SAMPLE EXAM 2

SECTION A

1. Answer **ALL** parts of the question:

(a) Determine the constant a such that the vectors $\mathbf{A} = 2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ are perpendicular.

(4 Marks)

(b) Consider a body that rotates with constant angular velocity $\boldsymbol{\omega} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Calculate the tangential velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ at the point given by the position vector $\mathbf{r} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$.

(4 Marks)

(c) Define $div\mathbf{V} = \nabla.\mathbf{V}$, where $\mathbf{V} = \mathbf{V}(x, y, z)$ is a vector field.
Describe, in words, what properties of the field $\mathbf{V}$ are expressed by its *divergence*.

(5 Marks)

(d) Define $curl\mathbf{V} = \nabla \times \mathbf{V}$, where $\mathbf{V} = \mathbf{V}(x, y, z)$ is a vector field.
Describe, in words, what properties of the field $\mathbf{V}$ are expressed by its *curl*.

(5 Marks)

(e) Find the matrix products $\mathbf{AB}$ and $\mathbf{BA}$, and hence show that $\mathbf{AB} \neq \mathbf{BA}$, when

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

(4 Marks)

(f) By considering matrix determinants, determine the *rank* of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(4 Marks)

QUESTION 1. IS CONTINUED ON NEXT PAGE

(g) The matrix $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ has *eigenvalues* given by $\lambda_1 = -2$ and $\lambda_2 = 5$.

Hence, verify that:

(i) the product of the eigenvalues of $\mathbf{B}$ is equal to the value of the determinant of $\mathbf{B}$; and

(ii) the sum of the eigenvalues of $\mathbf{B}$ is equal to the sum of the diagonal elements of $\mathbf{B}$.

(4 Marks)

(h) Use the *integrating factor method* to show that the general solution of

$\dfrac{dy}{dx} + py = q$, where p and q are constants, can be written as

$$y(x) = \frac{q}{p} + Ce^{-px} \; ,$$

where C is an arbitrary constant.

(5 Marks)

(i) Use the *chain rule* to show that $v = f(u)$ is a solution of the partial differential equation:

$$\frac{\partial v}{\partial x} + \frac{1}{c}\frac{\partial v}{\partial t} = 0 \, ,$$

when $u = x - ct$, c is a constant, and f is an arbitrary differentiable function.

(5 Marks)

SECTION B

2. An electrostatic charge distribution gives rise to a scalar potential given by:

$$\phi(x,y,z) = y^2 \sin x + xz^3 + 2z + 4y \ ,$$

where x, y, and z are space coordinates (physical units have been omitted for simplicity). Show that the associated vector field $\mathbf{E} = -\nabla\phi = -\mathrm{grad}\phi$ is given by:

$$\mathbf{E}(x,y,z) = -\left(y^2 \cos x + z^3\right)\mathbf{i} - \left(2y \sin x + 4\right)\mathbf{j} - \left(3xz^2 + 2\right)\mathbf{k} \ .$$

(10 Marks)

Hence, prove that the original charge distribution density, given by $\rho = \varepsilon_0 \nabla \bullet \mathbf{E} = \varepsilon_0 \mathrm{div}\,\mathbf{E}$, is:

$$\rho(x,y,z) = \varepsilon_0\left(y^2 - 2\right)\sin x - 6\varepsilon_0 xz \ .$$

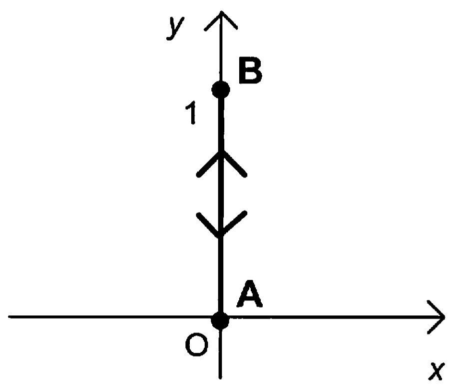
(10 Marks)

For the above vector field $\mathbf{E}(x,y,z)$, verify that $\oint_C \mathbf{E}.\mathbf{dr} = 0$ by considering line elements, $\mathbf{dr}$, along the closed path C (up and down the y-axis, in the $z = 0$ plane) defined as:

(i) from point A at the origin, where $(x, y) = (0,0)$, to point B, where $(x, y) = (0,1)$,

and then

(ii) from point B back to point A (see diagram below).



