'Tutorial A'

First supplement to 'Main Tutorials 1 → 4'

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES — Intermediate Mathematics

Gradients and Directional Derivatives

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The aim of this package is to provide a short self assessment programme for students who want to obtain an ability in vector calculus to calculate gradients and directional derivatives.

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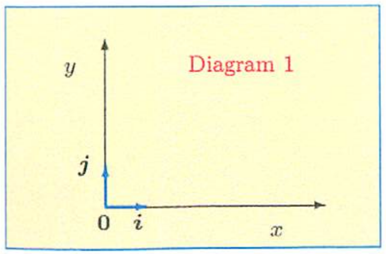
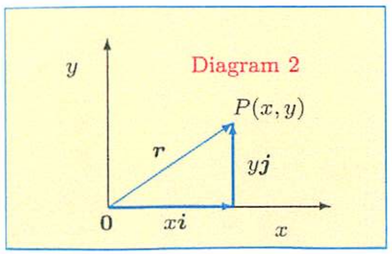
The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction (Vectors)

The base vectors in two dimensional Cartesian coordinates are the unit vector $\boldsymbol{i}$ in the positive direction of the x axis and the unit vector $\boldsymbol{j}$ in the y direction. See Diagram 1. (In three dimensions we also require $\boldsymbol{k}$, the unit vector in the z direction.)

The position vector of a point $P(x, y)$ in two dimensions is $x\boldsymbol{i} + y\boldsymbol{j}$. We will often denote this important vector by $\boldsymbol{r}$. See Diagram 2. (In three dimensions the position vector is $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.)





The vector differential operator ∇ , called "del" or "nabla", is **defined** in three dimensions to be:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} .$$

Note that these are *partial derivatives*!
This vector operator may be applied to (differentiable) scalar functions (scalar fields) and the result is a special case of a vector field, called a gradient vector field.

Here are two warming up exercises on partial differentiation.

Quiz Select the following partial derivative, $\frac{\partial}{\partial z}(xyz^x)$.

(a) x^2yz^{x-1} , (b) 0 , (c) $xy\log_x(z)$, (d) yz^{x-1} .

Quiz Choose the partial derivative $\frac{\partial}{\partial x}(x\cos(y) + y)$.

(a) $\cos(y)$, (b) $\cos(y) - x\sin(y) + 1$,
(c) $\cos(y) + x\sin(y) + 1$, (d) $-\sin(y)$.

2. Gradient (Grad)

The **gradient** of a function, $f(x, y)$, in two dimensions is defined as:

$$\text{grad} f(x,y) = \nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\,.$$

The gradient of a function is a vector field. It is obtained by applying the vector operator ∇ to the scalar function $f(x, y)$. Such a vector field is called a gradient (or conservative) vector field.

Example 1 The gradient of the function $f(x, y) = x + y^2$ is given by:

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\\ &= \frac{\partial}{\partial x}(x+y^2)\boldsymbol{i} + \frac{\partial}{\partial y}(x+y^2)\boldsymbol{j}\\ &= (1+0)\boldsymbol{i} + (0+2y)\boldsymbol{j}\\ &= \boldsymbol{i} + 2y\boldsymbol{j}\,.\end{aligned}$$

Quiz Choose the gradient of $f(x, y) = x^2y^3$.

(a) $2x\boldsymbol{i} + 3y^2\boldsymbol{j}$, (b) $x^2\boldsymbol{i} + y^3\boldsymbol{j}$,
(c) $2xy^3\boldsymbol{i} + 3x^2y^2\boldsymbol{j}$, (d) $y^3\boldsymbol{i} + x^2\boldsymbol{j}$.

The definition of the **gradient** may be extended to functions defined in three dimensions, $f(x, y, z)$:

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\,.$$

EXERCISE 1. Calculate the gradient of the following functions (click on the **green** letters for the solutions).

