THEORETICAL PHYSICS I *FORMULA SHEETS*

CONTENTS

Dr Graham S McDonald : last revised 10-01-02

BASIC ALGEBRA

Equations and identities

Equations are usually only true for particular values of the variable(s) involved

Identities are equations that are true for all values of the variable(s) involved
An identity has the left-hand side of the equations *identical* to the right-hand side of the equation

Examples of identities are

$(x+2)^2 = x^2 + 4x + 4$ (true for all values of x)

$x^2 - a^2 = (x-a)(x+a)$ (true for all values of x)

$x^2 - y^2 = (x-y)(x+y)$ (true for all values of x and y)

Inverse functions

For $y = f(x)$ we need f to be a one-to-one function to define the inverse function f^{-1} such that $x = f^{-1}(y)$

Changing the subject of the formula from $y = f(x)$ to $x = f^{-1}(y)$ can give the form of the inverse function

A sketch of the curve $y = f^{-1}(x)$ can be obtained by reflecting the curve $y = f(x)$ in the line $y = x$

Indices and logarithms

In the term $5a^3$, 5 is the *coefficient*, a is the *base*, and 3 is the *index, exponent*, or *logarithm to base a*
In general, if $a^n = N$ then $n = \log_a N$

Three basic laws of indices	Three basic laws of logarithms
$a^m a^n = a^{m+n}$	$\log AB = \log A + \log B$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\log \dfrac{A}{B} = \log A - \log B$
$(a^m)^n = a^{nm}$	$\log A^n = n \log A$

These sets of laws are related by identifying $A = a^m$ and $B = a^n$

We also have

$a^0 = 1$ $\qquad \log 1 = 0$

$\dfrac{1}{a^n} = a^{-n}$ $\qquad \log_a N = \dfrac{\log_b N}{\log_b a}$

$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{a^m}$ $\qquad$ (the rule for changing from base a to base b)

Natural logarithms: If $e^n = N$ then $n = \log_e N$, also written as $n = \ln N$

FORMULA SHEET 2

POLYNOMIALS

Polynomial evaluation by nesting

$$a_2x^2 + a_1x + a_0 = (a_2x + a_1)x + a_0$$
$$a_3x^3 + a_2x^2 + a_1x + a_0 = [(a_3x + a_2)x + a_1]x + a_0$$
$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = \{[(a_4x + a_3)x + a_2]x + a_1\}x + a_0$$

Factor theorem

If $f(x)$ is a polynomial and substituting $x = a$ gives zero, i.e. $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$

Polynomial division

If the highest power of x in the numerator is equal to or greater than the highest power of x in the denominator then an algebraic fraction is said to be *improper*

Three steps for polynomial division:

(i) Add a multiple of the denominator to the top line to give the term with the highest power of x

(ii) Compensate for any new terms that have been added to the top line

(iii) Write the result as two algebraic fractions. If one still has an improper algebraic fraction then go back to step (i)

Factorising a quadratic expression

When $ax^2 + bx + c = 0$,

If discriminant $b^2 - 4ac$ is
- positive then have real different roots
- zero then have real equal roots
- negative then have no real roots

If discriminant $b^2 - 4ac = 0^2, 1^2, 2^2, 3^2, \ldots$ (i.e. a perfect square) then expect simple linear factors

When $a = 1$, factorisation ' by inspection ' is straightforward:

$$x^2 + bx + c = (x - \alpha)(x - \beta), \quad \text{where } \alpha + \beta = -b \text{ and } \alpha\beta = c$$

When $a \neq 1$, $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, where $\alpha . \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

FORMULA SHEET 3

Completing the square (quadratic equations)

Consider the general quadratic equation $ax^2 + bx + c = 0$

When $a = 1$,

$$\begin{aligned}
x^2 + bx \qquad + c &= 0 \\
x^2 + bx + \left(\frac{b}{2}\right)^2 + c &= \left(\frac{b}{2}\right)^2 \\
\left(x + \frac{b}{2}\right)^2 &= \left(\frac{b}{2}\right)^2 - c \\
x + \frac{b}{2} &= \pm\sqrt{\left(\frac{b}{2}\right)^2 - c}
\end{aligned}$$

When $a \neq 1$,

$$\begin{aligned}
ax^2 + bx \qquad + c &= 0 \\
a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c &= a\left(\frac{b}{2a}\right)^2 \\
a\left(x + \frac{b}{2a}\right)^2 &= a\left(\frac{b}{2a}\right)^2 - c \\
x + \frac{b}{2a} &= \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}
\end{aligned}$$

Finding partial fractions

In general, there are three steps in finding partial fractions.
However, you may only have to do step (iii), for example.

(i) Ensure that the given expression involves only *proper* algebraic fractions. Otherwise, perform polynomial division until one has only proper algebraic fractions
(ii) Factorise the denominator(s) as far as possible
(iii) Express in terms of partial fractions using the following rules:

	Expression in the denominator	Form of partial fraction(s)
Linear factor	$(ax + b)$	$\dfrac{A}{ax + b}$
Repeated factor	$(ax + b)^2$	$\dfrac{A}{ax + b} + \dfrac{B}{(ax + b)^2}$
Quadratic factor (that does not factorise)	$(ax^2 + bx + c)$	$\dfrac{Ax + B}{ax^2 + bx + c}$

where A and B are constants to be determined

FORMULA SHEET 4

Binomial theorem (positive integer index n)

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$$

where $p! = p(p-1)(p-2)\ldots2.1$, e.g $4!=4.3.2.1=24$

For example,

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

When n is not too large, one can read the binomial coefficients from ` Pascals's triangle `

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

For example,

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

TRIGONOMETRY

Circles and angles

Area of circle (with radius r), $A = \pi r^2$. Circumference of circle, $C = 2\pi r$

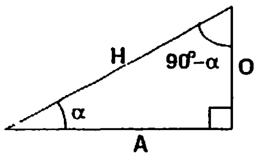
1 revolution = 360° = 2π radians , where angle (in radians) = (length of circle arc) / r

∴ Length of circular arc = r θ , and Area of sector of a circle = (1/2) r^2 θ , where θ is in radians.

