HANDOUT 10

- Ordinary Differential Equations
  - Review of 1st and 2nd order linear odes
  - Higher order linear odes
- Partial Differential Equations
  - Arbitrary functions
  - Similarities with the solution of ode's
  - Separation of variables
    - Finding <u>a</u> solution
    - Superposition to get <u>the</u> required solution

Definition of a differential equation

– an equation involving derivatives or differentials ...

e.g. 1 $\quad (y'')^2 + 3x = 2(y')^3 \quad$ where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$

e.g. 2 $\quad \frac{dy}{dx} + \frac{y}{x} = y^2$

e.g. 3 $\quad \frac{d^2Q}{dt^2} - 3\frac{dQ}{dt} + 2Q = 4\sin 2t$

e.g. 4 $\quad \frac{dy}{dx} = \frac{x+y}{x-y}$ or equivalently $(x+y)\,dx + (y-x)\,dy = 0$

e.g. 5 $\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

1.–4. have ONE independent variable and are <u>ordinary</u> differential equations

5. has more than one independent variable (x <u>and</u> y) and is a <u>partial</u> differential equation

- The highest order of derivative defines the <u>order</u> of the differential equation. In the above, 1 is second order, 2 is first order, 3 is second order, 4 is first order and 5 is second order.

Solutions of differential equations

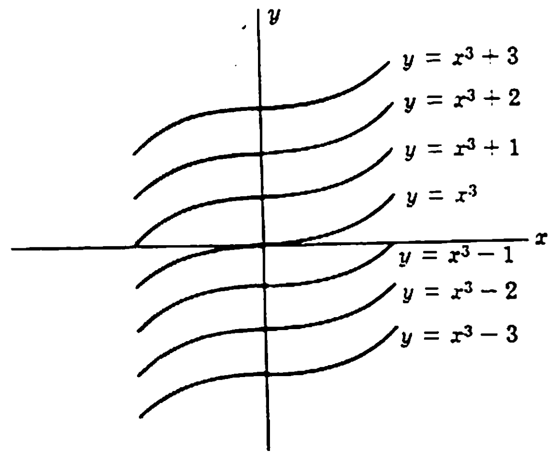
– a relation between the variables which is free of derivatives and which satisfies the differential equation

eg. $\frac{dy}{dx} = 3x^2$ has <u>general</u> solution $y = x^3 + c$

where c is an arbitrary constant. The general solution of an nth order differential equation n arbitrary constants.

$\frac{dy}{dx} = 3x^2$ has a <u>particular</u> solution $y = x^3 + 1$.

This can be found by assigning a value to the arbitrary constant c.



- General solution gives a <u>family</u> of curves.
- Particular solution is one of those curves.

ORDINARY DIFFERENTIAL EQUATIONS (O.D.E.'s) SUMMARY

Introduction

$\boxed{\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3 = x^7}$ is an example of an **ordinary** differential equation since it contains only **ordinary** derivatives such as $\frac{dy}{dx}$ and not **partial** derivatives such as $\frac{\partial y}{\partial x}$.

The dependent variable is y while the independent variable is x (an o.d.e. has only one independent variable while a partial differential equation has more than one independent variable).

It is a **second order** equation since the highest order of derivative involved is **two** i.e. the presence of the $\frac{d^2y}{dx^2}$ term.

An o.d.e. is **linear** when each term has y and its derivatives only appearing to the power one. The appearance of a term involving any product of y and $\frac{dy}{dx}$ would also make an equation **non-linear**.

In the above example, the term $\left(\frac{dy}{dx}\right)^3$ makes the equation **non-linear**.

The **general solution** of an n^{th} order o.d.e. has n arbitrary constants that can take any values.

In an **initial value problem**, one solves an n^{th} order o.d.e. to find the general solution and <u>then</u> applies n **boundary conditions** ("initial values/conditions") to find a **particular solution** that does not have any arbitrary constants.

Solving O.D.E.'s

- $\boxed{\frac{dy}{dx} = f(x)}$ $\longrightarrow$ $y = \int f(x)\,dx$ by **"direct integration"**

Ex Find the general solution of $\frac{d^2x}{dt^2} = -\sin\omega t$, ω = constant.

Ans Integrate w.r.t. t: $\frac{dx}{dt} = -\int \sin\omega t\,dt = -\left(-\frac{1}{\omega}\cos\omega t\right) + A = +\frac{1}{\omega}\cos\omega t + A$ (A = arbitrary constant)

Integrate w.r.t. t: $x = \int \frac{1}{\omega}\cos\omega t + A\,dt = \frac{1}{\omega^2}\sin\omega t + At + B$.

Note: 2 arbitrary constants A and B since equation is second order.

