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Universal Snell's law & bright spatial soliton refraction

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The behaviour of a scalar optical beam at the boundary between two dissimilar Kerr media is of fundamental interest in nonlinear photonics. Since this class of problem is intrinsically nonparaxial, the limitations of conventional (paraxial) theory must be observed with care. Recently, we proposed the first Helmholtz model of Kerr spatial soliton refraction that is valid across the entire range of incidence, reflection and refraction angles. Here, we report the first systematic generalization of these novel analyses to a wider class of power-law materials. Soliton refraction laws will be given, and excellent agreement shown between theoretical predictions and computer simulations. New qualitative phenomena are also uncovered in non-Kerr regimes.

1. Historical Context

Light beams impinging on the interface between two dissimilar dielectric materials is one of the most straightforward optical geometries (see Fig. 1). The seminal papers of Aceves, Moloney and Newell [1,2] considered perhaps the simplest scenario, where a spatial soliton was incident on the planar boundary between two different Kerr-type materials. Their intuitive approach reduced the full complexity of the electromagnetic interface problem to something far more tractable – namely, the solution of a scalar equation of the inhomogeneous nonlinear Schrödinger (NLS) type. Over the past two decades, their investigations of single [1] and double [2] interface geometries have paved the way to deeper understandings of how light behaves inside patterned nonlinear structures such as coupled waveguide arrays and photonics crystals.

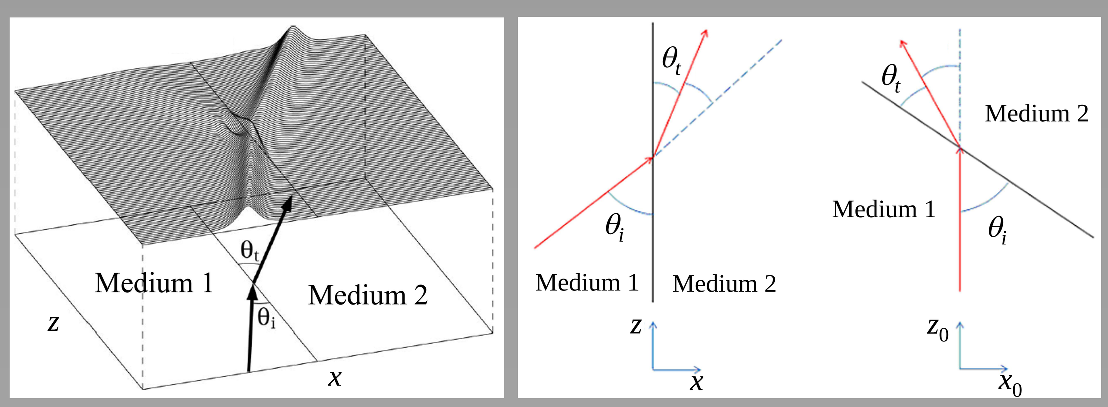

FIG. 1. Left: Schematic diagram illustrating the geometry used in the mathematical analysis of the Helmholtz interface problem. Right: In computations, one works in the frame of reference (x_0, z_0) in which the incident beam travels along the reference (longitudinal) axis (i.e., where the interface is rotated with respect to the incident beam). The two representations are physically equivalent, as they must be.

It is true to say that the analyses of Aceves and coworkers have provided an enormous level of insight into soliton behavior at interfaces [1–4], and that they have heralded new research fields in nonlinear photonics. **However, there is an intrinsic limitation in the use of paraxial (i.e., NLS-based) models to describe geometries that are, by their very nature, highly nonparaxial.** For instance, the paraxial approximation restricts their domain of validity to regimes were the angles of incidence, reflection and refraction are negligibly small. Hence, there is an obvious gap that needs to be filled – it is desirable to be able to describe angles of *any* size, while keeping an intuitive scalar model.

2. Power-Law Interfaces

In recent works [5,6], we have analyzed the refraction of spatial solitons incident on the boundary between dissimilar Kerr-type materials at *arbitrary angles*. By deploying the formalism of Helmholtz soliton theory, the angular limitation of paraxial models was lifted, and a manageable envelope equation emerged (by retaining the full generality of the in-plane Laplacian). Applying appropriate field continuity conditions at the interface led to a Snell's law for Kerr spatial solitons. At first glance, this new law strongly resembles the classic refraction rule for *plane waves* at the interface between *linear* media. However, there appears a factor (denoted by γ) that captures the interplay between finite-beam effects and (linear and nonlinear) medium discontinuities.