The 3 main trig ratios

For a positive acute angle α ,

cos α = A / H
sin α = O / H
tan α = O / A



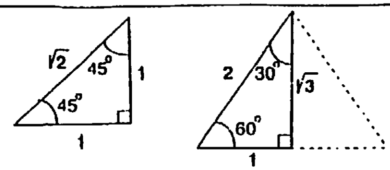
We also have:

Pythagoras theorem: $A^2 + O^2 = H^2$

Complementary angles: cos α = sin (90°−α) and sin α = cos (90°−α)

FORMULA SHEET 5

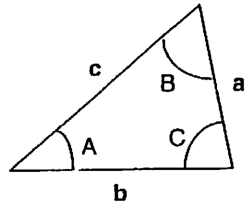
Special triangles



General triangles

Sine Rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

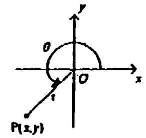
Cosine Rule:
$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

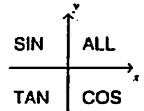


Recall also that: A + B + C = 180°

Evaluation of cos θ , sin θ and tan θ (any angle θ)

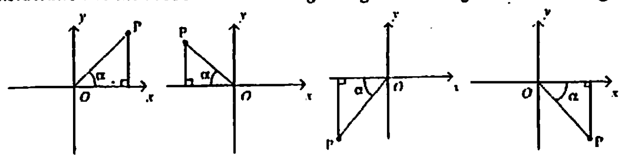
For any angle θ ,
$\cos\theta = x/r$
$\sin\theta = y/r$
$\tan\theta = y/x$



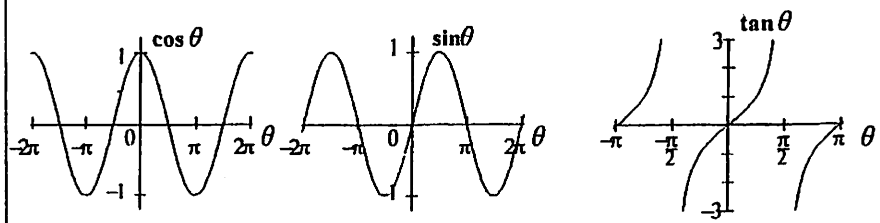


summarises where the ratios are positive, i.e. gives the *sign* of the trig ratio

Consideration of the *associated acute angle α* gives the *magnitude* of the trig ratio



Graphs of cos θ , sin θ and tan θ



$-1 \le \cos\theta \le 1$, period of 2π

$-1 \le \sin\theta \le 1$, period of 2π

$\tan\theta$ has unlimited range, period of π

Graphs of related functions (a , k , α are constants)

- $a\cos\theta$ has period 2π and amplitude a [similarly for $a\sin\theta$]
- $a+\cos\theta$ is $\cos\theta$ shifted upwards by a [similarly for $a+\sin\theta$]
- $\cos k\theta$ and $\sin k\theta$ have period $2\pi/k$; $\tan k\theta$ has period π/k
- $\cos(\theta-\alpha)$ is $\cos\theta$ shifted right by α [similarly for $\sin(\theta-\alpha)$, $\tan(\theta-\alpha)$]
- $\cos(k\theta-\alpha)$ has period $2\pi/k$ and is $\cos k\theta$ shifted right by amount α/k [similarly for $\sin(k\theta-\alpha)$]

Principal and Secondary Values

In solving the equations $\sin\theta=s$ and $\tan\theta=t$ (where s and t are constants),

the principal value (PV) is the solution $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
(i.e. in the 1st and 4th quadrants)

In solving the equation $\cos\theta=c$ (where c is a constant),

the principal value (PV) is the solution $0\le\theta\le\pi$
(i.e. in the 1st and 2nd quadrants)

For any of the above equations, a secondary value (SV) may exist in the other quadrants

General solutions

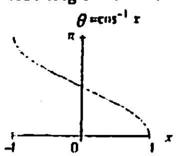
$\sin\theta=s$ has general solution $\theta=\begin{cases}\text{PV}+2n\pi\\ \text{SV}+2n\pi\end{cases}$

$\cos\theta=c$ has general solution $\theta=\pm\text{PV}+2n\pi$

$\tan\theta=t$ has general solution $\theta=\text{PV}+n\pi$

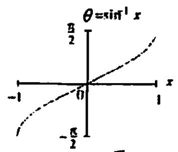
(where $n=0,\pm1,\pm2,\pm3,\ldots$)

Inverse trig functions



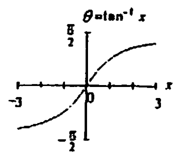
Principal Value $0\le\theta\le\pi$

θ is the angle whose cosine is x



Principal Value $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

θ is the angle whose sine is x



Principal Value $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

θ is the angle whose tangent is x

Small angle approximations (when $\theta<<1$ and θ measured in *radians*)

$\sin\theta\approx\theta$, $\tan\theta\approx\theta$, $\cos\theta\approx1-\frac{\theta^2}{2}$

Trig identities and further trig equations

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

(reciprocal trig functions)

$\cos^2\theta+\sin^2\theta=1$

$1+\tan^2\theta=\sec^2\theta$

$\cot^2\theta+1=\text{cosec}^2\theta$

where

$\sec\theta=1/\cos\theta$

$\text{cosec}\theta=1/\sin\theta$

$\cot\theta=1/\tan\theta$

For example, solving equations $a\cos\theta+b\sin^2\theta=c$, where a, b, c constants,
substitute $\sin^2\theta=1-\cos^2\theta$ to give a quadratic equation in $\cos\theta$.

For example, solving equations $a\cos^2\theta+b\sin\theta=c$, where a, b, c constants,
substitute $\cos^2\theta=1-\sin^2\theta$ to give a quadratic equation in $\sin\theta$.

(compound angle formulae)

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

For example, solving equations $a\cos\theta+b\sin\theta=c$, where a, b, c constants :

- compare $a\cos\theta+b\sin\theta$ with a compound angle formula to write the left-hand-side as $r\cos(\theta\pm\alpha)$ or $r\sin(\theta\pm\alpha)$, where r, α are constants
- match the signs of coefficients a and b with those of a compound angle formula to give α in the 1st quadrant

Note that a graph of $a\cos\theta+b\sin\theta$ versus θ would have amplitude r and phase shift α

***t*-substitution method** : an alternative method of solving $a\cos\theta+b\sin\theta=c$ is to

- let $t=\tan\frac{\theta}{2}$ to give $\cos\theta=\frac{1-t^2}{1+t^2}$ and $\sin\theta=\frac{2t}{1+t^2}$
- substitute for $\cos\theta$ and $\sin\theta$ and rearrange resulting equation into a quadratic in t
- solve the quadratic equation for t and re-express in terms of θ

·Trig identities and further trig equations — continued

(double angle formulae)

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

For example, solving equations $a\cos 2\theta + b\sin\theta = c$, where a, b, c constants :
substitute $\cos 2\theta = 1 - 2\sin^2\theta$ to give a quadratic equation in $\sin\theta$

For example, solving equations $a\cos 2\theta + b\cos\theta = c$, where a, b, c constants :
substitute $\cos 2\theta = 2\cos^2\theta - 1$ to give a quadratic equation in $\cos\theta$

(half angle formulae)

$$\cos^2 A = \frac{(1+\cos 2A)}{2}$$
$$\sin^2 A = \frac{(1-\cos 2A)}{2}$$

(sum formulae – "factorisation")

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

(product formulae)

$$\sin A\cos B = +\frac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos A\cos B = +\frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$$

HYPERBOLIC FUNCTIONS

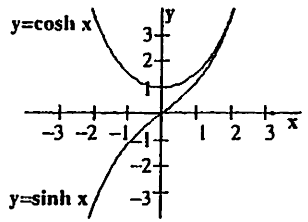
Basic Definitions

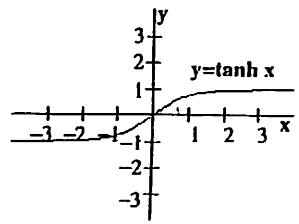
In a Maths 1A exam, you would only be asked to work from the following three basic definitions :

$$\cosh x = \frac{(e^x + e^{-x})}{2}, \quad \sinh x = \frac{(e^x - e^{-x})}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}$$

Notes
- Hyperbolic functions are closely related to the trig functions
- The name of each hyperbolic function comes from that of the corresponding trig function with an "h" appended. For example, cos becomes cosh , sin becomes sinh , etc .