- $\boxed{\dfrac{dy}{dx} = f(x)g(y)}$ $\longrightarrow$ $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$ by **"separation of variables"**

Ex $\quad \dfrac{dx}{dt} + \dfrac{x}{\tau} = 0$ DECAY / DAMPING WITH LIFETIME τ (a constant)

Note that one could also write this as $\dfrac{d^2x}{dt^2} + \dfrac{1}{\tau}\dfrac{dx}{dt} = 0$. 286

Rearrange as $\quad \dfrac{dx}{x} = -\dfrac{dt}{\tau}$

Integrate $\quad \displaystyle\int_{x(0)}^{x(t)} \frac{dx}{x} = -\frac{1}{\tau}\int_{t=0}^{t} dt$

i.e. $\left[\ln x\right]_{x(0)}^{x(t)} = -\frac{1}{\tau}\left[t\right]_0^t$ i.e. $\ln\left[x(t)\right] - \ln\left[x(0)\right] = -\dfrac{t}{\tau}$

i.e. $\ln\left[\dfrac{x(t)}{x(0)}\right] = -\dfrac{t}{\tau}$ i.e. $\boxed{x(t) = x(0)\,e^{-t/\tau}}$

Value of x at time t

Value of x at time $t=0$

Note that

- At time $t=\tau$, $x(\tau) = \dfrac{x(0)}{e}$ i.e. dropped to $\frac{1}{e}$ times the initial value.
- Evolution / dynamics are transient i.e. $x(t) \to 0$ as $t \to \infty$.

Ex Radioactive decay

Rate of decay is proportional to number of remaining atoms N 287

i.e. $\boxed{\dfrac{dN}{dt} = -\lambda N, \quad \text{where } \lambda = \text{constant}}$

gives $\boxed{N(t) = N(0)\,e^{-\lambda t}}$

Check for yourself that finding this solution is just the same as the previous example when one identifies
$$\left\{\begin{array}{l} x(t) \longleftrightarrow N(t) \\ x(0) \longleftrightarrow N(0) \\ \frac{1}{\tau} \longleftrightarrow \lambda \end{array}\right\}$$

Definition $M(x,y) = 3x^2 + xy$ is a **homogeneous function** since the sum of the powers of x and y in each term is the same (i.e. x^2 is x to the power 2 and $xy = x^1y^1$ giving total power of $1+1=2$). The **degree** of this homogeneous function is 2.

- $\boxed{\dfrac{dy}{dx} = \dfrac{M(x,y)}{N(x,y)}}$ where M and N are **homogeneous functions of the same degree**

Change the dependent variable from y to v where $y = vx$ then

LHS $= \dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$ and RHS $= \dfrac{M(x,y)}{N(x,y)}$ becomes function of v only.

Solve the resulting equation by separating the variables v and x, then re-express the solution in terms of x and y.

Note that this method also works for equations of the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$.

288

Ex $\quad \dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$.

Ans Set $y = vx$ and thus $\dfrac{dy}{dx} = \dfrac{dv}{dx}.x + v$

i.e. $\dfrac{dv}{dx}.x + v = \dfrac{x^2+y^2}{xy} = \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{1}{v} + v$

cancel out v's to get $\quad x\dfrac{dv}{dx} = \dfrac{1}{v}$

$\therefore \int v\,dv = \int \dfrac{dx}{x}$ i.e. $\dfrac{v^2}{2} = \ln x + A$

and $\quad y^2 = 2x^2(\ln x + A)$.

289

- $\boxed{\dfrac{dy}{dx} + P(x)y = Q(x)}$ **First order linear** o.d.e. – use the **integrating factor method**

Multiply the equation by integrating factor $IF = e^{\int P(x)\,dx}$ to give $\dfrac{d}{dx}(IF\ y) = IF\ Q(x)$. Then integrate both sides with respect to x, giving $IF\ y = \int IF\ Q(x)\,dx$. Finally, divide by IF to get y.

Ex $x\dfrac{dy}{dx} - y = x^2$ subject to $y(1) = 3$

Ans $\dfrac{dy}{dx} - \left(\dfrac{1}{x}\right)y = x$ i.e. $P(x) = -\dfrac{1}{x}$, $Q(x) = x$. $IF = e^{\int P(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x}$

$= e^{\ln x^{-1}} = x^{-1} = \dfrac{1}{x}$.

Multiply equation $\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x^2}y = 1$ i.e. $\dfrac{d}{dx}\left\{\dfrac{1}{x}y\right\} = 1$

Integrate $\dfrac{1}{x}y = x + C$ i.e. $y = x^2 + Cx$. (general solution) 290

Particular solution with $y(1) = 3$ i.e. $y = 3$ when $x = 1$

$\therefore$ $3 = 1^2 + C.1$ i.e. $3 = 1 + C$ i.e. $C = 2$.

