TUTORIAL 11

This tutorial covers PARTIAL DIFFERENTIATION.

In particular, when given a function of two or more independent variables $f(x,y)$, for example, then ....

- $\dfrac{\partial f}{\partial x} \equiv \left[\dfrac{df}{dx}\right]_{\text{treating } y \text{ as a constant}}$, $\dfrac{\partial f}{\partial y} \equiv \left[\dfrac{df}{dy}\right]_{\text{treating } x \text{ as a constant}}$

- higher-order derivatives in partial differentiation are defined in a similar manner to those in ordinary differentiation, i.e. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$

  and $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$,

  but we now also have $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

  WHICH EQUALS $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

- one also has the chain rule for "functions of a function"

  e.g. for $f(u(x,y))$ then

  $$\boxed{\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}} \text{ and } \boxed{\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}}$$

- and the product rule, e.g. for $f(x,y) = u(x,y)v(x,y)$

  $$\boxed{\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}} \text{ and } \boxed{\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y}}$$

- and the quotient rule, e.g. for $f(x,y) = \dfrac{u(x,y)}{v(x,y)}$

  $$\boxed{\frac{\partial f}{\partial x} = \frac{\frac{\partial u}{\partial x}.v - u\frac{\partial v}{\partial x}}{v^2}} \text{ and } \boxed{\frac{\partial f}{\partial y} = \frac{\frac{\partial u}{\partial y}.v - u\frac{\partial v}{\partial y}}{v^2}}$$

The second part of this tutorial deals with showing that a given function of two or more variables is a solution of the given partial differential equation.

One evaluates the derivatives that appear in the equation and then substitutes these derivatives into the equation to verify that the function 'solves the equation'.

Exercises

1. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ when:

   (a) $f(x,y) = 1 + x + y$, (b) $f(x,y) = x^2 + 2y^2 + 3xy - x + 1$,

   (c) $f(x,y) = \sin(x-y)$, (d) $f(x,y) = \frac{y}{x}$,

   (e) $f(x,y) = e^{2x+3y}$, (f) $f(x,y) = \frac{1}{x} + \frac{1}{y}$.

2. Show that the following are solutions of the given equations:

   (a) $z = e^x (x\cos y - y\sin y)$ of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$,

   (b) $y = \sin\left(\frac{px}{c}\right)\sin(pt + a)$ of $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$,

   where p, c and a are all <u>constants</u>.

   (c) $V = e^{-kx}(A\cos ky + B\sin ky)$

   of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$,

   where k, A and B are all <u>constants</u>.

1. (a) $\boxed{f(x,y) = 1 + x + y}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1+y) + \frac{\partial}{\partial x}(x) = 0 + 1 = 1.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(1+x) + \frac{\partial}{\partial y}(y) = 0 + 1 = 1.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(1) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(1) = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(1) = 0.$$

(b) $\boxed{f(x,y) = x^2 + 2y^2 + 3xy - x + 1}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy - x) + \frac{\partial}{\partial x}(2y^2 + 1)$$

$$= 2x + 3y - 1 + 0 = 2x + 3y - 1.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2y^2 + 3xy) + \frac{\partial}{\partial y}(x^2 - x + 1)$$

$$= 4y + 3x + 0 = 4y + 3x.$$

1. (b) continued

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 3y - 1)$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial x}(3y - 1) = 2 + 0 = 2.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(4y + 3x) = 4.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(4y + 3x) = 3.$$

(c) $\boxed{f(x,y) = \sin(x-y)}$

Let $u = x - y$ for chain rule
i.e. $\sin(x-y) = \sin u$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \cos u \frac{\partial (x-y)}{\partial x} = \cos u \,.\, (1)$$

$$= \cos(x-y).$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = \cos u \,.\, \frac{\partial}{\partial y}(x-y) = \cos u \,.\, (-1) = -\cos(x-y).$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left[\cos(x-y)\right] \equiv \frac{\partial}{\partial u}(\cos u)\frac{\partial u}{\partial x}$$

(chain rule again)

$$= -\sin u \,.\, \frac{\partial}{\partial x}(x-y) = -\sin u \,.\, (1) = -\sin(x-y).$$

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1\. (c) continued $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left[-\cos(x-y)\right]$

$$\equiv \frac{\partial}{\partial u}\left[-\cos u\right]\frac{\partial u}{\partial y} = +\sin u \cdot \frac{\partial}{\partial y}(x-y)$$

$$= \sin u \cdot (-1) = -\sin(x-y).$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[\cos(x-y)\right] \equiv \frac{\partial}{\partial u}(\cos u)\frac{\partial u}{\partial y}$$

$$= -\sin u \cdot \frac{\partial}{\partial y}(x-y)$$

$$= -\sin u \cdot (-1) = +\sin(x-y).$$

(d) $\boxed{f(x,y) = \dfrac{y}{x}}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = y\frac{\partial}{\partial x}(x^{-1})$$, since $\frac{\partial}{\partial x}$ "sees" y as a constant.