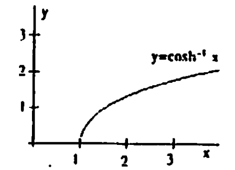
Graphs of cosh x , sinh x and tanh x



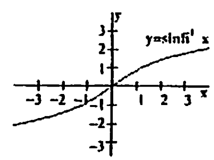


Inverse hyperbolic functions

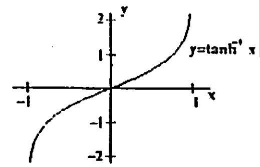
Graphs



$\cosh^{-1} x$ is the number $y \geq 0$ that satisfies $\cosh y = x$



$\sinh^{-1} x$ is the number y that satisfies $\sinh y = x$



$\tanh^{-1} x$ is the number y that satisfies $\tanh y = x$

Evaluation

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad (\text{where } x \geq 1)$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (\text{where } |x| < 1)$$

Hyperbolic function identities

$$\tanh x = \frac{\sinh x}{\cosh x}$$

(reciprocal hyperbolic functions)

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \text{sech}^2 x$$
$$\coth^2 x - 1 = \text{cosech}^2 x$$

where

$$\text{sech}\, x = 1/\cosh x$$
$$\text{cosech}\, x = 1/\sinh x$$
$$\coth x = 1/\tanh x$$

$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$$

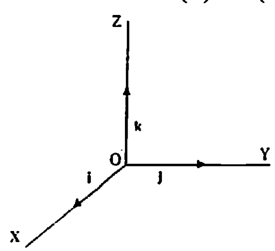
$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

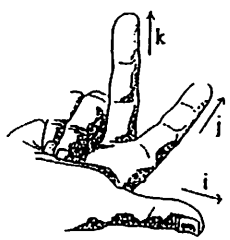
$$\cosh^2 x = \frac{(1 + \cosh 2x)}{2}$$
$$\sinh^2 x = \frac{(-1 + \cosh 2x)}{2}$$

VECTORS

Introduction

- A *scalar* quantity has magnitude only. A *vector* quantity has both magnitude *and* direction
- The Cartesian axes of reference OX , OY and OZ are chosen so that they form a *right-handed set*, whereby OX rotates towards OY in a clockwise manner when one looks along OZ
- The symbols $\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ denote unit vectors in directions OX, OY and OZ, respectively





Vectors introduction — continued

- A 3D vector $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ can be written in terms of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
- The *magnitude* of vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- The *unit vector* in the direction of $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$

Position vectors

- The *position vector* $\mathbf{r}$ of a point P with Cartesian coordinate (x_1, y_1, z_1) extends from the origin O to the point P
- The position vector of the point dividing the line AB in the ratio $m : n$ is $\mathbf{r} = \frac{n\mathbf{a} + m\mathbf{b}}{m + n}$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of points A and B , respectively
- The position vector of the *midpoint* of the line AB is $\mathbf{r} = \frac{\mathbf{a} + \mathbf{b}}{2}$
- The position vector of the *centroid* of the triangle ABC is $\mathbf{r} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of points A , B and C , respectively

Scalar product

The *scalar product* (or "dot product") of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3,$$

where θ is the *angle* between $\mathbf{a}$ and $\mathbf{b}$

The *resolved part* (or "projection") of vector $\mathbf{a}$ on vector $\mathbf{b}$ is then $|\mathbf{a}|\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{b}|}$

Vector product

The *vector product* (or "cross product") of vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

has *magnitude* $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$,

and *direction* perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ form a right-handed set

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = \mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$
$$= \mathbf{i}\,(a_2b_3 - a_3b_2) - \mathbf{j}\,(a_1b_3 - a_3b_1) + \mathbf{k}\,(a_1b_2 - a_2b_1)$$

COMPLEX NUMBERS

Introduction

- The purely imaginary number $j=\sqrt{-1}$, whereby $j^2=-1$, $j^3=-j$ and $j^4=1$
- Multiplying a position vector by a factor of j rotates the vector by $+90°$ (anti-clockwise direction)
- A complex number $z=a+jb$ has *real part* $=a$ and *imaginary part* $=$ b
- The *complex conjugate* of $z=a+jb$ is $a-jb$
- The product of two complex conjugate numbers is always real since $(a+jb)(a-jb)=a^2+b^2$
- Complex numbers $z_1=a+jb$ and $z_2=c+jd$ are *equal* if $a=c$ and $b=d$

Polar and exponential forms

To express a complex number in polar form : $\boxed{z=a+jb=r(\cos\theta+j\sin\theta)}$

- Sketch the position vector representing $a+jb$ in the complex plane (Argand diagram)
- Draw the right-angled triangle that involves the associated acute angle α (FORMULA SHEET 5)
- The *modulus* of z is $r=|z|=\sqrt{a^2+b^2}$
- The *argument* of z is $\theta=\arg z=$ 'angle made with the positive direction of the real axis'

To express a complex number in exponential form : $\boxed{z=a+jb=r(\cos\theta+j\sin\theta)=re^{j\theta}}$

- Express the complex number in polar form
- Use the values of r and θ from polar form (where θ MUST be in radians)

The *complex conjugate* of $a+jb=r(\cos\theta+j\sin\theta)=re^{j\theta}$
is $a-jb=r(\cos\theta-j\sin\theta)=re^{-j\theta}$

The *logarithm* of $z=re^{j\theta}$ is $\ln z=\ln\left(re^{j\theta}\right)=\ln r+j\theta$

Multiplication, division and powers (complex numbers in polar form)

If $\boxed{\begin{aligned}z_1&=r_1(\cos\theta_1+j\sin\theta_1)\\ z_2&=r_2(\cos\theta_2+j\sin\theta_2)\end{aligned}}$ and $\boxed{z=r(\cos\theta+j\sin\theta)}$

then ,
by considering the exponential forms of z_1 , z_2 and z ,

1[st] and 2[nd] laws of indices $\Rightarrow$ $\boxed{\begin{aligned}z_1z_2&=r_1r_2[\cos(\theta_1+\theta_2)+j\sin(\theta_1+\theta_2)]\\ \frac{z_1}{z_2}&=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+j\sin(\theta_1-\theta_2)]\end{aligned}}$

