SAMPLE EXAM 3

SECTION A

1. Answer **ALL** parts of the question:

(a) Determine the constant a such that the vectors $\mathbf{F} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = a\mathbf{i} - 7\mathbf{j} - 18\mathbf{k}$ are perpendicular.

(4 Marks)

(b) The position of a moving particle is given by a time-dependent position vector $\mathbf{r}(t)$. Derive an expression for the velocity of the particle, $\mathbf{v}(t) = \dfrac{d\,\mathbf{r}(t)}{dt}$,

when:

$$\mathbf{r}(t) = \mathrm{e}^{-t}\ \mathbf{i} + 2\cos(3t)\ \mathbf{j} + 2\sin(3t)\ \mathbf{k}.$$

(4 Marks)

(c) Describe what is meant by: (a) a *scalar field*; and (b) a *vector field.*
With reference to a typical weather forecast map, give one example of each type of field.

(6 Marks)

(d) Describe the property of the vector field $\mathbf{A}$ that $\operatorname{div}\mathbf{A}$ (i.e. $\nabla \bullet \mathbf{A}$) represents (make reference to the divergence theorem and give one, or more, physical examples in your answer).

(5 Marks)

(e) Describe the property of the vector field $\mathbf{A}$ that $\operatorname{curl}\mathbf{A}$ (i.e. $\nabla \times \mathbf{A}$) represents (make reference to Stokes' theorem and give one, or more, physical examples in your answer).

(5 Marks)

QUESTION A IS CONTINUED ON THE NEXT PAGE

(f) By considering matrix determinants, show that the *rank* of the matrix $\mathbf{A}$ is equal to 1 when:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}.$$

(4 Marks)

(g) Relate your answer for part *(c)* to a description of the solution of the following two simultaneous equations:

$$2x + 6y = 1$$

$$3x + 9y = 2.$$

Illustrate your answer with a sketch that includes lines in the x-y plane.

(4 Marks)

(h) Two simultaneous linear equations, with constant coefficients a , b , c and d , take the form:

$$ax + by = e$$

$$cx + dy = f$$

where e and f are also constants. Verify that the homogeneous system ($e = f = 0$) always has the trivial solution ($x = y = 0$).

(4 Marks)

(i) Use the *chain rule* to show that $v = f(u)$ is a solution of the partial differential equation:

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0,$$

where $u = y - 3x$ and f is an arbitrary differentiable function.

(4 Marks)

SECTION B

2. A particular scalar field is defined by the equation:

$$\phi(x,y,z) = x^2 z + 2xy^2 + yz^2 \ .$$

Show that $\nabla\phi$ for this field is given by:

$$\nabla\phi = \left(2xz + 2y^2\right)\mathbf{i} + \left(4xy + z^2\right)\mathbf{j} + \left(x^2 + 2yz\right)\mathbf{k} \ .$$

(7 marks)

Hence, calculate the *magnitude* of the rate of change of $\phi(x,y,z)$ at the point $(x,y,z) = (1,\ 2, -1)$ in the direction of the vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

(13 marks)

Use the given form of $\nabla\phi$ (in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$) to prove that $\nabla\phi$ is a *conservative field.*

(10 marks)

3. Answer **BOTH** parts of the question:

(a) For a vector field $\mathbf{A}(x, y, z)$, Stokes' theorem can be stated as:

$$\int_S (curl\ \mathrm{A})\cdot \mathrm{dS} = \oint_C \mathrm{A}\cdot \mathrm{dr}.$$

Explain the meaning of each of the symbols used in this equation and the interpretation of the quantity $curl\ \mathbf{A}$ that is implied by this theorem.

(8 Marks)

(b) The fluid velocity of a particular uniform flow is given by $\mathbf{V}_2 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. An example of a non-uniform flow is given by $\mathbf{V}_3 = 2y\,\mathbf{i}$. Evaluate $curl\ \mathbf{V}$ for each of the flows $\mathbf{V}_2$ and $\mathbf{V}_3$ and interpret the results. What does *Stokes' theorem* imply regarding the character of the vector fields representing the uniform and the non-uniform flows?