(a) $f(x, y) = x + 3y^2$, (b) $f(x, y) = \sqrt{x^2 + y^2}$,
(c) $f(x, y, z) = 3x^2\sqrt{y} + \cos(3z)$, (d) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$,
(e) $f(x, y) = \dfrac{4y}{(x^2 + 1)}$, (f) $f(x, y, z) = \sin(x)e^y \ln(z)$.

3. Directional Derivatives

To interpret the gradient of a scalar field

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\,,$$

note that its component in the $\boldsymbol{i}$ direction is the partial derivative of f with respect to x. This is the rate of change of f in the x direction since y and z are kept constant. In general, the component of ∇f in any direction is the rate of change of f in that direction.

Example 2 Consider the scalar field $f(x, y) = 3x + 3$ in two dimensions. It has no y dependence and it is linear in x. Its gradient is given by

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}(3x+3)\boldsymbol{i} + \frac{\partial}{\partial y}(3x+3)\boldsymbol{j}\\ &= 3\boldsymbol{i} + 0\boldsymbol{j}\,.\end{aligned}$$

As would be expected the gradient has zero component in the y direction and its component in the x direction is constant (3).

Quiz Select a point from the answers below at which the scalar field $f(x, y, z) = x^2yz - xy^2z$ *decreases* in the y direction.

(a) $(1, -1, 2)$, (b) $(1, 1, 1)$,
(c) $(-1, 1, 2)$, (d) $(1, 0, 1)$.

Definition: if $\hat{\boldsymbol{n}}$ is a unit vector, then $\hat{\boldsymbol{n}}\cdot\nabla f$ is called the directional derivative of f in the direction $\hat{\boldsymbol{n}}$. The directional derivative is the rate of change of f in the direction $\hat{\boldsymbol{n}}$.

Example 3 Let us find the directional derivative of $f(x, y,) = x^2yz$ in the direction $4\boldsymbol{i} - 3\boldsymbol{k}$ at the point $(1, -1, 1)$.
The vector $4\boldsymbol{i} - 3\boldsymbol{k}$ has magnitude $\sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$. The unit vector in the direction $4\boldsymbol{i} - 3\boldsymbol{k}$ is thus $\hat{\boldsymbol{n}} = \frac{1}{5}(4\boldsymbol{i} - 3\boldsymbol{k})$.
The gradient of f is

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}(x^2yz)\boldsymbol{i} + \frac{\partial}{\partial y}(x^2yz)\boldsymbol{j} + \frac{\partial}{\partial z}(x^2yz)\boldsymbol{k}\\ &= 2xyz\boldsymbol{i} + x^2z\boldsymbol{j} + x^2y\boldsymbol{k}\,,\end{aligned}$$

and so the required directional derivative is

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\nabla f &= \frac{1}{5}(4\boldsymbol{i}-3\boldsymbol{k})\cdot(2xyz\boldsymbol{i}+x^2z\boldsymbol{j}+x^2y\boldsymbol{k})\\ &= \frac{1}{5}\left[4\times 2xyz+0-3\times x^2y\right].\end{aligned}$$

At the point $(1,-1,1)$ the desired directional derivative is thus

$$\hat{\boldsymbol{n}}\cdot\nabla f = \frac{1}{5}\left[8\times(-1)-3\times(-1)\right] = -1\,.$$

EXERCISE 2. Calculate the directional derivative of the following functions in the given directions and at the stated points (click on the green letters for the solutions).

(a) $f = 3x^2-3y^2$ in the direction $\boldsymbol{j}$ at (1,2,3).

(b) $f=\sqrt{x^2+y^2}$ in the direction $2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ at (0,−2,1).

(c) $f=\sin(x)+\cos(y)+\sin(z)$ in the direction $\pi\boldsymbol{i}+\pi\boldsymbol{j}$ at (π,0,π).

We now state, without proof, **two useful properties** of the directional derivative and gradient.