3[rd] law of indices $\Rightarrow$ $\boxed{z^n=r^n(\cos n\theta+j\sin n\theta)}$

(DeMoivre's theorem)

Roots of a complex number

Any real or complex number can be written in the form :

$$z=re^{j\theta}=re^{j(\theta+2n\pi)} \quad , \text{ where } n=0,\pm1,\pm2\ldots.$$

whereby the m^{th} *roots* of z are given by :

$$z^{\frac{1}{m}}=r^{\frac{1}{m}}e^{j\left(\frac{\theta}{m}+\frac{2n\pi}{m}\right)}$$

In polar form , one finds the m distinct m^{th} *roots* of z are thus:

$$\boxed{\begin{aligned}z^{\frac{1}{m}}&=r^{\frac{1}{m}}\left[\cos\left(\frac{\theta}{m}+\frac{2\pi n}{m}\right)+j\sin\left(\frac{\theta}{m}+\frac{2\pi n}{m}\right)\right]\\ &\equiv r^{\frac{1}{m}}\left[\cos\left(\frac{\theta}{m}+\frac{360n^{\circ}}{m}\right)+j\sin\left(\frac{\theta}{m}+\frac{360n^{\circ}}{m}\right)\right],\\ &\text{where } n=0,1,2,\ldots,m-1\end{aligned}}$$

ORDINARY DIFFERENTIATION

Chain Rule (for a function of a function)

If $y = y(u(x))$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Product Rule

If $y = u(x)v(x)$ then $y' = u'v + uv'$ (dashes denoting differentiation with respect to x)

Quotient Rule

If $y = \frac{u(x)}{v(x)}$ then $y' = \frac{u'v - uv'}{v^2}$

Logarithmic Differentiation

For example, if $y = \frac{u(x)v(x)}{w(x)}$ then $\ln y = \ln u + \ln v - \ln w$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}$$ (after differentiation),

therefore $\frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}\right]$

and $\frac{dy}{dx} = \frac{uv}{w}\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}\right]$

ORDINARY DIFFERENTIATION (continued)

Implicit Differentiation

When y is only given implicitly in terms of x : $f(x, y) = 0$,
then to find $\frac{dy}{dx}$ differentiate each side of the equation with respect to x
and solve for $\frac{dy}{dx}$. For example, if $f(x, y) = 0$ has a term y^2

then $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (using the Chain Rule).

Parametric Differentiation

When x and y are given by parametric equations, such as $x = x(\theta)$, $y = y(\theta)$.

then $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$. If one sets $Y = \frac{dy}{dx}$ then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{dY}{d\theta}}{\frac{dx}{d\theta}}$

Turning points and their classification

For a given curve $y = f(x)$, $\frac{dy}{dx} = 0$ gives the x-coordinates of turning points.
The original equation $y = f(x)$ can then be used to find their y-coordinates.

Classification: $\frac{d^2y}{dx^2}$ is
- positive for a minimum
- zero for a point of inflection
- negative for a maximum

STANDARD DERIVATIVES

f(x)	f ′(x)	f(x)	f ′(x)
x^n	$n\,x^{n-1}$	$[\,g(x)\,]^n$	$n\,[\,g(x)\,]^{n-1}\,g'(x)$
e^x	e^x	a^x	$a^x \ln a \quad (a>0)$
$\ln x$	$\dfrac{1}{x} \quad (x>0)$	$\ln g(x)$	$\dfrac{1}{g(x)}g'(x) \quad (g(x)>0)$
$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$\sec^2 x$	$\tanh x$	$\mathrm{sech}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$	$\mathrm{cosech}\, x$	$-\mathrm{cosech}\, x \coth x$
$\sec x$	$\sec x \tan x$	$\mathrm{sech}\, x$	$-\mathrm{sech}\, x \tanh x$
$\cot x$	$-\mathrm{cosec}^2 x$	$\coth x$	$-\mathrm{cosech}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}} \quad (-1<x<1)$	$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}} \quad (-1<x<1)$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}} \quad (x>1)$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\tanh^{-1} x$	$\dfrac{1}{1-x^2} \quad (-1<x<1)$

where

$\mathrm{cosec}\, x = 1/\sin x$,	$\sec x = 1/\cos x$,	$\cot x = 1/\tan x$
$\mathrm{cosech}\, x = 1/\sinh x$,	$\mathrm{sech}\, x = 1/\cosh x$,	$\coth x = 1/\tanh x$

PARTIAL DIFFERENTIATION

Meaning of Partial Differentiation

When a given function is of *two or more* independent variables, $f(x, y)$ for example,

then $\dfrac{\partial f}{\partial x}$ means $\left[\dfrac{df}{dx}\right]$ while treating y as a constant

$\dfrac{\partial f}{\partial y}$ means $\left[\dfrac{df}{dy}\right]$ while treating x as a constant

$\dfrac{\partial^2 f}{\partial x^2}$ means $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$ and $\dfrac{\partial^2 f}{\partial y^2}$ means $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$

also
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Chain Rule (for a function of a function)

For $f(u(x, y))$, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}$

Product Rule

If $f(x, y) = u(x, y)\,v(x, y)$ then $\dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x}v + u\dfrac{\partial v}{\partial x}$

and $\dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y}v + u\dfrac{\partial v}{\partial y}$

Quotient Rule

If $f(x, y) = \dfrac{u(x, y)}{v(x, y)}$ then $f' = \dfrac{u'v - uv'}{v^2}$

(dashes denoting partial differentiation with respect to x , and similarly for y)

INTEGRATION

Techniques of integration

Functions of a linear function of x

For integrals of the form $\int f(ax+b)\,dx$, where a and b are constants ,

Let $u = ax+b$, then $\frac{du}{dx} = a$ and hence $dx = \frac{du}{a}$, giving :

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du$$

Integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$

Letting $u = f(x)$ gives $\frac{du}{dx} = f'(x)$ and $du = f'(x)\,dx$, thus :

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|f(x)| + C$$

Integrals of the form $\int [f(x)]^n f'(x)\,dx$

Letting $u = f(x)$ gives $\frac{du}{dx} = f'(x)$ and hence $du = f'(x)\,dx$, thus :

$$\int [f(x)]^n f'(x)\,dx = \int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{[f(x)]^{n+1}}{n+1} + C$$

For example, $n = 1$ gives

$$\int f(x)\,f'(x)\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{[f(x)]^2}{2} + C$$

Techniques of integration — continued

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx$$

This technique is useful for integrals of the following four forms:

(i) $\int x^n \begin{Bmatrix}\sin bx\\ \cos bx\end{Bmatrix} dx$, with $x^n \uparrow u$ and $\begin{Bmatrix}\sin bx\\ \cos bx\end{Bmatrix} \uparrow \frac{dv}{dx}$

(ii) $\int x^n e^{ax}\,dx$, with $x^n \uparrow u$ and $e^{ax} \uparrow \frac{dv}{dx}$

(iii) $\int x^n \ln(ax)\,dx$, with $x^n \uparrow \frac{dv}{dx}$ and $\ln(ax) \uparrow u$

(iv) $\int e^{ax}\begin{Bmatrix}\sin bx\\ \cos bx\end{Bmatrix} dx$, with $e^{ax} \uparrow u$ and $\begin{Bmatrix}\sin bx\\ \cos bx\end{Bmatrix} \uparrow \frac{dv}{dx}$

Integration using partial fractions

For example, given a proper algebraic fraction whose denominator is factorised as far as possible, and whose numerator cannot be written as a multiple of the derivative of the denominator,

then FORMULA SHEET 3 gives the prescription for writing this algebraic fraction in terms of partial fractions that can be individually integrated.