(14 Marks)

The *circulation* of a vector field $\mathbf{V}(x, y, z) = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$ around a closed path C, can be written as: $\oint_C \mathbf{V}.\ \mathbf{dr} = \oint_C V_x\,\mathrm{dx} + V_y\,\mathrm{dy} + V_z\,\mathrm{dz}$. Hence, verify *Stokes' theorem* for the uniform flow $\mathbf{V}_2$ by considering the closed path C in the *x-y* plane (around the four sides of a square) given by:

$$(x, y, z) = (0, 0, 0) \rightarrow (0, 1, 0) \rightarrow (1, 1, 0) \rightarrow (1, 0, 0) \rightarrow (0, 0, 0).$$

(8 Marks)

4. Answer **BOTH** parts of the question:

(a) If

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{bmatrix},$$

show that the *matrix of cofactors* of $\mathbf{A}$ is:

$$\mathbf{C} = \begin{bmatrix} -8 & 7 & -6 \\ 1 & -4 & 2 \\ 5 & -5 & 5 \end{bmatrix}.$$

Show also that the determinant of $\mathbf{A}$ is $\det(\mathbf{A}) = -5$. Hence, find the matrix $\mathbf{A}^{-1}$ (that is the inverse of the matrix $\mathbf{A}$).

(15 Marks)

(b) Prove that the *eigenvalues* of the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ are given by $\lambda_1 = 1$ and $\lambda_2 = 4$. Hence, find *either one* of the two linearly independent *eigenvectors* of the matrix $\mathbf{B}$.

(15 Marks)

5. Answer **BOTH** parts of the question:

(a) An electrical circuit consists of a resistance R and an inductance L connected in series to a battery of constant voltage V. The voltage dropped across R and L, and the current $I(t)$, are related through the ordinary differential equation:

$$L\frac{dI}{dt} + RI = V ,$$

where t is time.

Show that the general solution of this differential equation is given by:

$$I(t) = I(0) \ \exp\left[-\left(\frac{R}{L}\right)t\right] + \frac{V}{R}\left\{1 - \exp\left[-\left(\frac{R}{L}\right)t\right]\right\} .$$

Find the particular solution when $I(0) = 0$ and identify the long-term (steady-state) and transient components of this particular solution.

(18 Marks)

(b) Use the method of *separation of variables* to prove that a solution of the partial differential equation $3\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ is given by $u(x, y) = K\, e^{c(x-3y)}$, where K and c are constants.

(12 Marks)

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SAMPLE EXAM 3 SOLUTIONS

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
4	1. (a) $\underset{\sim}{F} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{r} = a\underset{\sim}{i} - 7\underset{\sim}{j} - 18\underset{\sim}{k}$ $\therefore \underset{\sim}{F}\cdot\underset{\sim}{r} = (2a) + (2).(-7) + (-1).(-18)$ $= 2a - 14 + 18 = 2a + 4$ ie $\underset{\sim}{F}.\underset{\sim}{r} = 0$ when $2a+4=0$ i.e. when $a = -2$.
4	1(b) $\underset{\sim}{r}(t) = e^{-t}\underset{\sim}{i} + 2\cos 3t\,\underset{\sim}{j} + 2\sin 3t\,\underset{\sim}{k} = (r_x(t), r_y(t), r_z(t))$ $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{dr_x}{dt}\underset{\sim}{i} + \frac{dr_y}{dt}\underset{\sim}{j} + \frac{dr_z}{dt}\underset{\sim}{k}$ $\therefore \underset{\sim}{v} = -e^{-t}\underset{\sim}{i} + (-6\sin 3t)\underset{\sim}{j} + 6\cos 3t\,\underset{\sim}{k}$.
6	1(c) Scalar field ≡ Region of space with unique scalar value associated with each point Vector field ≡ as above, with vector value. e.g. scalar: temperature, pressure, vector: wind ~~speed~~ velocity