- The maximal directional derivative of the scalar field $f(x,y,z)$ is in the direction of the gradient vector ∇f.
- If a surface is given by $f(x,y,z)=c$ where c is a constant, then the normals to the surface are the vectors $\pm\nabla f$.

Example 4 Consider the surface $xy^3=z+2$. To find its unit normal at $(1,1,-1)$, we need to write it as $f=xy^3-z=2$ and calculate the gradient of f:

$$\nabla f = y^3\boldsymbol{i}+3xy^2\boldsymbol{j}-\boldsymbol{k}\,.$$

At the point $(1,1,-1)$ this is $\nabla f=\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$. The *magnitude* of this maximal rate of change is

$$\sqrt{1^2+3^2+(-1)^2}=\sqrt{11}\,.$$

Thus the unit normals to the surface are $\pm\frac{1}{\sqrt{11}}(\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k})$.

Quiz Which of the following vectors is normal to the surface $x^2yz=1$ at (1,1,1)?

(a)	$4\boldsymbol{i}+\boldsymbol{j}+17\boldsymbol{k}\,,$	(b)	$2\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}\,,$
(c)	$\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\,,$	(d)	$-2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}\,.$

Quiz Which of the following vectors is a unit normal to the surface $\cos(x)yz=-1$ at (π,1,1)?

(a)	$-\frac{1}{\sqrt{2}}\boldsymbol{j}+\frac{1}{\sqrt{2}}\boldsymbol{k}\,,$	(b)	$\pi\boldsymbol{i}+\boldsymbol{j}+\frac{2}{\sqrt{\pi}}\boldsymbol{k}\,,$
(c)	$\boldsymbol{i}\,,$	(d)	$-\frac{1}{\sqrt{2}}\boldsymbol{j}-\frac{1}{\sqrt{2}}\boldsymbol{k}\,.$

Quiz Select a unit normal to the (spherically symmetric) surface $x^2+y^2+z^2=169$ at (5,0,12).

(a)	$\boldsymbol{i}+\frac{1}{6}\boldsymbol{j}-\frac{1}{6}\boldsymbol{k}\,,$	(b)	$\frac{1}{3}\boldsymbol{i}+\frac{1}{3}\boldsymbol{j}+\frac{1}{3}\boldsymbol{k}\,,$
(c)	$\frac{5}{13}\boldsymbol{i}+\frac{12}{13}\boldsymbol{k}\,,$	(d)	$-\frac{5}{13}\boldsymbol{i}+\frac{12}{13}\boldsymbol{k}\,.$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. What is the gradient of $f(x,y,z)=xyz^{-1}$?

   (a) $\boldsymbol{i}+\boldsymbol{j}-z^{-2}\boldsymbol{k}\,,$ (b) $\frac{y}{z}\boldsymbol{i}+\frac{x}{z}\boldsymbol{j}-\frac{xy}{z^2}\boldsymbol{k}\,,$

   (c) $yz^{-1}\boldsymbol{i}+xz^{-1}\boldsymbol{j}+xyz^{-2}\boldsymbol{k}\,,$ (d) $-\frac{1}{z^2}\,.$

2. If n is a constant, choose the gradient of $f(\boldsymbol{r})=1/r^n$, where $r=|\boldsymbol{r}|$ and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$.

   (a) 0, (b) $-\frac{n}{2}\frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{r^{n+1}}\,,$ (c) $-\frac{n\boldsymbol{r}}{r^{n+2}}\,,$ (d) $-\frac{n}{2}\frac{\boldsymbol{r}}{r^{n+2}}\,.$

3. Select the unit normals to the surface of revolution, $z=2x^2+2y^2$ at the point (1,1,4).

   (a) $\pm\frac{1}{\sqrt{3}}(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})\,,$ (b) $\pm\frac{1}{\sqrt{3}}(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\,,$

   (c) $\pm\frac{1}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})\,,$ (d) $\pm\frac{1}{\sqrt{2}}(2\boldsymbol{i}+2\boldsymbol{j}-4\boldsymbol{k})\,.$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The function $f(x, y) = x + 3y^2$, has gradient