Integrals of the form " $\int \sin px \cos qx\,dx$ "

The product formulae from trig identities give three results for dealing with products of $\sin px$ and/or $\cos qx$:

$$\int \sin A \cos B\,dx = \frac{1}{2}\int \sin(A+B) + \sin(A-B)\,dx$$

$$\int \cos A \cos B\,dx = \frac{1}{2}\int \cos(A+B) + \cos(A-B)\,dx$$

$$\int \sin A \sin B\,dx = -\frac{1}{2}\int \cos(A+B) - \cos(A-B)\,dx$$

STANDARD INTEGRALS I

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}$ $(n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a}$ $(a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\text{cosec}\, x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\text{cosech}\, x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\text{sech}\, x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\text{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

The following nine standard integrals are often used after two manipulations of more complicated integrals :

- completing the square to find an expression like $c(x+d)^2 \pm a^2$
- and, linear substitution such as $u = x + d$

STANDARD INTEGRALS II

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$ $(0 < \lvert x\rvert < a)$
		$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$ $(\lvert x\rvert > a > 0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert$ $(a > 0)$
		$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$ $(x > a > 0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
		$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

Introduction

$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=x^7$ is an example of an **ordinary differential equation** since it contains only ordinary derivatives such as $\frac{dy}{dx}$ and not **partial derivatives** such as $\frac{\partial y}{\partial x}$. The dependent variable is y while the independent variable is x (an o.d.e. has only one independent variable while a partial differential equation has more than one independent variable).

It is a **second order** equation since the highest order of derivative involved is two i.e. the presence of the $\frac{d^2y}{dx^2}$ term.

An o.d.e. is **linear** when each term has y and its derivatives only appearing to the power one. The appearance of a term involving any product of y and $\frac{dy}{dx}$ would also make and equation **non-linear**.

In the above example, the term $\left(\frac{dy}{dx}\right)^3$ makes the equation **non-linear**.

The **general solution** of an n^{th} order o.d.e. has n arbitrary constants which can take any values.

In an **initial value problem**, one solves an n^{th} order o.d.e. to find the general solution and <u>then</u> applies n **boundary conditions** ("initial values/conditions") to find a **particular solution** that does not have any arbitrary constants.

Solving O.D.E.'s

- $\frac{dy}{dx}=f(x)$ $\longrightarrow$ $y=\int f(x)\,dx$ by "**direct integration**"

- $\frac{dy}{dx}=f(x)g(y)$ $\longrightarrow$ $\int\frac{dy}{g(y)}=\int f(x)\,dx$ by "**separation of variables**"

<u>Definition</u> $M(x,y)=3x^2+xy$ is a **homogeneous function** since the sum of the powers of x and y in each term is the same (i.e. x^2 is x to the power 2 and $xy=x^1y^1$ giving total power of 1+1=2). The degree of this homogeneous function is 2.

- $\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)}$ where M and N are **homogeneous functions of the same degree**

  Change the dependent variable from y to v where $y=vx$ then

  LHS $=\frac{dy}{dx}=x\frac{dv}{dx}+v$ and RHS $=\frac{M(x,y)}{N(x,y)}$ becomes function of v only.

  Solve the resulting equation by separating the variables v and x, then re-express the solution in terms of x and y.

  Note that this method also works for equations of the form $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$.

- $\frac{dy}{dx}+P(x)y=Q(x)$ **First order linear** o.d.e. – use the **integrating factor method**

  Multiply the equation by integrating factor $IF=e^{\int P(x)\,dx}$ to give $\frac{d}{dx}(IF\ y)=IF\ Q(x)$. Then integrate both sides with respect to x, giving $IF\ y=\int IF\ Q(x)\,dx$. Finally, divide by IF to get y.

- $\frac{dy}{dx}+P(x)y=Q(x)y^n$ **Bernoulli's differential equation**

  Change the dependent variable from y to z where $z=y^{1-n}$.
  This makes the equation linear and we can use the integrating factor method.

  Dividing by y^n gives $\frac{1}{y^n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x)$

  i.e. $\frac{1}{1-n}\frac{dz}{dx}+P(x)\,z=Q(x)$.

  (using $\frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$).

- $\boxed{P(x,y)\,dx + Q(x,y)\,dy = 0}$ [this method of solution will not be examined]

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ then the o.d.e. is said to be exact.

This means that a function $u(x,y)$ exists such that $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$,

$$= P\ dx + Q\ dy = 0.$$

Ones solves $\dfrac{\partial u}{\partial x} = P$ and $\dfrac{\partial u}{\partial y} = Q$ to find $u(x,y)$.

Then $du = 0$ gives $u(x,y) = constant$ (this is the general solution of $Pdx + Qdy = 0$).

- $\boxed{a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0}$ Second order linear o.d.e. with **constant coefficients** a,b,c

It is called a **homogeneous equation** because the RHS = 0.

Setting $y = A\,e^{mx}$ gives $am^2 + bm + c = 0$ (the "auxiliary equation")

Then $m = \dfrac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$ gives

i) real different roots m_1 , m_2 and $y = Ae^{m_1x} + Be^{m_2x}$.

or ii) real equal roots $m_1 = m_2$ and $y = (A + Bx)e^{m_1x}$.

or iii) complex roots $m_{1,2} = p \pm iq$ and $y = e^{px}(A\cos qx + B\sin qx)$.

- $\boxed{a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)}$ **Second order linear** o.d.e. with **constant coefficients** a,b,c

It is not homogeneous since RHS is not zero.

<u>Step One</u> Solve the corresponding homogeneous equation to get $y = y_{CF}$
This is called the "complementary function".

<u>Step Two</u> The general solution of the full equation is $y = y_{CF} + y_{PS}$.
Where y_{PS} is a particular solution of the full equation.

Find y_{PS} by substituting a trial form into the full equation and equate the coefficients of the functions involved (e.g. e^{2x} , x^2 , $\cos x$, etc.).

$f(x)$	**Trial form of y_{PS}**
k	C
kx ...	$Cx+D$
kx^2 ...	$Cx^2 + Dx + E$
$k\cos ax$ OR $k\sin ax$	$C\cos ax + D\sin ax$
ke^{ax}	Ce^{ax}
Sum/product of the above	Sum/product of the above
(k, a are given constants)	(C, D, E are constants to be determined)

<u>Note</u> If the suggested form of y_{PS} already appears in y_{CF} then multiply the trial form of y_{PS} by x until it does not appear in y_{CF} .