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
5	1(d) $\underline{\operatorname{div} \underset{\sim}{A}}$ ≡ net volume density of sources and sinks of flux ("net outflow") $\int_V \underbrace{\operatorname{div} \underset{\sim}{A}}_{\text{volume density}}\, dV = \underbrace{\oint_S \underset{\sim}{A} \cdot d\underset{\sim}{S}}_{\text{net flux through closed surface } S}$ $\boxed{\text{sufficient selection for full marks}}$ $\underline{\text{e.g.}}$ ± point charges → $\underset{\sim}{E}$-flux, lack of magnetic monopoles → $\operatorname{div} \underset{\sim}{B} = 0$, incompressible fluid with no sources/sinks → $\operatorname{div} \underset{\sim}{v} = 0$
5	1(e) $\underline{\operatorname{curl} \underset{\sim}{A}}$ ≡ twist/swirl/rotation/vorticity/circulation about a point $\int_S \operatorname{curl} \underset{\sim}{A} \cdot d\underset{\sim}{S} = \underbrace{\oint_C \underset{\sim}{A} \cdot d\underset{\sim}{l}}_{\text{circulation around } C}$ $\boxed{\text{sufficient selection for full marks}}$ $\underline{\text{e.g.}}$ electrostatics : $\underset{\sim}{\nabla} \times \underset{\sim}{E} = \underset{\sim}{0}$ (no vortices, conservative), solenoid, current-carrying wire : loops/vortices in $\underset{\sim}{B}$-field, $\underset{\sim}{\nabla} \times \underset{\sim}{B} \neq \underset{\sim}{0}$, non-uniform fluid-flow ("paddlewheel"), tornadoes, hurricanes, plug-holes, fluid vortices...
4	1(f) $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$: $\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 2.9 - 6.3 = 18 - 18 = 0$ $\therefore \text{rank}(A) < 2$ Existence of one 1×1 submatrix with non-zero determinant is sufficient to give A a rank of 1. For 1×1 matrices, determinant = element value. $\underline{\text{Any of } [2], [6], [3], [9]}$ give non-zero determinant, $\therefore \text{rank}(A) = 1$.

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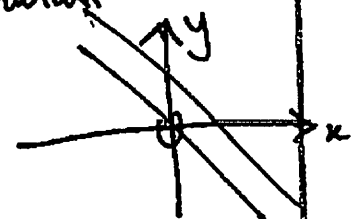
Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
4	1(g) System is $A\underset{\sim}{x} = \underset{\sim}{b}$, where $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$ and $\lvert A \rvert = 0$
	Augmented coeff. matrix: $A_b = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 9 & 2 \end{bmatrix}$ and, e.g., $\begin{vmatrix} 6 & 1 \\ 9 & 2 \end{vmatrix} \neq 0$
	$\Rightarrow \text{rank}(A_b) = 2$.
	$\therefore \text{rank}(A) < \text{rank}(A_b) \Rightarrow$ no solution
	Sketch: $\left.\begin{aligned} y &= -\tfrac{1}{3}x + \tfrac{1}{6} \\ y &= -\tfrac{1}{3}x + \tfrac{2}{9} \end{aligned}\right\}$ parallel lines do not cross $\rightarrow$ no solution 
	(sufficient amount of above for full marks)
4	1(h) $\begin{aligned} ax + by &= e = 0 \\ cx + dy &= f = 0 \end{aligned}$ The trivial solution is $x = y = 0$.
	Verify this general result by direct substitution:
	$x = 0, y = 0$ gives $\begin{aligned} a.0 + b.0 &= 0 \\ c.0 + d.0 &= 0 \end{aligned}$ ✓ ✓

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 1

SOLUTION

MARKS	TEXT
4	1(i) $v = f(u)$, $u = y - 3x$, $\boxed{\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0}$ (*) To show this (*), $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial u}\cdot(-3)$ $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial u}\cdot(1)$ $\therefore \frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\cdot(-3) + 3\frac{\partial v}{\partial u}\cdot(1)$ $= -3\frac{\partial v}{\partial u} + 3\frac{\partial v}{\partial u} = 0$ $\therefore$ $v = f(u)$ is a solution irrespective of the particular form of (arbitrary differentiable) function f.