(10 Marks)

3. For a vector field $\mathbf{A}(x, y, z)$, Stokes' theorem can be stated as:

$$\int_S (\mathbf{curl} \quad \mathbf{A}) \cdot \mathbf{dS} = \oint_C \mathbf{A} \cdot \mathbf{dr} .$$

Explain the meaning of each of the symbols used in this equation and the interpretation of the quantity $\mathbf{curl} \quad \mathbf{A}$ that is implied by this theorem.

(8 marks)

For the particular vector field:

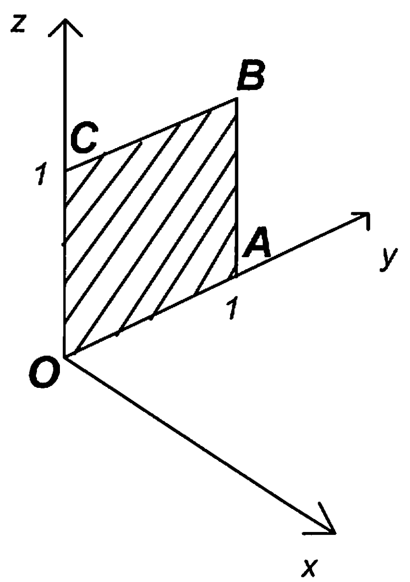
$$\mathbf{A}(x, y, z) = xy\,\mathbf{i} + (2y - xz)\mathbf{j} + xz\,\mathbf{k} ,$$

show that:

$$\mathbf{curl} \quad \mathbf{A} = x\mathbf{i} - z\mathbf{j} - (x + z)\mathbf{k} .$$

(10 marks)

Hence, show that $\int_S (\mathbf{curl} \quad \mathbf{A}) \cdot \mathbf{dS} = 0$ when S is the square area in the $x = 0$ plane whose corners are O (0,0,0), A (0,1,0), B (0,1,1) and C (0,0,1). This area is illustrated (shaded) in the figure below. In your calculation, assume that $\mathbf{dS}$ points in the *positive* x-direction.



(12 marks)

4. Answer **BOTH** parts of the question:

(a) Show that the *eigenvalues* of the matrix $\mathbf{A} = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix}$ are given by

$\lambda_1 = 4$ and $\lambda_2 = 6$.

(8 Marks)

Hence, find <u>either one</u> of the two linearly independent *eigenvectors* of the matrix $\mathbf{A}$.

(8 Marks)

(b) Use a method of your choice, to find the *inverse* of the matrix $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

(14 Marks)

5. Answer **BOTH** parts of the question:

(a) An electrical circuit consists of a resistance R and a capacitance C connected in series to a battery of constant voltage V. By considering the voltage dropped across R and C, one arrives at an ordinary differential equation for the charge stored $Q(t)$:

$$R\frac{dQ}{dt}+\frac{Q}{C}=V \ ,$$

where t is time. Show that the general solution of this differential equation is given by:

$$Q(t)=Q(0)\,\mathrm{e}^{-t/RC}+VC\left(1-\mathrm{e}^{-t/RC}\right).$$

Also find the particular solution when $Q(0)=0$ and identify the long-term steady-state and transient components of this particular solution. Give a physical interpretation of your results.

(20 Marks)

(b) Use the (partial differential equation) method of *separation of variables* to prove that a solution of $\dfrac{\partial u}{\partial x}=4\dfrac{\partial u}{\partial y}$ is given by $u(x,y)=K\,\mathrm{e}^{c(4x+y)}$, where K and c are constants.

(10 Marks)

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SAMPLE EXAM 2 SOLUTIONS

EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	(a) $\underset{\sim}{A} = 2\underset{\sim}{i} + a\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{B} = 4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$
	$\underset{\sim}{A}\cdot\underset{\sim}{B} = (2\underset{\sim}{i} + a\underset{\sim}{j} + \underset{\sim}{k})\cdot(4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k})$
	$= 8 - 2a - 2 = 6 - 2a$
4	Perpendicular when $\underset{\sim}{A}\cdot\underset{\sim}{B} = 0$ i.e. when $6 - 2a = 0$ i.e. when $a = 3$.
	(b) $\underset{\sim}{v} = \underset{\sim}{w} \times \underset{\sim}{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix}$
	$= \underset{\sim}{i}\begin{vmatrix} 3 & -1 \\ -6 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix}$
	$= \underset{\sim}{i}(-9-6) - \underset{\sim}{j}(-12+2) + \underset{\sim}{k}(-24-6)$
4	$= -15\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\sim}{k}$.
	(c) $\operatorname{div}\underset{\sim}{V} = \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}\right)$
	$= \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z$, when $\underset{\sim}{V} = (V_x, V_y, V_z)$.
5	$\operatorname{div}\underset{\sim}{V}$ is the net outflow of flux of $\underset{\sim}{V}$ per unit volume (at a point). Equivalently, it is the volume density of 'sources' and 'sinks' of flux of $\underset{\sim}{V}$.