$$= y(-x^{-2}) = -\frac{y}{x^2}.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}\frac{\partial}{\partial y}(y)$$, since $\frac{\partial}{\partial y}$ "sees" x as a constant.

$$= \frac{1}{x} \cdot 1 = \frac{1}{x}.$$

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1\. (d) continued

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\right) = -y\frac{\partial}{\partial x}\left(\frac{1}{x^2}\right)$$

$$= -y\frac{\partial}{\partial x}(x^{-2}) = -y \cdot (-2x^{-3}) = +\frac{2y}{x^3}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0.$$

{ $\frac{\partial}{\partial y}$ "sees" x, and hence $\frac{1}{x}$, as a constant }.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$ { you can do it <u>either</u> way }

e.g. $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{\partial}{\partial x}(x^{-1})$$

$$= -\frac{1}{x^2}.$$

(e) $\boxed{f(x,y) = e^{2x+3y}}$

Let $u = 2x+3y$ and use the chain rule where $\frac{\partial u}{\partial x} = 2$, $\frac{\partial u}{\partial y} = 3$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial u}(e^u)\frac{\partial u}{\partial x} = e^u \cdot 2 = 2e^{2x+3y}.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial u}(e^u)\frac{\partial u}{\partial y} = e^u \cdot 3 = 3e^{2x+3y}.$$

1\. (e) continued

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2e^u\right)$$

$$= 2\frac{\partial}{\partial u}(e^u)\frac{\partial u}{\partial x} = 2.e^u.2$$

$$= 4e^{2x+3y}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left[3e^u\right] = \frac{\partial}{\partial u}(3e^u)\frac{\partial u}{\partial y}$$

$$= 3.e^u.3 = 9e^{2x+3y}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2e^u\right) = \frac{\partial}{\partial u}(2e^u)\frac{\partial u}{\partial y}$$

$$= 2.e^u.3 = 6e^{2x+3y}$$

(f) $\boxed{f(x,y) = \frac{1}{x} + \frac{1}{y}}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{x^2}\right) = -\left(\frac{-2}{x^3}\right) = +\frac{2}{x^3}.$$

1\. (f) continued

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{y^2}\right) = +\frac{2}{y^3}.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right) = 0.$$

2\. (a) $z = e^x(x\cos y - y\sin y)$ of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

We need to work out $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ and substitute the results in the given equation to see if it is satisfied.

$$z = e^x(x\cos y - y\sin y) = xe^x\cos y - e^x y\sin y$$

x dependencies --- y dependencies

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(xe^x\cos y - e^x y\sin y\right)$$

$$= \cos y\frac{\partial}{\partial x}(xe^x) - y\sin y\frac{\partial}{\partial x}(e^x)$$

$$= (e^x + xe^x)\cos y - e^x y\sin y$$

$$= e^x(\cos y + x\cos y - y\sin y).$$

NB

$$\frac{\partial}{\partial x}(xe^x) = e^x + xe^x$$

u v

using the product rule

.(a) continued

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(xe^x\cos y - e^x y\sin y\right)$$

$$= xe^x\frac{\partial}{\partial y}(\cos y) - e^x\frac{\partial}{\partial y}(y\sin y)$$

$$= -xe^x\sin y - e^x(\sin y + y\cos y)$$ {using product rule on last term}

$$\therefore \frac{\partial z}{\partial y} = -e^x(x\sin y + \sin y + y\cos y).$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(e^x(\cos y + x\cos y - y\sin y)\right)$$

$$= \frac{\partial}{\partial x}\left(e^x\cos y\right) + \frac{\partial}{\partial x}\left(xe^x\cos y\right) - \frac{\partial}{\partial x}\left(e^x y\sin y\right)$$

$$= \cos y\frac{\partial}{\partial x}(e^x) + \cos y\frac{\partial}{\partial x}(xe^x) - y\sin y\frac{\partial}{\partial x}(e^x)$$

$$= \cos y \cdot e^x + \cos y\underbrace{\left(e^x + xe^x\right)}_{\text{product rule used here}} - y\sin y \cdot e^x$$

2.(a) continued

$$\therefore \frac{\partial^2 z}{\partial x^2} = e^x\left(\cos y + \cos y + x\cos y - y\sin y\right)$$

$$= e^x\left[(2+x)\cos y - y\sin y\right].$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(-e^x(x\sin y + \sin y + y\cos y)\right)$$

$$= -e^x\frac{\partial}{\partial y}\left(x\sin y + \sin y + y\cos y\right)$$

$$= -e^x\left[x\frac{\partial}{\partial y}(\sin y) + \frac{\partial}{\partial y}(\sin y) + \frac{\partial}{\partial y}(y\cos y)\right]$$

$$= -e^x\left[x\cos y + \cos y + \underbrace{\cos y - y\sin y}_{\text{product rule used}}\right]$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = -e^x\left[(2+x)\cos y - y\sin y\right].$$

. (a) continued

i.e. $$\frac{\partial^2 z}{\partial y^2} = -e^x \left\{ (x+2)\cos y - y\sin y \right\}$$

and $$\frac{\partial^2 z}{\partial x^2} = +e^x \left\{ (x+2)\cos y - y\sin y \right\}$$

We wish to show that z satisfies $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Substitution of the above expressions for $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ gives zero on the left-hand side of the equation.