Here, we report the first steps toward extending our Kerr analyses to regimes involving wider classes of materials. In our systematic approach, we first consider media whose nonlinear refractive index n_{NL} has a generic power law-type dependence on the electric field amplitude E,

$$n_{\mathrm{NL}}(E) = \frac{\alpha}{2n_0}|E|^q$$

where α is a (small) positive coefficient, n_0 is the linear index ($n_0 >> \alpha|E|^q$), and the exponent $q > 0$ [7]. The single power-law model is perhaps the simplest non-Kerr nonlinearity one might care to consider [8]. Materials that fall within this category include some semiconductors (e.g., InSb and GaAs/GaAlAs), doped filter glasses (e.g., CdS_xSe_{x-2}) and liquid crystals. **Non-Kerr regimes (i.e., where q deviates from the value of 2) have been found to give rise to a diverse range of new qualitative phenomena.**

3. Helmholtz Modelling

We consider the TE-polarized continuous-wave scalar electric field $E(x,z,t) = E(x,z)\exp(-i\omega t) + E^*(x,z)\exp(+i\omega t)$, where x and z are the spatial coordinates in the medium, t is the time coordinate and ω is the angular frequency. This representation makes sure that the field remains real, as should be the case. If the spatial part of the field varies slowly (in the transverse direction) on the scale of the free-space optical wavelength λ, then $E(x,z)$ satisfies a nonlinear Helmholtz equation on each side of the material boundary:

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)E(x,z)+\frac{\omega^2}{c^2}n_j^2(E)E(x,z)=0,$$

where $j = 1, 2$ denotes the medium and c is the vacuum speed of light. The total refractive index n_j is routinely taken to be the sum of two terms: $n_j = n_{0j} + n_{\mathrm{NL}j}(E)$, where n_{0j} is the linear index of medium j at frequency ω, and $n_{\mathrm{NL}j}(E)$ is a small field-dependent correction. Since the wave equation is quadratic in n_j, one may make the approximation $n_j^2 \sim n_{0j}^2 + 2n_{0j}n_{\mathrm{NL}j}(E) = n_{0j}^2 + \alpha_j|E|^q$. The carrier-wave component of $E(x,z)$ can be factored out according to $E(x,z) = E_0u(x,z)\exp(ik_1z)$, so that z and x denote the longitudinal and transverse coordinates, respectively. Here, $E_0 = (n_0/\alpha_1 kL_D)^{1/q}$ is a (real) constant, $k_1 = (\omega/c)n_{01}$, and $u(x,z)$ is the dimensionless envelope. Arbitrarily, the carrier wave in medium 1 has been factored out of $E(x,z)$ [equally, one could have stripped out the complex-exponential factor $\exp(ik_2z)$]. After substitution into the above field equation, it can be shown that u satisfies the dimensionless inhomogeneous equation,

$$\boxed{\kappa\frac{\partial^2 u}{\partial\zeta^2}+i\frac{\partial u}{\partial\zeta}+\frac{1}{2}\frac{\partial^2 u}{\partial\xi^2}+|u|^q u=\left[\frac{\Delta}{4\kappa}+(1-\alpha)|u|^q\right]H(\xi)u.} \quad \ldots(1)$$

The longitudinal and transverse coordinates are $\zeta = z/L_{D1}$ and $\xi = 2^{1/2}x/w_0$, where $L_{D1} = k_1w_0^2/2$ and w_0 are the diffraction length and waist of a reference Gaussian beam. The nonparaxial parameter $k = 1/(k_1w_0)^2 = \varepsilon^2/4\pi^2n_{01}^2$ quantifies the (inverse) beam width. The validity of the Helmholtz modelling approach requires $\varepsilon \equiv \lambda/w_0 << O(1)$, so that beam waists are much larger than the free-space light wavelength. Hence, κ is always taken to be a small parameter throughout: $\kappa << O(1)$. The Heaviside unit function $H(\xi)$ is defined so that $H(\xi < 0) = 0$ and $H(\xi > 0) = +1$ (see Fig. 1 – in this configuration, the interface is aligned along the z axis). Equation (1) contains the interface parameters that describe the mismatch between the linear and nonlinear characteristics of the two media:

MISMATCH IN **_LINEAR_** PART OF THE REFRACTIVE INDEX ← $\Delta \equiv 1-\left(\frac{n_{02}}{n_{01}}\right)^2$

$\alpha \equiv \frac{\alpha_2}{\alpha_1}$ → MISMATCH IN **_NONLINEAR_** PART OF THE REFRACTIVE INDEX

From these relations, three distinct scenarios emerge: (*i*) linear interfaces (defined by $\alpha = 1$ so that there is no mismatch in the nonlinear coefficients), (*ii*) nonlinear interfaces (defined by $\Delta = 0$, so the two media have the same linear index), (*iii*) mixed interfaces (with arbitrary choices of α and Δ).

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4. Snell's Law for Spatial Solitons

Model (1) in medium 1 is just the conventional power-law Helmholtz equation, for which the exact analytical bright soliton solutions are now known [8]. Far away from the interface, the solutions have a $\mathrm{sech}^{2/q}$ profile,

$$u(\xi,\zeta)=\eta_0\,\mathrm{sech}^{2/q}\left(a\frac{\xi-V\zeta}{\sqrt{1+2\kappa V^2}}\right)\exp\left[\pm i\sqrt{\frac{1+4\kappa\beta}{1+2\kappa V^2}}\left(V\xi+\frac{\zeta}{2\kappa}\right)\right]\exp\left(-i\frac{\zeta}{2\kappa}\right),$$

where η_0 is the peak amplitude, V is the conventional transverse velocity, and ± flags a forward- or backward-propagating beam. The additional parameters are $\beta = 2\eta_0^q/(2 + q)$ and $a = q[\eta_0^q/(2 + q)]^{1/2}$. Here, V is related to the propagation angle θ in the laboratory (x,z) frame (measured with respect to the longitudinal, i.e., z, axis) through the trigonometric relation $\tan\theta = (2\kappa)^{1/2}V$. By deploying the power-law Helmholtz solitons in tandem with the interfaces formalism [5,6], one can arrive at the following **Helmholtz-Snell law** predicting soliton refraction:

$$\boxed{\gamma n_{01}\cos\theta_i = n_{02}\cos\theta_t} \qquad \text{where} \qquad \gamma \equiv \left[\frac{1+8\kappa\eta_0^q(2+q)^{-1}}{1+8\kappa\eta_0^q\alpha(2+q)^{-1}(1-\Delta)^{-1}}\right]^{1/2}.$$

Here, θ_i and θ_t are the angles of incidence and transmission, respectively (see Fig. 1). The factor γ allows for the interplay between finite-width optical beams, system nonlinearity, and mismatched medium properties. The critical angle θ_C is defined to be the value of θ_i at which $\theta_t = 0$ (i.e., where the refracted beam, in principle, travels along the material boundary).

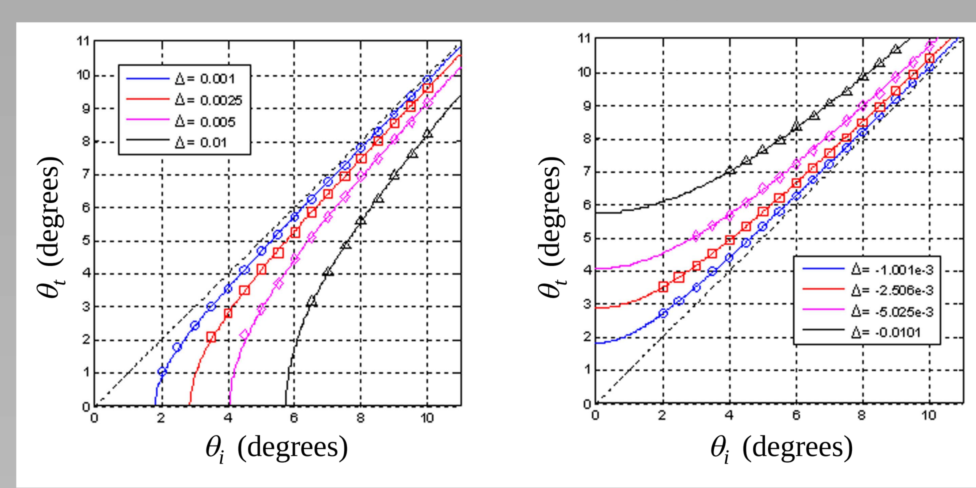
5. Solitons at Linear Interfaces

It is convenient to introduce the dimensionless parameter δ through [5,6]