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
5	(d) curl $\underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ V_y & V_z \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ V_x & V_z \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ V_x & V_y \end{vmatrix}$ etc. curl $\underset{\sim}{V}$ gives the 'circulation' of the field $\underset{\sim}{V}$ about each point. In other words, it gives the degree (and orientation) of 'twist/swirl/rotation/vorticity' of the field.
4	(e) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$. $BA = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix} \neq AB$ (here, or in general)
4	(f) $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = (0\times 0) - (1\times 0) = 0 - 0 = 0$ $\therefore$ rank < 2 while submatrix $[1]$ can be found with non-zero determinant $(\lvert 1\rvert \neq 0)$. $\therefore$ rank $= 1$.

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
	(g) $B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ $\lambda_1 = -2$, $\lambda_2 = 5$ (given eigenvalues)
	(i) $\det B = \lvert B \rvert = (2\times1 - 3\times4) = 2-12 = -10$
	while $\lambda_1\lambda_2 = (-2)\times(5) = -10$; (an example of a $\underline{\text{general}}$ result).
	(ii) $\lambda_1+\lambda_2 = -2+5 = 3$ ✓
	while "trace of B" is sum of diagonal elements.
4	Here, trace(B) $= 2+1 = 3$ ✓, also ; (another $\underline{\text{general}}$ result).
	(h) $\frac{dy}{dx} + py = q$ ✓ (p, q constants) : of standard form : $\frac{dy}{dx} + P(x)y = Q(x)$
	where $P(x)=p$, $Q(x)=q$. Integrating factor $= e^{\int P dx} = e^{\int p dx} = e^{p\int dx}$
	$= e^{px}$.
	$\underline{\text{Multiply equation:}}$
	$e^{px}\frac{dy}{dx} + e^{px}py = e^{px}q$ to get : $\frac{d}{dx}\left(e^{px}y\right) = e^{px}q$ ✓
	$\underline{\text{Integrate:}}$ $e^{px}y = \int e^{px}q\,dx + C$ $\Rightarrow$ $e^{px}y = q\int e^{px}dx + C$
5	$\Rightarrow e^{px}y = \frac{q}{p}e^{px} + C$ $\Rightarrow$ $y = \frac{q}{p} + Ce^{-px}$.

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT

(i) $\boxed{\frac{\partial v}{\partial x} + \frac{1}{c}\frac{\partial v}{\partial t} = 0}$. Verify $v = f(u)$ is solution where $u = x - ct$.

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u} \cdot \frac{\partial u}{\partial x}$ (chain rule)

$= \frac{\partial v}{\partial u} \cdot 1$ $\left[\text{since } \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x - ct)\right]$.

$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial u} \cdot \frac{\partial u}{\partial t}$ (chain rule)

$= \frac{\partial v}{\partial u} \cdot (-c)$

$\Rightarrow$ Substitute into equation (left-hand-side):

$\frac{\partial v}{\partial u} \cdot 1 + \frac{1}{c} \cdot \frac{\partial v}{\partial u} \cdot (-c) = \frac{\partial v}{\partial u} - \frac{\partial v}{\partial u} = 0$ = RHS of equation

$\therefore$ we have proved this is a solution, without needing to ~~specific~~ specify the function "f".