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT

$$\phi(x,y,z) = x^2z + 2xy^2 + yz^2$$

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$$

$$\Rightarrow \underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial}{\partial x}(x^2z+2xy^2+yz^2) + \underset{\sim}{j}\frac{\partial}{\partial y}(x^2z+2xy^2+yz^2) + \underset{\sim}{k}\frac{\partial}{\partial z}(x^2z+2xy^2+yz^2)$$

$$= \underset{\sim}{i}(2xz+2y^2+0) + \underset{\sim}{j}(0+4xy+z^2) + \underset{\sim}{k}(x^2+0+2yz)$$

7 $$\therefore \underset{\sim}{\nabla}\phi = (2xz+2y^2)\underset{\sim}{i} + (4xy+z^2)\underset{\sim}{j} + (x^2+2yz)\underset{\sim}{k}.$$

At point $(1,2,-1)$ $$\underset{\sim}{\nabla}\phi = (-2+8)\underset{\sim}{i} + (8+1)\underset{\sim}{j} + (1-4)\underset{\sim}{k}$$

$$= 6\underset{\sim}{i} + 9\underset{\sim}{j} - 3\underset{\sim}{k}.$$

Direction derivative $\frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi \cdot \underset{\sim}{\hat{u}}$, where $\underset{\sim}{\hat{u}}$ is a unit vector in the specified direction

In the direction of $2\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k}$,

$$\underset{\sim}{\hat{u}} = \frac{2\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k}}{|2\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k}|}$$

where $|2\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k}| = (2^2+3^2+(-4)^2)^{\frac{1}{2}}$

$$= (4+9+16)^{\frac{1}{2}} = \sqrt{29}.$$

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EXAMINATION SOLUTION CONTINUATION SHEET

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Paper Code:	Question No. 2

SOLUTION

MARKS	TEXT
	$\therefore \ \hat{\underset{\sim}{n}} = \frac{1}{\sqrt{29}}(2,3,-4)$
	and $\frac{d\phi}{ds} = \underset{\sim}{\nabla}\phi \cdot \hat{\underset{\sim}{n}} = (6\underset{\sim}{i}+9\underset{\sim}{j}-3\underset{\sim}{k})\cdot\frac{1}{\sqrt{29}}(2\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k})$
	$= \frac{1}{\sqrt{29}}(12+27+12)$
13	$= \frac{51}{\sqrt{29}}$.
	Let $\underset{\sim}{V} = \underset{\sim}{\nabla}\phi \equiv V_x\underset{\sim}{i}+V_y\underset{\sim}{j}+V_z\underset{\sim}{k}$,
	where $V_x = 2xz+2y^2$, $V_y = 4xy+z^2$, $V_z = x^2+2yz$
ii	$\underset{\sim}{\nabla}\phi$ conservative $\Leftrightarrow$ $\underset{\sim}{\nabla}\times\underset{\sim}{V} = \underset{\sim}{0}$,
	where $\underset{\sim}{\nabla}\times\underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$
	$= \underset{\sim}{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \underset{\sim}{j}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \underset{\sim}{k}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$

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SOLUTION

MARKS	TEXT
	$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(x^2+2yz) - \frac{\partial}{\partial z}(4xy+z^2)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(x^2+2yz) - \frac{\partial}{\partial z}(2xz)\right]$
	$+\underset{\sim}{k}\left[\frac{\partial}{\partial x}(4xy+z^2) - \frac{\partial}{\partial y}(2xz+2y^2)\right]$
	$= \underset{\sim}{i}\left[2z - 2z\right] - \underset{\sim}{j}\left[2x - 2x\right] + \underset{\sim}{k}\left[4y - 4y\right]$
	$= 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} = \underset{\sim}{0}$ (for all x, y, z)
10	$\therefore \underset{\sim}{\nabla}\phi = \underset{\sim}{V}$ is a conservative vector field