$\therefore$ Particular solution is $y = x^2 + 2x$.

Ex Exactly the same example in terms of a variable $x(t)$ i.e. $t\dfrac{dx}{dt} - x = t^2$; $x(t=1) = 3$

Ans $\dfrac{dx}{dt} - \left(\dfrac{1}{t}\right)x = t$ i.e. $P(t) = -\dfrac{1}{t}$ and $Q(t) = t$

$IF = e^{\int P(t)dt} = e^{-\int \frac{dt}{t}} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1} = \dfrac{1}{t}$.

Multiply equation $\dfrac{1}{t}\dfrac{dx}{dt} - \dfrac{1}{t^2}x = 1$ i.e. $\dfrac{d}{dt}\left\{\dfrac{1}{t}x\right\} = 1$

Integrate $\dfrac{1}{t}x = t + C$ i.e. $x(t) = t^2 + Ct$. (general solution) 291

Particular solution with $x(1) = 3$ i.e. $x = 3$ when $t = 1$

$\therefore$ $3 = 1^2 + C.1$ i.e. $3 = 1 + C$ i.e. $C = 2$.

$\therefore$ Particular solution is $x(t) = t^2 + 2t$.

- $\boxed{\dfrac{dy}{dx}+P(x)y=Q(x)y^n}$ **Bernoulli's differential equation**

Change the dependent variable from y to z where $z=y^{1-n}$.
This makes the equation linear and we can use the integrating factor method.

Dividing by y^n gives $\dfrac{1}{y^n}\dfrac{dy}{dx}+P(x)y^{1-n}=Q(x)$

i.e. $\dfrac{1}{1-n}\dfrac{dz}{dx}+P(x)\,z=Q(x)$.

(using $\dfrac{dz}{dx}=(1-n)y^{-n}\dfrac{dy}{dx}$).

292

Ex $\dfrac{dy}{dx}+\dfrac{y}{x}=y^2$ Ans This is of the form $\dfrac{dy}{dx}+P(x)y=Q(x)y^n$ with $P(x)=\frac{1}{x}$, $Q(x)=1$, $n=2$

Setting $z=y^{1-n}=y^{1-2}=\frac{1}{y}$ gives $\dfrac{dz}{dx}=-\dfrac{1}{y^2}\dfrac{dy}{dx}$ {using the chain rule}

$\therefore \dfrac{dy}{dx}=-y^2\dfrac{dz}{dx}$ i.e. $-y^2\dfrac{dz}{dx}+\dfrac{y}{x}=y^2$ i.e. $\dfrac{dz}{dx}-\dfrac{1}{x}\dfrac{1}{y}=-1$

i.e. $\dfrac{dz}{dx}-\dfrac{1}{x}z=-1$.

For $\dfrac{dz}{dx}-\dfrac{1}{x}z=-1$ we use the integrating factor (IF) method

i.e. $IF=e^{-\int\frac{dx}{x}}=e^{\ln x^{-1}}=\dfrac{1}{x}$ (as before).

Multiply equation $\dfrac{1}{x}\dfrac{dz}{dx}-\dfrac{1}{x^2}z=-\dfrac{1}{x}$ i.e. $\dfrac{d}{dx}\left\{\dfrac{z}{x}\right\}=-\dfrac{1}{x}$

Integrate w.r.t. x $\dfrac{z}{x}=-\int\dfrac{dx}{x}=-\ln x+C$.

293

Re-express in terms of x and y $z=\dfrac{1}{y}$ gives $\dfrac{1}{xy}=-\ln x+C$.

- $\boxed{P(x,y)\,dx + Q(x,y)\,dy = 0}$ [this method of solution will **not** be examined]

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ then the o.d.e. is said to be **exact**.

This means that a function $u(x,y)$ exists such that $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$

$$= P\ dx + Q\ dy = 0.$$

Ones solves $\dfrac{\partial u}{\partial x} = P$ and $\dfrac{\partial u}{\partial y} = Q$ to find $u(x,y)$.

Then $du = 0$ gives $u(x,y) = constant$ (this is the general solution of $Pdx + Qdy = 0$).

294

Ex

$$\boxed{(8e^{4x} + 2xy^2)\,dx + (4\cos 4y + 2x^2y)\,dy = 0}$$

$P = 8e^{4x} + 2xy^2$, $Q = 4\cos 4y + 2x^2y$

$\therefore \dfrac{\partial P}{\partial y} = 4xy$; $\dfrac{\partial Q}{\partial x} = 4xy = \dfrac{\partial P}{\partial y}$ $\therefore$ o.d.e. is exact.