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \\ &= \frac{\partial}{\partial x}(x+3y^2)\boldsymbol{i} + \frac{\partial}{\partial y}(x+3y^2)\boldsymbol{j} \\ &= (1+0)\boldsymbol{i} + (0 + 3\times 2y^{2-1})\boldsymbol{j} \\ &= \boldsymbol{i} + 6y\boldsymbol{j}\,.\end{aligned}$$

Click on the green square to return □

Exercise 1(b) The gradient of the function

$$f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$$

is given by:

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} = \frac{\partial}{\partial x}(x^2+y^2)^{\frac{1}{2}}\boldsymbol{i} + \frac{\partial}{\partial y}(x^2+y^2)^{\frac{1}{2}}\boldsymbol{j} \\ &= \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}\times\frac{\partial}{\partial x}(x^2)\boldsymbol{i} \\ &\quad + \frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}\times\frac{\partial}{\partial y}(y^2)\boldsymbol{j} \\ &= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\times 2x^{2-1}\boldsymbol{i} + \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\times 2y^{2-1}\boldsymbol{j} \\ &= (x^2+y^2)^{-\frac{1}{2}}x\boldsymbol{i} + (x^2+y^2)^{-\frac{1}{2}}y\boldsymbol{j} \\ &= \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j}\,.\end{aligned}$$

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Exercise 1(c) The gradient of the function

$$f(x,y,z) = 3x^2\sqrt{y} + \cos(3z) = 3x^2y^{\frac{1}{2}} + \cos(3z)\,,$$

is given by:

$$\begin{aligned}\nabla f(x,y,z) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} \\ &= 3y^{\frac{1}{2}}\frac{\partial}{\partial x}(x^2)\,\boldsymbol{i} + 3x^2\frac{\partial}{\partial y}(y^{\frac{1}{2}})\,\boldsymbol{j} + \frac{\partial}{\partial z}(\cos(3z))\,\boldsymbol{k} \\ &= 3y^{\frac{1}{2}}\times 2x^{2-1}\,\boldsymbol{i} + 3x^2\times\frac{1}{2}y^{\frac{1}{2}-1}\,\boldsymbol{j} - 3\sin(3z)\,\boldsymbol{k} \\ &= 6y^{\frac{1}{2}}x\,\boldsymbol{i} + \frac{3}{2}x^2y^{-\frac{1}{2}}\,\boldsymbol{j} - 3\sin(3z)\,\boldsymbol{k} \\ &= 6x\sqrt{y}\,\boldsymbol{i} + \frac{3}{2}\frac{x^2}{\sqrt{y}}\boldsymbol{j} - 3\sin(3z)\,\boldsymbol{k}\,.\end{aligned}$$

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Exercise 1(d) The partial derivative of the function

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}\,,$$

with respect to the variable x is

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}\times\frac{\partial(x^2)}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

and similarly the derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are

$$\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,,\quad \frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,.$$

Therefore the gradient is

$$\nabla f(x,y,z) = -\frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,.$$

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Exercise 1(e) The gradient of the function

$$f(x,y) = \frac{4y}{(x^2+1)} = 4y(x^2+1)^{-1},$$

is:

$$\begin{aligned}\nabla f(x,y) &= 4y \times \frac{\partial}{\partial x}(x^2+1)^{-1}\boldsymbol{i} + (x^2+1)^{-1} \times \frac{\partial}{\partial y}4y\,\boldsymbol{j}\\ &= 4y \times (-1)(x^2+1)^{-1-1}\frac{\partial}{\partial x}(x^2+1)\,\boldsymbol{i} + 4(x^2+1)^{-1}\boldsymbol{j}\\ &= -4y(x^2+1)^{-2} \times 2x\,\boldsymbol{i} + \frac{4}{(x^2+1)}\boldsymbol{j}\\ &= -\frac{8xy}{(x^2+1)^2}\boldsymbol{i} + \frac{4}{(x^2+1)}\boldsymbol{j}.\end{aligned}$$