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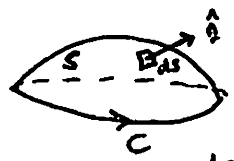
Paper Code:	Question No. 3

SOLUTION

MARKS	TEXT
	(a) $\displaystyle\int_S (\text{curl}\,\underset{\sim}{A})\cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{A}\cdot d\underset{\sim}{r}$



Surface integral over open surface S of dot product of curl $\underset{\sim}{A}$ $\left(= \left|\begin{matrix} \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot \end{matrix}\right|\right)$ and $d\underset{\sim}{S}$ and closed line/curve integral along bounding curve C (clockwise sense with respect to $\hat{\underset{\sim}{n}}$) of dot product of $\underset{\sim}{A}$ and $d\underset{\sim}{r}$ element (along C).

Projection of $\underset{\sim}{\nabla}\times\underset{\sim}{A}$ gives surface density that integrates over S to give total circulation of $\underset{\sim}{A}$ around C

curl $\underset{\sim}{A}$ measures circulation/rotation/vorticity around a point

[sufficient selection for full marks]

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EXAMINATION SOLUTION CONTINUATION SHEET 1

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Paper Code:	Question No. 3.

MARKS

(b) $\underset{\sim}{v}_2 = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ (a uniform flow)

$$\underset{\sim}{\nabla} \times \underset{\sim}{v}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2 & 3 & 1 \end{vmatrix} = \underset{\sim}{i}(0-0) - \underset{\sim}{j}(0-0) + \underset{\sim}{k}(0-0) = \underset{\sim}{0}$$

$\rightarrow$ Irrotational field (no circulation/vortices anywhere).

$\underset{\sim}{v}_3 = 2y\,\underset{\sim}{i}$

$$\underset{\sim}{\nabla} \times \underset{\sim}{v}_3 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & 0 & 0 \end{vmatrix} = \underset{\sim}{i}(0-0) - \underset{\sim}{j}(0-0) + \underset{\sim}{k}(0-2) = -2\underset{\sim}{k}.$$

$\rightarrow$ Rotational field (const. circulation / degree of rotation at each point).

- Stokes' theorem $\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{r} = \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{v})\cdot d\underset{\sim}{S}$

$\underset{\sim}{\nabla}\times\underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \oint_C \underset{\sim}{v}\cdot d\underset{\sim}{r} = 0$ (no circulation around any closed path)
(— provided it's a simple path)!

also implies path independence of $\int_A^B \underset{\sim}{v}\cdot d\underset{\sim}{r}$ and other 'conservative properties'.

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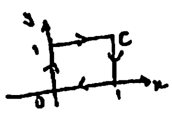

$$\oint_C \underset{\sim}{v}\cdot d\underset{\sim}{r} = \oint_C v_x dx + v_y dy + v_z dz$$

$$= \oint_C 2dx + 3dy + 1.dz \quad \text{, using } \underset{\sim}{v}_2.$$

$$\therefore \oint_C \underset{\sim}{v}_2\cdot d\underset{\sim}{r} = \underset{(dx=dz=0)}{\int_0^1 3\,dy} + \underset{(dy=dz=0)}{\int_0^1 2dx} + \underset{(dx=dz=0)}{\int_1^0 3\,dy} + \underset{(dy=dz=0)}{\int_1^0 2dx}$$

$$= \int_0^1 3dy + \int_0^1 2dx - \int_0^1 3dy - \int_0^1 2dx = 0$$

(from above, $\underset{\sim}{\nabla}\times\underset{\sim}{v}_2 = 0 \Rightarrow \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{v})\cdot d\underset{\sim}{S} = 0$ and we have verified consistency with Stokes' theorem.)