$\therefore$ $u(x,y)$ exists such that $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$

$$= P\,dx + Q\,dy = 0$$

Giving $\dfrac{\partial u}{\partial x} = 8e^{4x} + 2xy^2$ (i)

$\dfrac{\partial u}{\partial y} = 4\cos 4y + 2x^2y$ (ii)

Integrate (i) : $\boxed{u = 2e^{4x} + x^2y^2 + \phi(y)}$

Differentiate $\dfrac{\partial u}{\partial y} = 2x^2y + \dfrac{d\phi}{dy} = 4\cos 4y + 2x^2y$ (using (ii))

i.e. $\dfrac{d\phi}{dy} = 4\cos 4y$ i.e. $\int d\phi = 4\int \cos 4y\,dy$

i.e. $\phi = \sin 4y + C'$ $\therefore u = 2e^{4x} + x^2y^2 + \sin 4y + C'$

du=0 gives u=C,

$$\therefore 2e^{4x} + x^2y^2 + \sin 4y = A.$$

Ex

$$\boxed{(6x^2 + 8xy^3)\,dx + (12x^2y^2 + 12y^3)\,dy = 0}$$

$P = 6x^2 + 8xy^3$, $Q = 12x^2y^2 + 12y^3$

$\dfrac{\partial P}{\partial y} = 24xy^2$, while $\dfrac{\partial Q}{\partial x} = 24xy^2$ $\therefore$ o.d.e. is exact.

$\therefore$ $u(x,y)$ exists such that $du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy$

$$= P\,dx + Q\,dy = 0$$

Giving, $\dfrac{\partial u}{\partial x} = 6x^2 + 8xy^3$ (i)

$\dfrac{\partial u}{\partial y} = 12x^2y^2 + 12y^3$ (ii)

Integrate (i) : $\boxed{u = 2x^3 + 4x^2y^3 + \phi(y)}$

Differentiate $\dfrac{\partial u}{\partial y} = 12x^2y^2 + \dfrac{d\phi}{dy} = 12x^2y^2 + 12y^3$ (using (ii))

i.e. $\dfrac{d\phi}{dy} = 12y^3$ i.e. $\int d\phi = 12\int y^3\,dy$ i.e. $\phi = 3y^4 + C'$.

$\therefore u = 2x^3 + 4x^2y^3 + 3y^4 + C'$.

du=0 gives u=C,

$$\therefore 2x^3 + 4x^2y^3 + 3y^4 = A.$$

295

- $\boxed{a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0}$ **Second order linear** o.d.e. with **constant coefficients** a,b,c

It is called a **homogeneous equation** because the RHS = 0.

Setting $y = A\,e^{mx}$ gives $am^2+bm+c=0$ (the "auxiliary equation")

Then $m=\dfrac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$ gives

i) real different roots m_1 , m_2 and $y=Ae^{m_1x}+Be^{m_2x}$,

or ii) real equal roots $m_1=m_2$ and $y=(A+Bx)e^{m_1x}$,

or iii) complex roots $m_{1,2}=p\pm iq$ and $y=e^{px}(A\cos qx+B\sin qx)$.

Ex $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ **Ans** Set $y = Ae^{mx}$ i.e. $\frac{dy}{dx} = Ame^{mx}$, $\frac{d^2y}{dx^2} = Am^2e^{mx}$

Aux. Equ^{n} (A.E.) is $Am^2e^{mx} + 5Ame^{mx} + 6Ae^{mx} = 0$

i.e. $m^2 + 5m + 6 = 0$ (dividing by Ae^{mx})

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2} = -\frac{4}{2} \text{ or } -\frac{6}{2}$$

i.e. $m = -2$ or $m = -3$ $\therefore$ let $m_1 = -2$ and $m_2 = -3$

$y = Ae^{m_1 x} + Be^{m_2 x}$ (real different roots)

i.e. $y = Ae^{-2x} + Be^{-3x}$.

297

Ex $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ **Ans** A.E. is $m^2 - 6m + 9 = 0$ $\therefore$ $m = \frac{+6 \pm \sqrt{36-36}}{2}$

i.e. $m = \frac{6 \pm 0}{2} = 3$

real equal roots, $m_1 = m_2 = 3$

$y = (A + Bx)e^{m_1 x}$

$\therefore$ $y = (A + Bx)e^{3x}$.

Ex $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ **Ans** A.E. is $m^2 + m + 1 = 0$ $\therefore$ $m = \frac{-1 \pm \sqrt{1-4}}{2}$

i.e. $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-1}\sqrt{3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

complex roots $m_{1,2} = p \pm iq$, where $p = -\frac{1}{2}$
$q = \frac{\sqrt{3}}{2}$

$\therefore$ $y = e^{px}(A\cos qx + B\sin qx)$ 299

i.e. $y = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$.

