi/8$, $a=1$ and $A_0=0.9$. All other parameters are determined from implicit conditions.

Reshaping of the dark-dark H-M soliton is expected to mirror that of the scalar Helmholtz dark soliton [6], in the same way that the bright-bright H-M solution reshapes like its scalar counterpart [5].

Conclusions

- Helmholtz-Manakov vector solitons have a rich dynamical behaviour afforded by the inclusion of a second field component and an extra degree of freedom.
- Focusing Kerr Non-Linearity
  - →Paraxial input conditions (equivalent to perturbed paraxial solitons in the rotated frame) can exhibit monotonically-decreasing oscillations in the peak amplitude (e.g. bright-bright). The corresponding H-M soliton is then referred to as a **STABLE FIXED POINT** of the system, in a non-linear dynamical sense.
  - → Alternatively, no steady-state asymptotic soliton (fixed peak amplitude) may emerge from the paraxial initial condition. Instead, oscillations persist and are the dominant feature of the long-term dynamics. The corresponding H-M soliton is then referred to as a **LIMIT CYCLE** of the system (e.g. bright-dark).
- Defocusing Non-Linearity
  - → These H-M solitons have strongly attracting properties and are generally **STABLE FIXED POINTS** of the system. The asymptotic soliton state is usually reached much more rapidly than in the focusing case. This may be linked to the modulational stability of CW solutions in a defocusing Kerr medium, with the primary *dark* component possessing greater stability than the anti-guiding (secondary) *bright* component.

BRIGHT-DARK (focusing non-linearity) and DARK-BRIGHT (defocusing non-linearity) ARE NOT EQUIVALENT!!

- For comparison: Bright scalar Helmholtz solitons are *weakly attracting*, with the reshaping oscillations gradually decaying to a fixed-amplitude steady-state [5].
  Dark scalar Helmholtz solitons are *strongly attracting*, with the asymptotic soliton state emerging after a relatively short propagation length [6].

References

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