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT

(a)

$$A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{pmatrix}$$

cofactors $A_{11} = +\begin{vmatrix} 2 & 1 \\ -2 & -5 \end{vmatrix} = -8$, $A_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} = 7$, $A_{13} = +\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -6$,

$A_{21} = -\begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1$, $A_{22} = +\begin{vmatrix} 2 & -3 \\ 2 & -5 \end{vmatrix} = -4$, $A_{23} = -\begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$,

$A_{31} = +\begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} = 5$, $A_{32} = -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$, $A_{33} = +\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$.

$\therefore$ Matrix of cofactors, $C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} -8 & 7 & -6 \\ 1 & -4 & 2 \\ 5 & -5 & 5 \end{pmatrix}$

e.g. $|A| = 2A_{11} - 1.A_{12} - 3.A_{13}$ (along top row)

$= -16 - 7 + 18 = -5.$

Then, $A^{-1} = \frac{1}{|A|} C^T = -\frac{1}{5}\begin{pmatrix} -8 & 1 & 5 \\ 7 & -4 & -5 \\ -6 & 2 & 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 8 & -1 & -5 \\ -7 & 4 & 5 \\ 6 & -2 & -5 \end{pmatrix}.$

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SOLUTION

MARKS	TEXT

(b) $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$: Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2 = 0$$

i.e. $6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

i.e. $\lambda^2 - 5\lambda + 4 = 0$

By inspection, $\left.\begin{matrix} \lambda_1 + \lambda_2 = +5 \\ \lambda_1\lambda_2 = 4 \end{matrix}\right\} \rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 4. \end{matrix}$

Eigenvectors Since $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$, these satisfy $A\underset{\sim}{x} - \lambda I\underset{\sim}{x} = \underset{\sim}{0}$

i.e. $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

i.e. $\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Denote $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ as the eigenvector associated with λ_1.

Then, $\begin{pmatrix} 3-\lambda_1 & 2 \\ 1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix} (3-\lambda_1)x_1 + 2y_1 = 0 \\ x_1 + (2-\lambda_1)y_1 = 0 \end{matrix}$

i.e. $\begin{matrix} (3-1)x_1 + 2y_1 = 0 \\ x_1 + (2-1)y_1 = 0 \end{matrix}$ (since $\lambda_1 = 1$)

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Paper Code:	Question No. 4

SOLUTION

MARKS	TEXT

i.e. $\left.\begin{aligned} 2x_1 + 2y_1 &= 0 \\ x_1 + y_1 &= 0 \end{aligned}\right\} \Rightarrow x_1 = -y_1$

Eigenvectors are only defined in terms of the ratio of the components (to within an undetermined scalar)

$\therefore \underset{\sim}{x}_1 = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, α = undetermined scalar.

OR

$$\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Denote $\underset{\sim}{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ as the eigenvector associated with $\lambda_2 = 4$.

Then, $\begin{pmatrix} 3-\lambda_2 & 2 \\ 1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{aligned} (3-\lambda_2)x_2 + 2y_2 &= 0 \\ x_2 + (2-\lambda_2)y_2 &= 0 \end{aligned}$

i.e. $\begin{aligned} (3-4)x_2 + 2y_2 &= 0 \\ x_2 + (2-4)y_2 &= 0 \end{aligned}$ (since $\lambda_2 = 4$)

i.e. $\begin{aligned} -x_2 + 2y_2 &= 0 \\ x_2 - 2y_2 &= 0 \end{aligned}$

Both equations imply $x_2 = 2y_2$.

$\therefore \underset{\sim}{x}_2 = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where β is an undetermined scalar.