$\therefore$ z satisfies this equation.

(b) $\boxed{y = \sin\left(\frac{px}{c}\right)\sin(pt+a)}$ and $\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0}$

We need to find $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ and then substitute into this equation.

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left\{ \sin\left(\frac{px}{c}\right)\sin(pt+a) \right\} = \sin(pt+a)\,\frac{\partial}{\partial x}\left\{ \sin\left(\frac{px}{c}\right) \right\}$$

let $u = \frac{px}{c}$, then $\frac{\partial}{\partial x}(\sin u) = \frac{\partial}{\partial u}(\sin u)\,\frac{\partial u}{\partial x}$ {chain rule}

$$= \cos u \cdot \left(\frac{p}{c}\right)$$

2. (b) continued

i.e. $$\frac{\partial y}{\partial x} = \sin(pt+a)\cdot\cos\left(\frac{px}{c}\right)\cdot\left(\frac{p}{c}\right) = \frac{p}{c}\sin(pt+a)\cos\left(\frac{px}{c}\right).$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left\{ \sin\left(\frac{px}{c}\right)\sin(pt+a) \right\} = \sin\left(\frac{px}{c}\right)\frac{\partial}{\partial t}\left\{ \sin(pt+a) \right\}$$

i.e. $$\frac{\partial y}{\partial t} = \sin\left(\frac{px}{c}\right)\cdot p\cdot\cos(pt+a).$$ {using chain rule again}

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{p}{c}\sin(pt+a)\cos\left(\frac{px}{c}\right) \right) = \frac{p}{c}\sin(pt+a)\,\frac{\partial}{\partial x}\left(\cos\left(\frac{px}{c}\right)\right)$$

$$= \frac{p}{c}\sin(pt+a)\cdot\left(\frac{p}{c}\right)\left(-\sin\left(\frac{px}{c}\right)\right)$$ {chain rule}

$$= -\frac{p^2}{c^2}\sin(pt+a)\sin\left(\frac{px}{c}\right).$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left(\sin\left(\frac{px}{c}\right)\cdot p\cdot\cos(pt+a) \right)$$

$$= p\sin\left(\frac{px}{c}\right)\frac{\partial}{\partial t}\left\{ \cos(pt+a) \right\}$$

$$= p\sin\left(\frac{px}{c}\right)\cdot p\cdot\left(-\sin(pt+a)\right)$$

$$= -p^2\sin\left(\frac{px}{c}\right)\sin(pt+a).$$

Now, substitute $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ into $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$

....

2 (b) continued

.... this gives

$$-\frac{p^2}{c^2}\sin(pt+a)\sin\left(\frac{px}{c}\right) - \frac{1}{c^2}\left\{-p^2\sin\left(\frac{px}{c}\right)\sin(pt+a)\right\} = 0$$

i.e. $$-\frac{p^2}{c^2}\sin(pt+a)\sin\left(\frac{px}{c}\right) + \frac{p^2}{c^2}\sin\left(\frac{px}{c}\right)\sin(pt+a) = 0$$

$\therefore$ i.e. the equation 'is satisfied' and y is a solution.

2.(c) $\boxed{V = e^{-kx}(A\cos ky + B\sin ky)}$ and $\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0}$

$$\frac{\partial V}{\partial x} = (A\cos ky + B\sin ky)\frac{\partial}{\partial x}(e^{-kx})$$

$$= (A\cos ky + B\sin ky)(-k)e^{-kx} = -kV$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(-kV) = -k\frac{\partial V}{\partial x} = -k(-kV) = +k^2V.$$

2. (c) continued

$$\frac{\partial V}{\partial y} = e^{-kx}\frac{\partial}{\partial y}(A\cos ky + B\sin ky)$$

$$= e^{-kx}(-kA\sin ky + kB\cos ky) \quad \left\{\text{chain rule used}\right\}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial}{\partial y}\left\{e^{-kx}(-kA\sin ky + kB\cos ky)\right\}$$

$$= e^{-kx}\frac{\partial}{\partial y}(-kA\sin ky + kB\cos ky)$$

$$= e^{-kx}(-k^2A\cos ky - k^2B\sin ky)$$

$$\therefore \frac{\partial^2 V}{\partial y^2} = -k^2e^{-kx}(A\cos ky + B\sin ky) = -k^2V.$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = +k^2V - k^2V = 0,$$ as required for V to be a solution of the given equation.