SERIES

Series involving constants

Sum of arithmetic series, $S_n = a + (a+d) + (a+2d) + \dots + [a+(n-1)d] = \frac{n}{2}[2a + (n-1)d]$

Sum of geometric series, $S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$.

For $|r|<1$, $\lim_{n \to \infty} S_n = \frac{a}{1-r}$

Series of powers of natural numbers, $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$, $\sum_{r=1}^{n} r^2 = \frac{n}{6}(2n+1)(n+1)$,

$\sum_{r=1}^{n} r^3 = \left[\frac{n}{2}(n+1)\right]^2$

L'Hopital's rule: When $f(a) = g(a) = 0$, $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x \to a}\left[\frac{f'(x)}{g'(x)}\right]$

Tests of convergence of the series $u_1 + u_2 + u_3 + u_4 + \dots + u_n + \dots$

1) $\lim_{n \to \infty} u_n = 0$ is required for, but does <u>not</u> guarantee, convergence

2) Useful series for comparison test: $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \begin{cases} \text{converges for } p > 1 \\ \text{diverges for } p \le 1 \end{cases}$

3) D'Alembert's ratio test: If $\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right|$ is $\begin{cases} < 1, \text{ then the series converges} \\ > 1, \text{ then the series diverges} \\ = 1, \text{ then convergence is undetermined} \end{cases}$

4) If $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ converges

Taylor series: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

Maclaurin series: $f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

SERIES (CONTINUED)

Taylor series: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

Maclaurin series: $f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

Power series

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$ $(-1 < x < 1$, when n not a positive integer)

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ $(-1<x<1)$.

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $(-1<x\le 1)$

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$ $(-\frac{\pi}{2}<x<\frac{\pi}{2})$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ (for all x)

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ (for all x).

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ (for all x).

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ (for all x).

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ (for all x).

FOURIER SERIES

Useful Results : For integer n, $\sin n\pi = 0$, $\cos n\pi = (-1)^n = \begin{cases} 1 & (n \text{ even}) \\ -1 & (n \text{ odd}) \end{cases}$

$$\sin n\frac{\pi}{2} = \begin{cases} 0 & (n \text{ even}) \\ 1 & (n=1,5,9,\ldots) \\ -1 & (n=3,7,11,\ldots) \end{cases} \qquad \cos n\frac{\pi}{2} = \begin{cases} 0 & (n \text{ odd}) \\ 1 & (n=0,4,8,\ldots) \\ -1 & (n=2,6,10,\ldots) \end{cases}$$

For integers n and m (not zero),

$$\int_{2\pi} \sin nx\,dx = \int_{2\pi} \cos nx\,dx = 0, \qquad \int_{2\pi} \sin^2 nx\,dx = \int_{2\pi} \cos^2 nx\,dx = \pi$$

$$\int_{2\pi} \sin nx \sin mx\,dx = \int_{2\pi} \cos nx \cos mx\,dx = \begin{cases} 0 & (n \neq m) \\ \pi & (n=m) \end{cases}, \qquad \int_{2\pi} \sin nx \cos mx\,dx = 0$$

An EVEN function $y = F(x)$ is symmetrical about the y - axis : $F(-x) = F(x)$ and $\int_{-\pi}^{\pi} F(x)\,dx = 2\int_0^{\pi} F(x)\,dx$.

An ODD function $y = F(x)$ is symmetrical about the origin : $F(-x) = -F(x)$ and $\int_{-\pi}^{\pi} F(x)\,dx = 0$.

Sums of even/odd functions : "even + even = even", "odd + odd = odd".

Products of even/odd functions : "even x even = even", "odd x odd = even", "odd x even = odd"

Functions with Period of 2π :

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] ,$$

where $a_0 = \frac{1}{\pi}\int_{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{2\pi} f(x)\sin nx\,dx$.

For example, $f(x)$ defined over $-\pi < x < \pi$ when $f(x)$ is an even or odd function :

Cosine series : $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$ (when $f(x)$ is even, ie $b_n = 0$).

Sine series : $f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$ (when $f(x)$ is odd, ie $a_0 = a_n = 0$).

For example, $f(x)$ defined over $0 < x < \pi$ when $f(x)$ is an even or odd function :

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$ and $b_n = 0$ (when $f(x)$ is even).

$a_0 = 0$, $a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ (when $f(x)$ is odd).

Fourier Series of Functions with Period of $L = 2\pi / k$:

$$f(x) = \frac{a_0}{2} + a_1 \cos kx + a_2 \cos 2kx + a_3 \cos 3kx + \cdots$$
$$+ b_1 \sin kx + b_2 \sin 2kx + b_3 \sin 3kx + \cdots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nkx + b_n \sin nkx] ,$$

where $a_0 = \frac{2}{L}\int_L f(x)\,dx$, $a_n = \frac{2}{L}\int_L f(x)\cos nkx\,dx$ and $b_n = \frac{2}{L}\int_L f(x)\sin nkx\,dx$.

For example, $f(x)$ defined over $-\frac{L}{2} < x < \frac{L}{2}$ when $f(x)$ is an even or odd function :

Cosine series : $f(x) = \frac{a_0}{2} + a_1 \cos kx + a_2 \cos 2kx + a_3 \cos 3kx + \cdots$ (when $f(x)$ is even, ie $b_n = 0$).

Sine series : $f(x) = b_1 \sin kx + b_2 \sin 2kx + b_3 \sin 3kx + \cdots$ (when $f(x)$ is odd, ie $a_0 = a_n = 0$).

For example, $f(x)$ defined over $0 < x < \frac{L}{2}$ when $f(x)$ is an even or odd function :

$a_0 = \frac{4}{L}\int_0^{L/2} f(x)\,dx$, $a_n = \frac{4}{L}\int_0^{L/2} f(x)\cos nkx\,dx$ and $b_n = 0$ (when $f(x)$ is even).

$a_0 = 0$, $a_n = 0$ and $b_n = \frac{4}{L}\int_0^{L/2} f(x)\sin nkx\,dx$ (when $f(x)$ is odd).

VECTOR CALCULUS

Triple Products

- $(\mathbf{a}.\mathbf{b})\,\mathbf{c} \neq \mathbf{a}\,(\mathbf{b}.\mathbf{c})$
- $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}||\mathbf{b}\times\mathbf{c}|\cos\theta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (= 0 when a, b and c are coplanar)

  (Scalar triple product)
- $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$

  $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$ (Vector triple product)

Differentiation

If $\mathbf{a}(t) = a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k}$

Then $\dfrac{d\,\mathbf{a}(t)}{dt} = \dfrac{d\,a_1(t)}{dt}\mathbf{i} + \dfrac{d\,a_2(t)}{dt}\mathbf{j} + \dfrac{d\,a_3(t)}{dt}\mathbf{k}$ (which is perpendicular to $\mathbf{a}(t)$)

Similar results hold for differentiation with respect to x and for partial differentiation of vectors.