Now, you are just as likely to meet o.d.e.s expressing $x(t)$.

So, we will re-do these examples in different variables, but firstly repeat the summary sheet in terms of $x(t)$ ....

- $$\boxed{a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0}$$

Substituting $x = Ae^{mt}$ gives $am^2 + bm + c = 0$ (Auxiliary equation)

Then $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives 298

i) real different roots m_1, m_2 and $x = Ae^{m_1 t} + Be^{m_2 t}$,

or ii) real equal roots $m_1 = m_2$ and $x = (A + Bt)e^{m_1 t}$,

or iii) complex roots $m_{1,2} = p \pm iq$ and $x = e^{pt}(A\cos qt + B\sin qt)$.

Ex $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$ Ans A.E. is $m^2 + 5m + 6 = 0$, giving $m_1 = -2$, $m_2 = -3$

$\to x = Ae^{-2t} + Be^{-3t}$ (real different roots)

Ex $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 0$ Ans A.E. is $m^2 - 6m + 9 = 0$, giving $m_1 = m_2 = 3$

$\to x = (A + Bt)e^{3t}$ (real equal roots) 300

Ex $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = 0$ Ans A.E. is $m^2 + m + 1 = 0$, giving $m_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\to x = e^{-\frac{t}{2}}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right).$

- $\boxed{a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=f(x)}$ **Second order linear** o.d.e. with **constant coefficients** a,b,c 301

  It is **not homogeneous** since RHS is not zero.

Step One Solve the corresponding homogeneous equation to get $y = y_{CF}$
This is called the "complementary function".

Step Two The general solution of the full equation is $y = y_{CF} + y_{PS}$.
Where y_{PS} is a particular solution of the full equation.

Find y_{PS} by substituting a trial form into the full equation and equate the coefficients of the functions involved (e.g. e^{2x}, x^2, $\cos x$, etc.).

$f(x)$	Trial form of y_{PS}
k	C
kx ...	$Cx+D$
kx^2 ...	$Cx^2 + Dx + E$
$k\cos ax$ OR $k\sin ax$	$C\cos ax + D\sin ax$
ke^{ax}	Ce^{ax}
Sum/product of the above	Sum/product of the above
(k, a are given constants)	(C, D, E are constants to be determined)

302

Note If the suggested form of y_{PS} already appears in y_{CF} then multiply the trial form of y_{PS} by x until it does not appear in y_{CF}.

Ex $\dfrac{d^2y}{dx^2} + 4y = 8\sin 2x$

Ans A.E. is $m^2+4=0$ giving $m^2=-4$ i.e. $m = \pm\sqrt{4}\sqrt{-1} = \pm 2i$.

$y_{cf} = e^{px}(A\cos qx + B\sin qx)$, where $p=0$ and $q=2$

i.e. $y_{cf} = e^{0}(A\cos 2x + B\sin 2x) = A\cos 2x + B\sin 2x$.

RHS is $8\sin 2x$. Try $y_{ps} = C\cos 2x + D\sin 2x$? No, this already appears in y_{cf}.

Try $y_{ps} = x(C\cos 2x + D\sin 2x)$ i.e. multiply by x until y_{ps} does not appear in y_{cf}.

$y_{ps}' = C\cos 2x + D\sin 2x + x(-2C\sin 2x + 2D\cos 2x)$ - - using product rule

$y_{ps}'' = -2C\sin 2x + 2D\cos 2x + (-2C\sin 2x + 2D\cos 2x) + x(-4C\cos 2x - 4D\sin 2x)$ 303

i.e. $y_{ps}'' = \sin 2x(-2C - 2C - 4xD) + \cos 2x(2D + 2D - 4xC)$

$\therefore\ y_{ps}'' = -4(C+xD)\sin 2x + 4(D-xC)\cos 2x$

Substitute into original equation

$$\left[-4(C+xD)\sin 2x + 4(D-xC)\cos 2x\right] + 4\left[xC\cos 2x + xD\sin 2x\right] = 8\sin 2x$$

cancels

cancels

leaving $-4C\sin 2x + 4D\cos 2x = 8\sin 2x$

Compare coefficients of $\sin 2x$, $\cos 2x$:

$\sin 2x$: $-4C = 8$
$\cos 2x$: $4D = 0$
$\Rightarrow$ $C=-2$, $D=0$

General solution is $y = y_{cf} + y_{ps}$

i.e. $y = A\cos 2x + B\sin 2x - 2x\cos 2x$.

$$\boxed{\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-3t}\ ;\ x=\tfrac{1}{2} \text{ and } \frac{dx}{dt} = -2 \text{ at } t=0}$$

A.E. is $m^2+4m+3=0$ i.e. $(m+3)(m+1)=0$ i.e. $m=-3$ or $m=-1$

$x_{cf} = Ae^{-3t} + Be^{-t}$.