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Exercise 1(f) The partial derivatives of the function

$$f(x,y,z) = \sin(x)e^y\ln(z)$$

are

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\sin(x))\,e^y\ln(z) = \cos(x)\,e^y\,\ln(z),\\ \frac{\partial f}{\partial y} &= \sin(x)\,\frac{\partial}{\partial y}(e^y)\,\ln(z) = \sin(x)e^y\,\ln(z),\\ \frac{\partial f}{\partial z} &= \sin(x)e^y\,\frac{\partial}{\partial z}(\ln(z)) = \sin(x)\,e^y\,\frac{1}{z}.\end{aligned}$$

Therefore the gradient is

$$\nabla f(x,y,z) = \cos(x)\,e^y\ln(z)\boldsymbol{i} + \sin(x)\,e^y\ln(z)\boldsymbol{j} + \sin(x)\,e^y\frac{1}{z}\,\boldsymbol{k}.$$

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Exercise 2(a) The directional derivative of the function

$$f = 3x^2 - 3y^2$$

in the unit vector $\boldsymbol{j}$ direction is given by the scalar product $\boldsymbol{j}\cdot\nabla f$. The gradient of the function $f = 3x^2 - 3y^2$ is

$$\nabla f = 6x\boldsymbol{i} - 6y\boldsymbol{j}$$

Therefore the directional derivative in the $\boldsymbol{j}$ direction is

$$\boldsymbol{j}\cdot\nabla f = \boldsymbol{j}\cdot(6x\boldsymbol{i} - 6y\boldsymbol{j}) = -6y$$

and at the point $(1,2,3)$ it has the value $-6\times 2 = -12$.

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Exercise 2(b) The directional derivative of the function $f = \sqrt{x^2+y^2}$ in the direction defined by vector $2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ is given by the scalar product $\hat{\boldsymbol{n}}\cdot\nabla f$, where the unit vector $\hat{\boldsymbol{n}}$ is

$$\hat{\boldsymbol{n}} = \frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}}{\sqrt{9}} = \frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{1}{3}\boldsymbol{k}.$$

The gradient of the function f is

$$\nabla f = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{j} + 0\boldsymbol{k} = \frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}$$

Therefore the required directional derivative is

$$\hat{\boldsymbol{n}}\cdot\nabla f = \left(\frac{2}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{1}{3}\boldsymbol{k}\right)\cdot\left(\frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}}\right) = \frac{2}{3}\frac{x+y}{\sqrt{x^2+y^2}}.$$

At the point $(0,-2,1)$ it is equal to $\frac{2}{3}\frac{0-2}{\sqrt{0^2+(-2)^2}} = \frac{2}{3}\times\frac{-2}{2} = -\frac{2}{3}$.

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Exercise 2(c) The directional derivative of the function

$$f = \sin(x) + \cos(x) + \sin(z)$$

in the direction defined by the vector $\pi\boldsymbol{i} + \pi\boldsymbol{j}$ is given by the scalar product $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} f$, where the unit vector $\hat{\boldsymbol{n}}$ is

$$\hat{\boldsymbol{n}} = \frac{\pi\boldsymbol{i} + \pi\boldsymbol{j}}{\sqrt{\pi^2 + \pi^2}} = \frac{\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}}.$$

The gradient of the function f is

$$\boldsymbol{\nabla} f = \cos(x)\boldsymbol{i} - \sin(y)\boldsymbol{j} + \cos(z)\boldsymbol{k}.$$

Therefore the directional derivative is

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} f = \frac{\boldsymbol{i} + \boldsymbol{j}}{\sqrt{2}} \cdot (\cos(x)\boldsymbol{i} - \sin(y) + \cos(z)\boldsymbol{k}) = \frac{\cos(x) - \sin(y)}{\sqrt{2}}$$

and at the point $(\pi, 0, \pi)$ it becomes $\dfrac{\cos(\pi) - \sin(0)}{\sqrt{2}} = -\dfrac{1}{\sqrt{2}}$.