$$\boxed{\delta \equiv \Delta + 8\kappa\frac{\eta_0^q}{2+q}(1-\alpha).}$$

Refraction thus tends to fall into two regimes: it may be either **external** (where $\delta < 0$, which implies $\theta_t > \theta_i$) or **internal** (where $\delta > 0$, which implies $\theta_t < \theta_i$). The boundary between these two regimes is determined by $\delta = 0$, in which case $\theta_t = \theta_i$ and the incident beam passes undeviated across the interface. Thus, $\delta = 0$ defines the **transparency condition**, in which linear and nonlinear mismatches in refractive index exactly cancel each other.

FIG. 2. Comparison of theoretical predictions made by the Helmholtz-Snell law (solid lines) and full numerical computations (points) for linear interfaces ($\alpha = 1$) with $\eta_0 = 1$ and $\kappa = 10^{-4}$. Excellent quantitative agreement has been uncovered (these particular plots are for $q = 1.0$) Left: internal refraction ($\delta = \Delta > 0$), where the four curves lie *below* the line $\theta_t = \theta_i$ and a critical angle clearly exists. Right: external refraction ($\delta = \Delta < 0$), where the four curves lie *above* the line $\theta_t = \theta_i$. Beams at larger (i.e., nonparaxial) incidence angles tend to suffer less deviation than at low angles because the interface perturbation is distributed over a much shorter propagation length.



6. Solitons at Nonlinear Interfaces

Solitons incident on nonlinear or mixed interfaces tend to undergo self-reshaping oscillations in the second medium because the focusing properties of the media are different on either side of the boundary. A small but illustrative range of simulations is shown below. One can see that the qualitative features of nonlinear refraction can depend strongly on the exponent q. When $\alpha > 1$, one tends to always find an externally-refracted beam (the threshold energy-flow for soliton formation in medium 2 is always exceeded – see Fig. 3). However, when $\alpha < 1$, beam stability is reduced (the threshold energy-flow requirement for soliton formation in medium 2, which is now higher than in medium 1, is not always crossed). This effect tends to become more pronounced as q increases, where a beam encountering a nonlinear interface may break up into radiation (see Fig. 4).