Try $x_{ps} = Ce^{-3t}$? No. Already in x_{cf}.

Try $x_{ps} = Cte^{-3t}$, $\dfrac{dx_{ps}}{dt} = Ce^{-3t} - 3tCe^{-3t} = (1-3t)Ce^{-3t}$.

$\dfrac{d^2x_{ps}}{dt^2} = -3.Ce^{-3t} - 3(1-3t)Ce^{-3t} = (9t-6)Ce^{-3t}$.

Substitute: $(9t-6)Ce^{-3t} + 4(1-3t)Ce^{-3t} + 3Cte^{-3t} = e^{-3t}$

Coeff. e^{-3t}: $9tC - 6C + 4C - 12tC + 3Ct = 1$ i.e. $C = -\tfrac{1}{2}$

$x_{ps} = -\tfrac{1}{2}te^{-3t}$,

general solution is $\boxed{x = x_{cf} + x_{ps} = Ae^{-3t} + Be^{-t} - \tfrac{1}{2}te^{-3t}}$

Boundary conditions $x=\tfrac{1}{2}$ when $t=0$: $\tfrac{1}{2} = A + B$ (i)

$\dfrac{dx}{dt} = -3Ae^{-3t} - Be^{-t} + \tfrac{3}{2}te^{-3t} - \tfrac{1}{2}e^{-3t}$

$\dfrac{dx}{dt} = -2$ when $t=0$: $-2 = -3A - B - \tfrac{1}{2}$

i.e. $-\tfrac{3}{2} = -3A - B$ (ii)

Solve (i) and (ii) for A and B : (i)+(ii) gives $\tfrac{1}{2} - \tfrac{3}{2} = A - 3A$ i.e. $A = \tfrac{1}{2}$.

Then, (i) gives $B = 0$.

Particular solution is $x = \tfrac{1}{2}e^{-3t} - \tfrac{1}{2}te^{-3t} = \tfrac{1}{2}(1-t)e^{-3t}$.

304

$$\boxed{\frac{d^2x}{dt^2}+4\frac{dx}{dt}+5x=6\sin t}$$

Identify transient and long-term ("steady-state") components of the general solution.

A.E. is $m^2+4m+5=0$ i.e. $m=\frac{1}{2}\left(-4\pm\sqrt{16-20}\right)=-2\pm i$

$x_{CF}=e^{-2t}(A\cos t+B\sin t)$.

[Since e^{-2t} multiplies $\sin t$ in x_{CF}, $\sin t$ is an independent function with respect to the components of x_{CF}.]

Try $x_{PS}=C\cos t+D\sin t$

$x_{PS}'=-C\sin t+D\cos t$

$x_{PS}''=-C\cos t-D\sin t$ (dash denoting $\frac{d}{dt}$, here)

Substitute: $-C\cos t-D\sin t+4(-C\sin t+D\cos t)+5(C\cos t+D\sin t)=6\sin t$

Coeff. $\cos t$: $-C+4D+5C=0$ i.e. $C+D=0$ (i)

Coeff. $\sin t$: $-D-4C+5D=6$ i.e. $-C+D=3/2$ (ii)

ADD $2D=3/2$

i.e. $D=\frac{3}{4}$ and $C=-3/4$.

General solution: $x=\underbrace{e^{-2t}(A\cos t+B\sin t)}-\underbrace{\frac{3}{4}(\cos t-\sin t)}$.

as $t\to\infty$, this component tends to zero i.e. it is the TRANSIENT component

as $t\to\infty$, this component continues to oscillate i.e. it is the LONG-TERM ("steady-state") component.

Solution of higher order linear differential equations

(with constant coefficients)

- HOMOGENEOUS EQUATIONS (RHS=0)

i.e. $$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_ny=0$$

Set $y=e^{mx}$,

$\to a_0m^n+a_1m^{n-1}+\dots+a_n=0$ characteristic / auxiliary equation

i.e. $a_0(m-m_1)(m-m_2)\dots(m-m_n)=0$

with roots $m_1, m_2, \dots, m_n$

$\to$ 3 cases

(i) Roots all real and distinct:

$$y=c_1e^{m_1x}+c_2e^{m_2x}+\dots+c_ne^{m_nx}$$

(ii) Repeated roots (k times)

If m_1 has multiplicity k then its contribution to the solution is

$$y=(c_1+c_2x+c_3x^2+\dots+c_kx^{k-1})e^{m_1x}$$

(iii) Complex roots always appear as conjugate pairs $p \pm iq$

Each pair of complex roots contributes $y = e^{px}(A\cos qx + B\sin qx)$ to the solution.