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Solutions to Quizzes

Solution to Quiz:
The partial derivative of xyz^x with respect to the variable z is

$$\frac{\partial}{\partial z}(xyz^x) = xy \times \frac{\partial}{\partial z}(z^x) = xy \times x \times z^{(x-1)} = x^2yz^{(x-1)}$$

End Quiz

Solution to Quiz:
Consider the function $f(x, y) = x\cos(y) + y$, its derivative with respect to the variable x is

$$\begin{aligned}
\frac{\partial}{\partial x} f(x,y) &= \frac{\partial}{\partial x}(x\cos(y) + y) \\
&= \frac{\partial}{\partial x}(x) \times \cos(y) + \frac{\partial}{\partial x}(y) \\
&= 1 \times \cos(y) + 0 = \cos(y).
\end{aligned}$$

End Quiz

Solution to Quiz:
The gradient of the function $f(x, y) = x^2y^3$ is given by:

$$\begin{aligned}
\boldsymbol{\nabla} f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x^2y^3)\boldsymbol{i} + \frac{\partial}{\partial y}(x^2y^3)\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x^2) \times y^3\boldsymbol{i} + x^2 \times \frac{\partial}{\partial y}(y^3)\boldsymbol{j} \\
&= 2x^{2-1} \times y^3\boldsymbol{i} + x^2 \times 3y^{3-1}\boldsymbol{j} \\
&= 2xy^3\boldsymbol{i} + 3x^2y^2\boldsymbol{j}.
\end{aligned}$$

End Quiz

Solution to Quiz: The partial derivative of the scalar function $f(x, y, z) = x^2yz - xy^2z$ with respect to y is

$$\frac{\partial f}{\partial y}(x, y, z) = x^2 - 2xyz\,.$$

Evaluating it at the point $(1, 1, 1)$ gives

$$\frac{\partial f}{\partial y}(1, 1, 1) = 1^2 - 2 \times 1 \times 1 \times 1 = 1 - 2 = -1\,.$$

This is negative and therefore the function f decreases in the y direction at this point.
It may be verified that the function does not decrease in the y direction at any of the other three points. End Quiz

Solution to Quiz: The surface is defined by the equation

$$x^2yz = 1.$$

To find its normal at $(1, 1, 1)$ we need to calculate the gradient of the function $f(x, y, z) = x^2yz$:

$$\nabla f = 2xyz\boldsymbol{i} + x^2z\boldsymbol{j} + x^2y\boldsymbol{k}\,.$$

At the point $(1, 1, 1)$ this is

$$\nabla f = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$$

Thus the required normals to the surface are $\pm(2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$. Hence (d) is a normal vector to the surface. End Quiz

Solution to Quiz: The surface is defined by the equation

$$\cos(x)yz = -1\,.$$

To find its unit normal at the point $(\pi, 1, 1)$, we need to evaluate the gradient of $f(x, y, z) = \cos(x)yz$:

$$\nabla f = -\sin(x)yz\boldsymbol{i} + \cos(x)z\boldsymbol{j} + \cos(x)y\boldsymbol{k}\,.$$

At the point $(\pi, 1, 1)$ this is

$$\nabla f = 0\boldsymbol{i} + (-1)\boldsymbol{j} + (-1)\boldsymbol{k} = -\boldsymbol{j} - \boldsymbol{k}$$

The magnitude of this vector is

$$\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}\,.$$

Therefore the unit normal is

$$\hat{n} = -\frac{1}{\sqrt{2}}\boldsymbol{j} - \frac{1}{\sqrt{2}}\boldsymbol{k}\,.$$