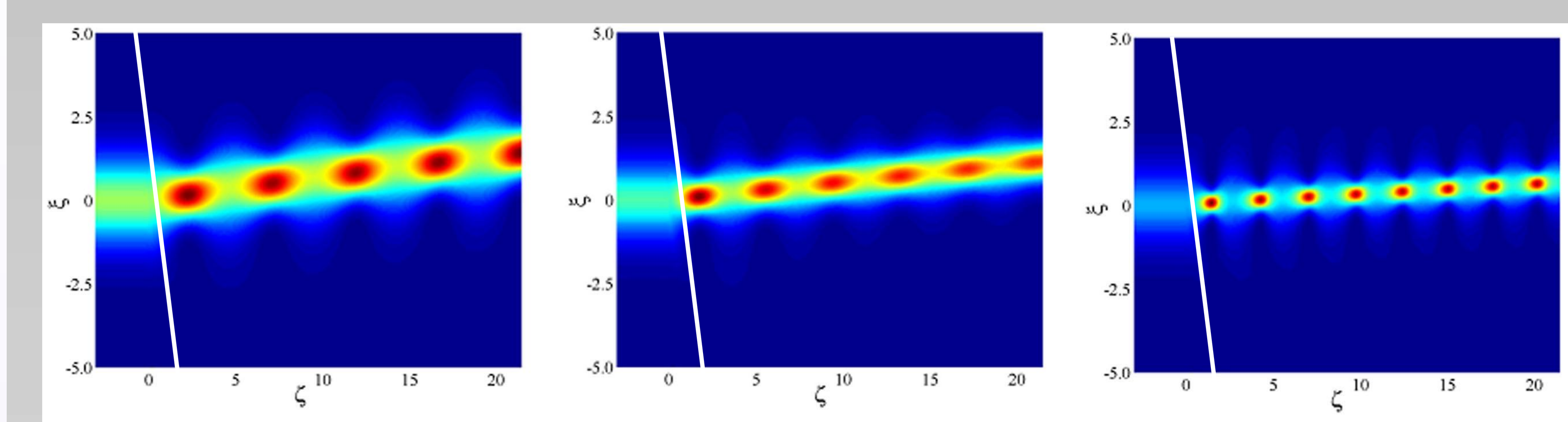


FIG. 3. Simulations showing **external refraction** for a nonparaxial incidence angle. A soliton of unit amplitude ($\eta_0 = 1$) is incident at θ_i = 30º on a nonlinear interface ($\Delta = 0$) with $\alpha = 2.0$, and is refracted *away from* the interface. Oscillations in the beam parameters (amplitude, width, and area) are more rapid (in z) as q increases. Left: $q = 1$, middle: $q = 2$ (Kerr nonlinearity); right: $q = 3$. White line marks the interface.

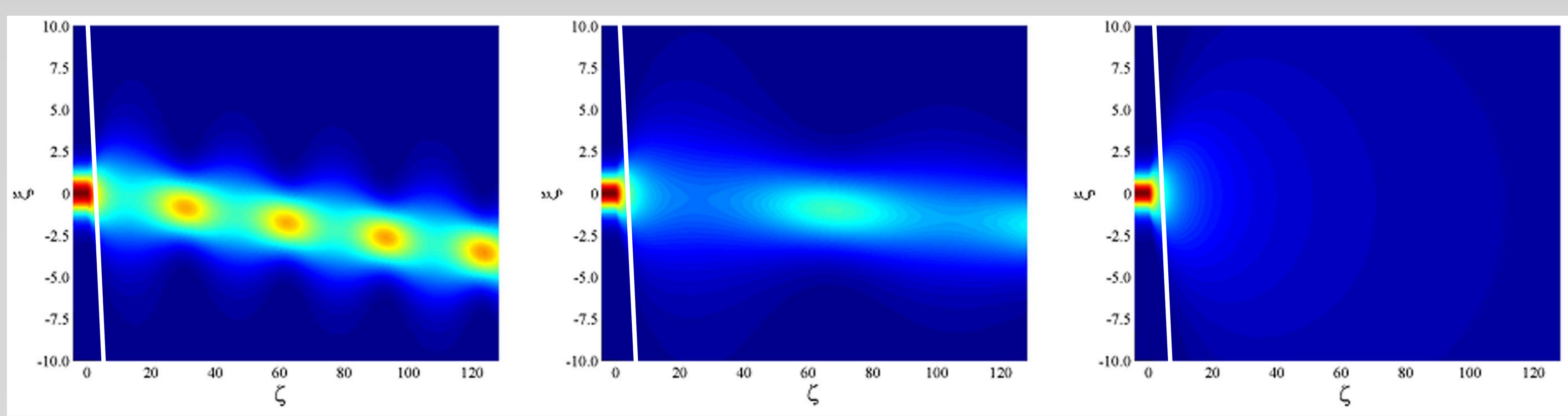


FIG. 4. Simulations showing **internal refraction** for a nonparaxial incidence angle. A soliton of unit amplitude ($\eta_0 = 1$) is incident at θ_i = 30º on a nonlinear interface ($\Delta = 0$) with $\alpha = 0.5$, and is refracted *toward* the interface. Oscillations in the beam parameters are slower (in z) as q increases (compare this with Fig. 3, where $\alpha > 1$). Left: $q = 1$, middle: $q = 2$ (Kerr nonlinearity); right: $q = 3$. White line marks the interface.

7. Conclusions

Our work here comprises **two-fold novelty**, combining for the first time Helmholtz propagation effects (i.e., oblique incidence of solitons at interfaces) with non-Kerr nonlinearities. Our model can also describe regimes where $\Delta < 0$, while paraxial theory [1–4] is confined to $\Delta > 0$. Extensive computations have tested the analytical predictions of the Helmholtz-Snell law against direct numerical integration of Eq. (1); excellent agreement has been found in a range of different of parameter regimes. Simulations have considered scenarios such as purely linear interfaces (defined by $\alpha = 1$), purely nonlinear interfaces (defined by $\Delta = 0$), and mixed interfaces (with arbitrary values of α and Δ). New qualitative phenomena have been uncovered, with beam robustness tending to decrease as q increases.

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