- INHOMOGENEOUS EQUATIONS (RHS $\neq 0$)

General solution $\boxed{y = Y_{cf}(x) + Y_{ps}(x)}$

where $Y_{cf}(x)$ = solution of the homogeneous equation

$Y_{ps}(x)$ = a particular solution of the full equation

To find $Y_{ps}(x)$

- Substitute a trial solution involving unknown constants $C, D, E, \ldots$
- Guess the trial solution from the form of the RHS (as before)

Ex Show that the general solution of the inhomogeneous system $\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = R(x)}$

is $y = Y_c(x) + Y_p(x)$ where $Y_c(x)$ = complementary solution, $Y_p(x)$ = particular solution

Ans Write system as $\phi(D)y = R(x)$

where $D = \frac{d}{dx}$ and $\phi(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_n$.

General complementary solution $Y_c(x)$ satisfies $\phi(D)y = 0$

and has the required n arbitrary constants. i.e. $\boxed{\phi(D)Y_c = 0}$ ①

$Y_p(x)$ is a particular solution of $\phi(D)y = R(x)$

ie $\boxed{\phi(D)Y_p = R(x)}$ ②

Add equations ① and ② to get

$$\phi(D)Y_c(x) + \phi(D)Y_p(x) = R(x)$$

$$\text{i.e.} \quad \phi(D)\left[Y_c(x) + Y_p(x)\right] = R(x)$$

(since $\phi(D)$ is a linear differential operator)

$\therefore$ $y = Y_c(x) + Y_p(x)$ is a solution of $\phi(D)y = R(x)$ with n arbitrary constants i.e. the general solution.

PARTIAL DIFFERENTIAL EQUATIONS

- Some important p.d.e.'s
- Role of arbitrary functions
- Connections with solution of o.d.e's
  - direct integration
  - homogeneous systems
  - inhomogeneous systems
- Separation of variables
  - finding a solution
  - superposition to get the required solution (Fourier analysis)

Some important partial differential equations

Laplace's equation: $\boxed{\nabla^2 u = 0}$

u represents a potential in absence of sources/sinks

e.g. gravitational potential (where there is no matter)
electrostatic potential (where there are no charges)
temperature (where no sources of heat): steady-state
velocity potential of incompressible fluid (when there are no vortices/sources/sinks)

Poisson's equation $\boxed{\nabla^2 u = f(x,y,z)}$

as above but $f(x,y,z)$ is the source density
e.g. electric charge density

Diffusion or heat flow equation $\boxed{\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}}$

no sources but diffusion in time
e.g. non-steady-state temperature evolution

Wave equation

$$\boxed{\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}}$$

v = wave speed

u = displacement of vibrating string
or amplitude of wave in gas, liquid, etc.
or electric/magnetic field

Role of arbitrary functions

A partial differential equation involves two or more independent variables and partial derivatives with respect to these variables.

As with ode's, the <u>order</u> of the equation is the order of the highest derivative present.

e.g. $\dfrac{\partial^2 u}{\partial x \partial y} = 2x - y$: second order p.d.e.
independent variables x, y

The arbitrary constants of general solutions of ode's become <u>arbitrary functions</u> in the general solution of p.d.e.'s. Particular solutions then have a particular choice of arbitrary function.

Why arbitrary functions?

Think about what a partial differential means.

For the equation $\boxed{\dfrac{\partial^2 u}{\partial x \partial y} = 2x - y}$,

there is the general solution $u = x^2 y - \frac{1}{2}xy^2 + \underbrace{F(x) + G(y)}_{\text{arbitrary functions.}}$

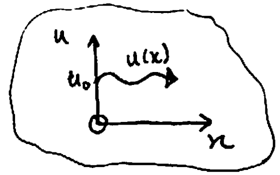
$$\frac{\partial u}{\partial y} = x^2 - xy + G'(y)$$

i.e. the whole function of x, $F(x)$, is treated as a constant in the operation $\dfrac{\partial}{\partial y}$. So if we integrated $\dfrac{\partial u}{\partial y}$ with respect to y, we would generally have to introduce a function of x (rather than just an integration constant).

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2 - xy + G'(y)\right) = 2x - y, \text{ as required.}$$

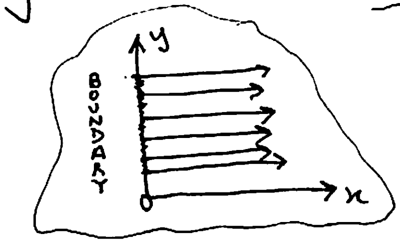
Another difference with odes is that initial-value problems

e.g. $\dfrac{du}{dx} = f(x, y)$ with $\underbrace{u(x=0) = u_0}_{\text{initial value}}$



tend to become <u>boundary-value problems</u>:

e.g. $\dfrac{\partial^2 u}{\partial x \partial y} = f(x,y)$ with $\underbrace{u(x=0,y) = g(y)}_{\text{boundary}}$



i.e. because we have more than one independent variable, boundary conditions are not specified at a point.