Differential operators grad and div

For a scalar field $\varphi(x,y,z)$ and for a vector field $\mathbf{V}(x,y,z) = V_x(x,y,z)\mathbf{i} + V_y(x,y,z)\mathbf{j} + V_z(x,y,z)\mathbf{k}$:

- $grad\varphi = \nabla\varphi = \mathbf{i}\dfrac{\partial\varphi}{\partial x} + \mathbf{j}\dfrac{\partial\varphi}{\partial y} + \mathbf{k}\dfrac{\partial\varphi}{\partial z}$ i.e. "*grad* (scalar) = vector"

  where $\nabla\varphi$ is a vector that is perpendicular to the surface/contour $\varphi(x,y,z) = \text{constant}$

  and $\nabla\varphi.\hat{\mathbf{a}}$ is the rate of change of $\varphi(x,y,z)$ in the direction of $\mathbf{a}$

  ($\hat{\mathbf{a}}$ is the unit vector in the direction of $\mathbf{a}$)

- $div\mathbf{V} = \nabla.\mathbf{V} = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right).\left(V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}\right)$

$\therefore$ $div\mathbf{V} = \nabla.\mathbf{V} = \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$ i.e. "*div* (vector) = scalar"

Differential operator curl

- $curl\,\mathbf{V} = \nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \mathbf{i}\left(\dfrac{\partial V_z}{\partial y} - \dfrac{\partial V_y}{\partial z}\right) - \mathbf{j}\left(\dfrac{\partial V_z}{\partial x} - \dfrac{\partial V_x}{\partial z}\right) + \mathbf{k}\left(\dfrac{\partial V_y}{\partial x} - \dfrac{\partial V_x}{\partial y}\right)$

i.e. "*curl* (vector) = vector"

Identities

(a) *curl grad* φ = $\nabla\times(\nabla\varphi) = 0$

(b) *div curl* $\mathbf{V} = \nabla.(\nabla\times\mathbf{V}) = 0$

(c) *div grad* $\varphi = \nabla^2\varphi = \nabla.(\nabla\varphi) = \dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\varphi}{\partial z^2}$

(Laplacian of φ , a similar result holds for $\nabla^2\mathbf{V}$)

(d) *curl curl* $\mathbf{V} = \nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla.\mathbf{V}) - \nabla^2\mathbf{V}$

Integration

For a constant force $\mathbf{F}$ and linear displacement $\mathbf{r}$, Work Done = $\mathbf{F}.\mathbf{r}$

For a space-dependent force $\mathbf{F}$ and/or a more involved path ,

Work Done by $\mathbf{F}$ along path ABC = $\displaystyle\int_A^C \mathbf{F}.d\mathbf{r} = \int_A^B \mathbf{F}.d\mathbf{r} + \int_B^C \mathbf{F}.d\mathbf{r}$

For example, when $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$

then $\mathbf{F}.d\mathbf{r} = F_x dx + F_y dy$

where F_x , F_y , dx and dy take appropriate forms according to path

Conservative fields

If $\displaystyle\int_A^C \mathbf{F}.d\mathbf{r}$ is independent of the path taken between A and C then $\mathbf{F}$ is conservative

(but one would need to check *all* paths!)

The *curl* test : $\nabla\times\mathbf{F} = \mathbf{0}$ if and only if $\mathbf{F}$ *is a conservative vector field*

Flux

Flux of vector field A through a surface $S = \int_S \mathbf{A}\cdot d\mathbf{S}$,

where the vector area element is $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ and $\hat{\mathbf{n}}$ is the unit normal to the (scalar) surface element dS ($\hat{\mathbf{n}}$ pointing outwards from a closed surface).

$$\therefore \quad \text{Flux of } \mathbf{A} \text{ through } S = \int_S \mathbf{A}\cdot\hat{\mathbf{n}}\, dS$$

Solid angles

Ω (steradians) $= \dfrac{S}{r^2}$, where S is the surface area subtended at a sphere of radius r

"Full solid angle" $= \dfrac{\text{sphere area}}{r^2} = 4\pi$

Divergence theorem

For a closed surface S containing volume V

$$\int_V div\, \mathbf{A}\, dV = \oint_S \mathbf{A}\cdot d\mathbf{S}$$

(the right hand side is the flux of A through S)

Stokes' theorem

For an open surface S with a bounding curve C

$$\int_S (curl\, \mathbf{A})\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{r}$$

(the right hand side is the circulation of A around C)

MATRICES

Simultaneous linear equations (a_{ij} and b_j are constants) :

$$\begin{array}{l} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array}$$

can be written as $\mathbf{A}\mathbf{x} = \mathbf{b}$

where coefficient matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, solution matrix $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Matrix multiplication (the "row dot column rule"):

The ij^{th} element of the matrix product $\mathbf{A}\,\mathbf{B} = \mathbf{C}$ is given by the dot product of two vectors defined by the i^{th} row of A and the j^{th} column of B. If A is a nxm matrix and B is mxp , then C is a nxp matrix.

Determinants

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\Delta = \det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

If $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then, expansion across the first row of A gives,

$$\Delta = \det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

In general,

$$\det(\mathbf{A}) = \sum_{row/column} a_{ij}(-1)^{i+j}\left(\text{minor of } a_{ij}\right) = \sum_{row/column} a_{ij}A_{ij}$$

In the last example, $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ is the minor of element a_{12} ,

$-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ is the signed minor of a_{12} i.e. the cofactor A_{12} ,

and the factor $(-1)^{i+j}$ generates the sign table $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Properties of determinants

1) if each element of one row (or column) is multiplied by K then $\Delta \to K\Delta$

2) $\Delta = 0$ if a) all elements in one row (or column) are zero
or b) two rows (or columns) are identical or proportional

3) if two rows (or two columns) are interchanged then $\Delta \to -\Delta$

4) Δ remains unchanged if a) $\mathbf{A} \to \mathbf{A}^T$
or if b) multiples of one row (or column) are added to another

Solution using Cramer's rule

For example, to solve the following system: $\begin{array}{|l|} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array}$, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

Cramer's rule gives :

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\det(A)}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\det(A)}, \quad z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\det(A)}$$

Independence of equations

Given a system of n simultaneous linear equations of which only m are independent:

<u>If the system is inhomogeneous ($b \neq 0$):</u>

- $|A| \neq 0$ and $m=n$ $\Rightarrow$ unique non-trivial solution
- $|A| = 0$ and $m<n$ $\Rightarrow$ infinite number of solutions
- $|A| = 0$ and $m=n$ $\Rightarrow$ no solution (inconsistency)

<u>If the system is homogeneous ($b = 0$):</u>

- $|A| \neq 0$ $\Rightarrow$ only the trivial solution $x = 0$ exists
- $|A| = 0$ $\Rightarrow$ infinite number of solutions

Rank of a matrix

The rank of a matrix is given by the largest non-zero determinant that can be formed from the elements (in the order that they appear within the original matrix)