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT

$$\phi = y^2 \sin x + xz^3 + 2z + 4y$$

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} \; ; \; \underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial\phi}{\partial x} + \underset{\sim}{j}\frac{\partial\phi}{\partial y} + \underset{\sim}{k}\frac{\partial\phi}{\partial z},$$

where $\frac{\partial\phi}{\partial x} = y^2\cos x + z^3$, $\frac{\partial\phi}{\partial y} = 2y\sin x + 4$,

$$\frac{\partial\phi}{\partial z} = 3xz^2 + 2$$

$$\Rightarrow \underset{\sim}{E} = -\underset{\sim}{\nabla}\phi = -\left\{\underset{\sim}{i}(y^2\cos x + z^3) + \underset{\sim}{j}(2y\sin x + 4) + \underset{\sim}{k}(3xz^2+2)\right\}$$

$$= -(y^2\cos x + z^3)\underset{\sim}{i} - (2y\sin x + 4)\underset{\sim}{j} - (3xz^2+2)\underset{\sim}{k}$$

10

$$\rho = \varepsilon_0 \underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \varepsilon_0\left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\left(E_x\underset{\sim}{i} + E_y\underset{\sim}{j} + E_z\underset{\sim}{k}\right)$$

$$= \varepsilon_0\left(\frac{\partial}{\partial x}E_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

where $\frac{\partial}{\partial x}E_x = \frac{\partial}{\partial x}\left\{-(y^2\cos x + z^3)\right\} = -\left(y^2(-\sin x)\right) = +y^2\sin x$

$$\frac{\partial}{\partial y}E_y = \frac{\partial}{\partial y}\left\{-(2y\sin x + 4)\right\} = -(2\sin x + 0) = -2\sin x$$

$$\frac{\partial}{\partial z}E_z = \frac{\partial}{\partial z}\left\{-(3xz^2+2)\right\} = -(6xz+0) = -6xz$$

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT
	$\therefore \rho = \varepsilon_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \varepsilon_0 \left\{ y^2 \sin x - 2\sin x - 6xz \right\}$
	$= \varepsilon_0 (y^2 - 2)\sin x - 6\varepsilon_0 xz.$
10	
	$\underset{\sim}{E} = -(y^2\cos x + z^3)\underset{\sim}{i} - (2y\sin x + 4)\underset{\sim}{j} + (3xz^2 + 2)\underset{\sim}{k}$
	$\underset{\sim}{E}(z=0) = -y^2\cos x\,\underset{\sim}{i} - (2y\sin x + 4)\underset{\sim}{j} + 2\underset{\sim}{k}$
	Here, $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} = dy\,\underset{\sim}{j}$ $\quad (dx = dz = 0)$
	$\underset{\sim}{E} \cdot d\underset{\sim}{r} = (E_x \underset{\sim}{i} + E_y \underset{\sim}{j} + E_z \underset{\sim}{k}) \cdot (dy\,\underset{\sim}{j})$
	$= E_y\,dy \quad (\underset{\sim}{j} \cdot \underset{\sim}{j} = 1,\ \underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{j} = 0)$
	$= -(2y\sin x + 4)\,dy$
	Also, $x = 0 \Rightarrow \underset{\sim}{E} \cdot d\underset{\sim}{r} = -4\,dy$
	$\therefore \oint_C \underset{\sim}{E} \cdot d\underset{\sim}{r} = \int_A^B \underset{\sim}{E} \cdot d\underset{\sim}{r} + \int_B^A \underset{\sim}{E} \cdot d\underset{\sim}{r} = \int_0^1 (-4)\,dy + \int_1^0 (-4)\,dy$
	$= -4[y]_0^1 + (-4).[y]_1^0 = -4 - 4.(-1)$
10	$= 4 - 4 = 0.$

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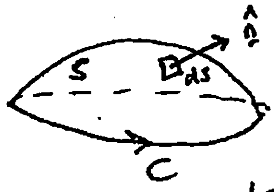
Paper Code:	Question No. 3

SOLUTION

MARKS	TEXT

$$\boxed{\int_S (\text{curl}\,\underset{\sim}{A}) \cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{A} \cdot d\underset{\sim}{r}}$$



Surface integral over open surface S of dot product of curl $\underset{\sim}{A}$ $\left(= \left|\begin{smallmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{smallmatrix}\right|\right)$ and $d\underset{\sim}{S}$ and closed line/curve integral along bounding curve C (clockwise sense with respect to $\hat{n}$) of dot product of $\underset{\sim}{A}$ and $d\underset{\sim}{r}$ element (along C).