End Quiz

Solution to Quiz: The surface is defined by the equation

$$x^2 + y^2 + z^2 = 169\,.$$

To find its unit normal at point $(5, 0, 12)$ we need to evaluate the gradient of $f(x, y, z) = x^2 + y^2 + z^2$:

$$\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}\,.$$

At the point $(5, 0, 12)$ this is

$$\nabla f = 2 \times 5\boldsymbol{i} + 0 \times \boldsymbol{j} + 2 \times 12\boldsymbol{k} = 10\boldsymbol{i} + 24\boldsymbol{k}$$

The magnitude of this vector is

$$\sqrt{(2 \times 5)^2 + (2 \times 12)^2} = \sqrt{4 \times (25 + 144)} = 2\sqrt{169} = 2 \times 13\,.$$

Therefore the unit normal is

$$\hat{n} = \frac{5}{13}\boldsymbol{i} + \frac{12}{13}\boldsymbol{k}\,.$$

End Quiz

Mini-Tutorial : GRADIENTS and DIRECTION DERIVATIVES

Theoretical Physics:
GS McDONALD

Solutions to 4. FINAL QUIZ

1.
$$\nabla f = \underset{\sim}{i}\frac{\partial}{\partial x}\left(\frac{xy}{z}\right) + \underset{\sim}{j}\frac{\partial}{\partial y}\left(\frac{xy}{z}\right) + \underset{\sim}{k}\frac{\partial}{\partial z}\left(\frac{xy}{z}\right)$$

$$= \frac{y}{z}\underset{\sim}{i} + \frac{x}{z}\underset{\sim}{j} - \frac{xy}{z^2}\underset{\sim}{k}$$

2.
$$\nabla f = \underset{\sim}{i}\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right) + \underset{\sim}{j}\frac{\partial}{\partial y}\left(\frac{1}{r^n}\right) + \underset{\sim}{k}\frac{\partial}{\partial z}\left(\frac{1}{r^n}\right)$$

$$= \underset{\sim}{i}\frac{\partial}{\partial r}\left(\frac{1}{r^n}\right)\frac{\partial r}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial r}\left(\frac{1}{r^n}\right)\frac{\partial r}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial r}\left(\frac{1}{r^n}\right)\frac{\partial r}{\partial z}$$

$$= -\frac{n}{r^{n+1}}\cdot\frac{x}{r}\underset{\sim}{i} - \frac{n}{r^{n+1}}\cdot\frac{y}{r}\underset{\sim}{j} - \frac{n}{r^{n+1}}\cdot\frac{z}{r}\underset{\sim}{k}$$

$$= -\frac{n}{r^{n+2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}\right) = -\frac{n}{r^{n+2}}\underset{\sim}{r}.$$

Chain Rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x}$$

where $r^2 = x^2+y^2+z^2$

$\rightarrow 2r\frac{\partial r}{\partial x} = 2x \rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

also $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

3. Surface equation : $2x^2+2y^2-z=0$: $f(x,y,z)$ = constant.

Normals given by
$$\nabla f = \underset{\sim}{i}\frac{\partial}{\partial x}(2x^2+2y^2-z) + \underset{\sim}{j}\frac{\partial}{\partial y}(2x^2+2y^2-z) + \underset{\sim}{k}\frac{\partial}{\partial z}(2x^2+y^2-z)$$

$$= 4x\underset{\sim}{i} + 4y\underset{\sim}{j} - \underset{\sim}{k}. = \underset{\sim}{n}.$$

(Unit normal $\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}$, where $|\underset{\sim}{n}| = \sqrt{\ }$)

At $(1,1,4)$ $\underset{\sim}{n} = 4\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$, where $|\underset{\sim}{n}| = \sqrt{4^2+4^2+(-1)^2}$

$x=1, y=1, z=4$

$= \sqrt{33}$

$$\rightarrow \hat{\underset{\sim}{n}} = \pm\frac{1}{\sqrt{33}}\left(4\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}\right)$$