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EXAMINATION SOLUTION SHEET

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Paper Code:	Question No. 5

SOLUTION	
MARKS	TEXT

a)

Standard form: $L\frac{dI}{dt} + RI = V \Rightarrow \frac{dI}{dt} + \left(\frac{R}{L}\right)I = \left(\frac{V}{L}\right)$

$$I.F. = e^{\int P\,dt} = e^{R/L\int dt} = e^{R/L\,t} \Rightarrow e^{R/L\,t}\frac{dI}{dt} + e^{R/L\,t}\left(\frac{R}{L}\right)I = e^{R/L\,t}\left(\frac{V}{L}\right)$$

$$\Rightarrow \frac{d}{dt}\left[e^{R/L\,t}.\,I\right] = e^{R/L\,t}.\left(\frac{V}{L}\right)$$

$$\Rightarrow e^{R/L\,t}.I = \frac{V}{L}\int e^{R/L\,t}dt + C'$$

$$= \frac{V}{L}\cdot\left(\frac{L}{R}\right)e^{R/L\,t} + C = \frac{V}{R}e^{R/L\,t} + C.$$

$\Rightarrow I(t) = V/R + Ce^{-R/L\,t}$, after dividing by $e^{R/L\,t}$.

C can be identified by noting that at $t=0$, $I(0) = \frac{V}{R} + Ce^{0} = \frac{V}{R} + C$

$$\rightarrow C = I(0) - \frac{V}{R}. \rightarrow I(t) = \frac{V}{R} + \left(I(0) - \frac{V}{R}\right)e^{-R/L\,t} = \frac{V}{R} + I(0)e^{-R/L\,t} - \frac{V}{R}e^{-R/L\,t}$$

i.e. $I(t) = I(0)e^{-R/L\,t} + \frac{V}{R}\left(1 - e^{-R/L\,t}\right).$

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$I(t) = I(0)e^{-R/L\,t} + \frac{V}{R}\left(1 - e^{-R/L\,t}\right).$

Particular solution when $I(0)=0$, i.e. switch closed at $t=0 \Rightarrow I=0$ at $t=0$,

$I(t) = 0 + \frac{V}{R}\left(1 - e^{-R/L\,t}\right)$, $\quad I(t) = \underbrace{\frac{V}{R}}_{\text{(constant/ steady-state)}} \underbrace{- \frac{V}{R}e^{-R/L\,t}}_{\text{(time-varying, tends to zero, transient)}}$

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Paper Code:	Question No. 5

SOLUTION

MARKS	TEXT

(b)

$$\boxed{3\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0}$$

let $u(x,y)=X(x).Y(y)$ and substitute ....

$3Y\frac{\partial X}{\partial x}+X\frac{\partial Y}{\partial y}=0$, rearrange as functions of x only, y only

$\Rightarrow \frac{3}{X}\frac{\partial X}{\partial x}=-\frac{1}{Y}\frac{\partial Y}{\partial y}$, introduce separation constant (c),

$\Rightarrow \frac{3}{X}\frac{\partial X}{\partial x}=c \Rightarrow \boxed{\frac{3}{X}\frac{dX}{dx}=c}$ and $-\frac{1}{Y}\frac{\partial Y}{\partial y}=c \Rightarrow \boxed{\frac{dY}{dy}=-cY}$

i.e. pdes → odes (only one indept. var. in each)

$\frac{dX}{dx}=\frac{c}{3}.X$

$\int\frac{dX}{X}=\frac{c}{3}\int dx$ (ode sep. of vars)

$\ln X=\frac{c}{3}x+A$

$X=e^{(\frac{c}{3}x+A)}$

$\int\frac{dY}{Y}=-c\int dy$

$\Rightarrow \ln Y=-cy+B$

$\Rightarrow Y=e^{(-cy+B)}$

A solution is then $u=XY=e^{(\frac{c}{3}x+A)}.e^{(-cy+B)}$

$=e^{c/3.x-cy}.e^{A+B}$

$=e^{c(\frac{x}{3}-y)}.e^{A+B}$

$=e^{c'(x-3y)}.e^{A+B}$, $c'=\frac{c}{3}$

$\therefore u(x,y)=ke^{c'(x-3y)}$, $k=e^{A+B}$.

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