Projection of $\underset{\sim}{\nabla} \times \underset{\sim}{A}$ gives surface density that integrates over S to give total circulation of $\underset{\sim}{A}$ around C

curl $\underset{\sim}{A}$ measures circulation/rotation/vorticity around a point

8 * [sufficient selection for full marks] *

$$\underset{\sim}{A} = xy\,\underset{\sim}{i} + (2y - xz)\,\underset{\sim}{j} + xz\,\underset{\sim}{k} \equiv A_x \underset{\sim}{i} + A_y \underset{\sim}{j} + A_z \underset{\sim}{k}$$

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ A_y & A_z \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ A_x & A_z \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ A_x & A_y \end{vmatrix}$$

$$= \underset{\sim}{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial}{\partial z}A_y\right) - \underset{\sim}{j}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) + \underset{\sim}{k}\left(\frac{\partial}{\partial x}A_y - \frac{\partial A_x}{\partial y}\right)$$

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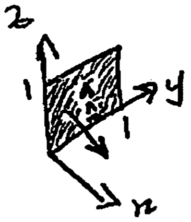
Paper Code:	Question No. 3

MARKS	TEXT

where $\frac{\partial A_z}{\partial y} = \frac{\partial}{\partial y}(xz) = 0$, $\frac{\partial A_y}{\partial z} = \frac{\partial}{\partial z}(2y - xz) = -x$, $\frac{\partial A_z}{\partial x} = \frac{\partial}{\partial x}(xz) = z$,

$\frac{\partial A_x}{\partial z} = \frac{\partial}{\partial z}(xy) = 0$, $\frac{\partial A_y}{\partial x} = \frac{\partial}{\partial x}(2y - xz) = -z$, $\frac{\partial A_x}{\partial y} = \frac{\partial}{\partial y}(xy) = x$

10 $\quad \therefore \underset{\sim}{\nabla} \times \underset{\sim}{A} = \underset{\sim}{i}\left[0 - (-x)\right] - \underset{\sim}{j}\left[z - 0\right] + \underset{\sim}{k}\left[(-z) - x\right] = x\underset{\sim}{i} - z\underset{\sim}{j} - (x+z)\underset{\sim}{k}$



$d\underset{\sim}{S} = dS\,\underset{\sim}{i}$ (+ve x-direction)

i.e. $d\underset{\sim}{S} = dy\,dz\,\underset{\sim}{i}$

AND on S, we have $x = 0$, therefore

$$\underset{\sim}{\nabla} \times \underset{\sim}{A} = 0\underset{\sim}{i} - z\underset{\sim}{j} - (0+z)\underset{\sim}{k} = -z\underset{\sim}{j} - z\underset{\sim}{k}.$$

$$\therefore \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{A}) \cdot d\underset{\sim}{S} = \int_S (-z\underset{\sim}{j} - z\underset{\sim}{k}) \cdot (\underset{\sim}{i}\,dy\,dz)$$

$$= \int_S \Big(-z\underbrace{\underset{\sim}{j} \cdot \underset{\sim}{i}}_{=0} - z\underbrace{\underset{\sim}{k} \cdot \underset{\sim}{i}}_{=0}\Big)\,dy\,dz$$

12 $\quad = \int_S 0 \,.\, dy\,dz = 0.$

[NB Circulation could be calculated instead but this takes longer]

NECESSARY

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 4

SOLUTION

MARKS

$A = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix}$. Eigenvalues are given by the characteristic equation: $\det|A-\lambda I| = 0$

Here, $\begin{vmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{vmatrix} = 0$.

i.e. $(8-\lambda)(2-\lambda) + 8 = 0$

i.e. $16 - 2\lambda - 8\lambda + \lambda^2 + 8 = 0$

i.e. $\lambda^2 - 10\lambda + 24 = 0$

8

i.e. $\left.\begin{array}{l}\lambda_1 + \lambda_2 = +10 \\ \lambda_1\lambda_2 = 24\end{array}\right\} \Rightarrow \begin{array}{l}\lambda_1 = 4 \\ \lambda_2 = 6\end{array}$.

Eigenvectors $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{x}_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$:

Satisfy $\begin{pmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

FIND EITHER EIGENVECTOR...

$\lambda_1 = 4$

$\begin{pmatrix} 8-\lambda_1 & -2 \\ 4 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $(8-\lambda_1)x_1 - 2y_1 = 0$
$4x_1 + (2-\lambda_1)y_1 = 0$..

i.e. $4x_1 - 2y_1 = 0$
$4x_1 - 2y_1 = 0$ $\left(\begin{array}{c}\text{since} \\ \lambda_1 = 4\end{array}\right)$

i.e. $2x_1 = y_1$.

$\therefore$ $\underset{\sim}{x}_1 = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, α undetermined scalar.

pt.