For example, if A is an nxn matrix and $|A| \neq 0$ then $\text{rank}(A) = n$, or if, for example, $\text{rank}(A) = m<n$ then $|A| = 0$ but there exists an mxm submatrix that can be formed having non-zero determinant

To use the concept of rank in determining the character and existence of solutions of simultaneous equations, consider both the coefficient matrix A and

the *augmented coefficient matrix* A_b that takes the form : $A_b = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

Then,

- $\text{rank}(A) = \text{rank}(A_b) = n$ $\Rightarrow$ unique solution
- $\text{rank}(A) = \text{rank}(A_b) < n$ $\Rightarrow$ infinite number of solutions
- $\text{rank}(A) < \text{rank}(A_b)$ $\Rightarrow$ no solution (inconsistency)

Note also that

- Inhomogeneous systems can only have non-trivial solutions
- Homogeneous systems are always consistent (have at least the trivial solution)
- $|A| = 0$ required for non-trivial solutions of homogeneous systems

Alternative way to determine the rank of a matrix

This procedure involves applying elementary row operations to the given matrix A to reduce it to echelon ("staircase") form. Once echelon form has been achieved then dependent equations have been eliminated and rank(A) is given by the number of non-zero rows.

The elementary row operations are :

(i) interchanging two rows
(ii) multiplying a row by a number
(iii) adding a multiple of one row to another

Matrix inversion

The inverse of an nxn matrix A is another nxn matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$, where I is the nxn identity matrix.

Note that

- If $Ax = b$ then $A^{-1}Ax = A^{-1}b$ and thus $x = A^{-1}b$
- The inverse of a product of matrices is given by $(BA)^{-1} = A^{-1}B^{-1}$

Finding A^{-1}

<u>The formal method</u> $\boxed{A^{-1} = \frac{1}{\det(A)} C^T}$ where C^T is the transpose of the matrix of cofactors

i.e. where $C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ and $C^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

(for definition of the cofactors, see formula sheet covering determinants)

<u>The row reduction method</u>

(i) Form the *combined coefficient matrix* $A \vdots I$, e.g. $A \vdots I = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$

(ii) Apply elementary row operations on each row of $A \vdots I$ until it reduces to the form $I \vdots A^{-1}$

Eigenvalues and eigenvectors

In $\boxed{A\mathbf{x} = \lambda \mathbf{x}}$ λ is the (scalar) eigenvalue and $\mathbf{x}$ is the corresponding eigenvector

Then, $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$ and thus $(A - \lambda I)\mathbf{x} = \mathbf{0}$ (a homogeneous system).

For non-trivial solutions $\mathbf{x}$ of this homogeneous system, we require: $\boxed{\det(A - \lambda I) = 0}$

(the characteristic equation)

When A is an nxn matrix, the characteristic equation gives an n^{th} degree polynomial in λ whose solution gives the required n eigenvalues

Once the eigenvalues are found, substitution of each value of λ into $\boxed{(A - \lambda I)\mathbf{x} = \mathbf{0}}$ yields each eigenvector (defined to within an undetermined scalar)

Example

If A is a 2x2 coefficient matrix then $\boxed{A\mathbf{x} = \lambda\mathbf{x}}$ takes the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

while $(A - \lambda I)\mathbf{x} = \mathbf{0}$ becomes

$$\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. the homogeneous system

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In this case, the characteristic equation is a quadratic in λ and is found from :

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

Some important pde's

$\boxed{\nabla^2 u = 0}$ Laplace's equation

$\boxed{\nabla^2 u = f(x, y, z)}$ Poisson's equation

$\boxed{\nabla^2 u = \frac{1}{a^2}\frac{\partial u}{\partial t}}$ Diffusion (or heat flow) equation

$\boxed{\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}}$ Wave equation

Verifying the solution of a pde

The general solutions of pde's involve arbitrary functions (rather than arbitrary constants)

Here is an example. Suppose you are asked to verify that $\boxed{v = F(u)}$, where $u = u(x, y)$, is a solution of a given pde such as

$$\boxed{a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y} = 0}$$, where a and b are constants

In this case, F is the arbitrary function, and it is a *function of the function* $u(x, y)$

Use the chain rule to work out $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, and then substitute the results into the given pde.

i.e. use the fact that $\boxed{\frac{\partial v}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}}$ and $\boxed{\frac{\partial v}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}}$

Solving pde's I

Many of the techniques for solving ode's may be used (such as the integrating factor method) provided that constants of integration (in the ode technique) are replaced with the appropriate *arbitrary functions* in the process of solving a pde. These arbitrary functions arise from "partial integration", i.e. integrating with respect to only *one* of the independent variables.

For example, direct integration (with respect to *only* x) of $\frac{\partial^2 u}{\partial x \partial y} = 2x - y$

yields $\frac{\partial u}{\partial y} = x^2 - xy + G(y)$

and $G(y)$ is an arbitrary function of y that disappears when $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$ is evaluated.

FORMULA SHEET 38

Solving pde's II

Continuing the theme of adapting ode methods to solve pde's, recall the solution of second order ode's with constant coefficients. This topic breaks down into two main parts:

Homogeneous equations

For homogeneous ode's (with a single independent variable x), one sets $y = Ae^{mx}$.

For a homogeneous pde that has two independent variables (x and y), set $y = Ae^{ax+by}$ and this allows one to guess the form of the general solution.

Inhomogeneous equations

As with ode's, one can find the general solution of the homogeneous system first and then use the method of undetermined coefficients to find a particular solution of the full equation. Addition of these two solutions yields the required general solution.

Solving pde's III

The important pde method of **separation of variables** has a similarity with the ode method of the same name, but it is more involved. Here, the method is illustrated by example.

Suppose we wish to solve the boundary-value problem $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0, y) = 8e^{-3y}$

- We assume the solution can be expressed as a product of unknown functions of each of the independent variables i.e. substitute the following into the pde

$$\boxed{u(x, y) = X(x)Y(y)}$$

- Rearrange the result so that the LHS depends only on x and the RHS depends only on y.
In this example, we find:

$$\boxed{\frac{1}{4X}\frac{\partial X}{\partial x} = \frac{1}{Y}\frac{\partial Y}{\partial y}}$$

- Equating LHS and RHS to the "separation constant" c, yields two odes's:

$$\boxed{\frac{dX}{dx} = c4X} \quad \text{and} \quad \boxed{\frac{dY}{dy} = cY}$$

with solutions: $\boxed{X = Ae^{4cx}}$ and $\boxed{Y = Be^{cy}}$

- Reconstruct $u = XY$ and apply the boundary condition(s) to u or to a *sum of solutions of this form*

i.e. $\boxed{u = XY = Ke^{c(4x+y)}, \text{ where } K = AB}$

and boundary condition $\boxed{u(0, y) = 8e^{-3y} \equiv Ke^{c(0+y)}}$ yields $\boxed{u(x, y) = 8e^{-3(4x+y)}}$