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SOLUTION

MARKS	TEXT

OR

$\lambda_2 = 6$

$$\begin{pmatrix} 8-\lambda_2 & -2 \\ 4 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{matrix} (8-\lambda_2)x_2 - 2y_2 = 0 \\ 4x_2 + (2-\lambda_2)y_2 = 0 \end{matrix}$$

i.e. $2x_2 - 2y_2 = 0$
$4x_2 - 4y_2 = 0$ (since $\lambda_2 = 6$)

i.e. $x_2 = y_2$.

$\therefore \ \underset{\sim}{x_2} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

where β is an undetermined scalar.

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METHOD I (b)

$B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$. Consider $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$ and work on column 1 first.

$r_2 \to r_2 + 2r_1$, $r_3 \to r_3 - r_1$ gives $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \end{array}\right]$. Now do column 2, $r_3 \to r_3 + r_2$ gives $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$.

Finally, do column 3: $r_1 \to r_1 + r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ and $r_2 \to r_2 + 2r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ i.e. $B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

2nd

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EXAMINATION SOLUTION CONTINUATION SHEET 3

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SOLUTION

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METHOD II (alternative)

Inverse of $B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ by the formal method: $B^{-1} = \frac{1}{|B|} C^T$

here $|B| = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} + 0 - \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}$ (expanding along row 1)

$= 2 - (2-1) = 1.$

Matrix of cofactors $A_{11} = +\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$, $A_{12} = -\begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = 4$, $A_{13} = +\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1$

$A_{21} = -\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = 1$, $A_{22} = +\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$, $A_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$

$A_{31} = +\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$, $A_{32} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$, $A_{33} = +\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$

$\therefore B^{-1} = \frac{1}{1} C^T = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$

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...ESSARY

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EXAMINATION SOLUTION SHEET

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SOLUTION

MARKS

$$\boxed{\frac{dQ}{dt} + \left(\frac{1}{RC}\right)Q = \frac{V}{R}}$$

$IF = e^{\frac{1}{RC}\int dt} = e^{\frac{t}{RC}}$. Now multiply equation

$$e^{t/RC}\frac{dQ}{dt} + \frac{e^{t/RC}}{RC}Q = e^{t/RC}\frac{V}{R}$$

$$\frac{d}{dt}\left[e^{t/RC}Q\right] = e^{t/RC}\cdot\frac{V}{R}$$

$$e^{t/RC}Q = \frac{V}{R}\cdot\frac{1}{\left(\frac{1}{RC}\right)}e^{t/RC} + A$$

i.e. $Q(t) = \frac{V}{R}(RC) + Ae^{-t/RC}$

(multiplying through by $e^{-t/RC}$)

i.e. $Q(t) = VC + Ae^{-t/RC}$

<u>Identify physical character of A</u>

at $t=0$, $Q(0) = VC + A$

$\therefore A = Q(0) - VC$

$\therefore$ General solution is $Q(t) = VC + \left[Q(0) - VC\right]e^{-t/RC}$

i.e. $Q(t) = VC + Q(0)e^{-t/RC} - VCe^{-t/RC}$

i.e. $Q(t) = Q(0)e^{-t/RC} + VC(1 - e^{-t/RC})$.

(+15)

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EXAMINATION SOLUTION CONTINUATION SHEET 1

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SOLUTION

MARKS	
	Particular solution: Uncharged at $t=0$ i.e. $Q(0)=0$ since switch closed
	$\therefore \quad Q(t) = VC\left(1-e^{-t/RC}\right)$
	$Q(t) = \underbrace{VC}_{\text{steady-state}} - \underbrace{VCe^{-t/RC}}_{\text{transient}}$ i.e. $\to 0$ as $t\to\infty$
(+5)	As $t\to\infty$, all voltage across C since voltage across R is IR
→ 20 SUBTOTAL	i.e. $\frac{dQ}{dt}R$ and requires time-varying charge.
	$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$
	Separation of variables: $u = X(x)Y(y)$
	gives $YX_x = 4XY_y$ (subscript denoting partial derivative)
	i.e. $\frac{X_x}{4X} = \frac{Y_y}{Y} = c$ (separation constant)
	Each equation can now be treated as an o.d.e. i.e. $\frac{dX}{dx} = 4cX$ and $\frac{dY}{dy} = cY$
	Solutions are $X = Ae^{4cx}$, $Y = Be^{cy}$.
10	A solution is thus $\boxed{u = XY = Ke^{c(4x+y)}}$, $K = AB$