We will deal here with linear partial differential equations that have constant coefficients.

e.g. $\boxed{a_0 \dfrac{\partial^2 u}{\partial x^2} + a_1 \dfrac{\partial^2 u}{\partial x \partial y} + a_2 \dfrac{\partial^2 u}{\partial y^2} + a_3 \dfrac{\partial u}{\partial x} + a_4 \dfrac{\partial u}{\partial y} + a_5 u = f(x,y)}$

is second order, linear in u and $a_1, a_2, \dots, a_5$ are constants

If $f(x,y) = 0$ then the equation is homogeneous.

Solution by direct integration

Ex Starting with $\dfrac{\partial^2 u}{\partial x \partial y} = 2xy$ derive the general solution.

Ans Consider the left-hand side as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ and integrate with respect to x ...

i.e. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 2xy$

gives $\dfrac{\partial u}{\partial y} = x^2 y + F(y)$

Then, integrate with respect to y ...

$$u = \frac{x^2 y^2}{2} + \int F(y)\,dy + G(x)$$

$$\Rightarrow \boxed{u = \frac{x^2 y^2}{2} + H(y) + G(x)}$$, where $H(y) = \int F(y)\,dy$

N.B. General solution of pde of order 2 has 2 arbitrary functions.

Homogeneous systems

Recall that for ode's one finds the solution of a homogeneous equation by setting $y = e^{mx}$ and then seeking the roots of the resulting characteristic equation, where x is the independent variable.

Now we may have two independent variables, x and t for example: $\boxed{\dfrac{\partial y}{\partial x} + \dfrac{1}{c}\dfrac{\partial y}{\partial t} = 0}$

Set $y = e^{ax+bt}$ i.e. $a.e^{ax+bt} + \frac{1}{c}.b.e^{ax+bt} = 0$

i.e. $\left(a + \frac{b}{c}\right)e^{ax+bt} = 0$ i.e. $a + \frac{b}{c} = 0$ i.e. $b = -ac$

A solution of $\boxed{\frac{\partial y}{\partial x} + \frac{1}{c}\frac{\partial y}{\partial t} = 0}$

is then $y = e^{ax+bt} = e^{ax-act} = e^{a(x-ct)}$,

for any a.

This is not the arbitrary function but it suggests an arbitrary function $\boxed{y = F(x-ct)}$.

Let's show that this is a solution ...

Let $u = x-ct$ i.e. $y = F(u)$.

$$\frac{\partial y}{\partial x} = \underbrace{\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}}_{\text{(CHAIN RULE)}} = \frac{\partial F}{\partial u} \quad ; \quad \frac{\partial y}{\partial t} = \underbrace{\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}}_{\text{(CHAIN RULE)}} = \frac{\partial F}{\partial u}.(-c)$$

$$\therefore \frac{\partial y}{\partial x} + \frac{1}{c}\frac{\partial y}{\partial t} = \frac{\partial F}{\partial u} + \frac{1}{c}.(-c)\frac{\partial F}{\partial u} = 0.$$

- This technique can allow one to quickly determine the general solution of homogeneous partial differential equations.

Another example ...

The wave equation in one space dimension

i.e. $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

where the Laplacian, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \rightarrow \frac{\partial^2}{\partial x^2}$, i.e one space dimension only

i.e. $\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$: homogeneous equation

Set $u = e^{ax+bt}$, $a^2 - \frac{1}{v^2}b^2 = 0$ i.e. $b = \pm av$

$\therefore u = e^{ax \pm avt} = e^{a(x \pm vt)}$, for any a.

General solution of the 1D wave equation is

$$\boxed{u = F(x+vt) + G(x-vt)},$$

where F and G are arbitrary functions.

Note Here we have employed the "superposition principle".

i.e. since the equation is linear and homogeneous, the sum of solutions is also a solution.

Inhomogeneous systems

To solve an inhomogeneous ode for the general solution, we added the general solution of the homogeneous ode to a particular solution of the full equation.

We can do the same for partial differential equations.

Ex $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = e^{2x+y}$

Ans Set $u = e^{ax+by}$ in $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = 0$

The general solution of the homogeneous equation can be written as $u = F(2x+y) + G(2x-y)$.

Try $u = Ce^{2x+y}$ as a particular solution and determine C? No. We already have $F(2x+y)$ in the complementary solution. Try $u = Cxe^{2x+y}$ (or $u = Cye^{2x+y}$).

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = 2Ce^{2x+y} + 2Ce^{2x+y} + 4Cxe^{2x+y}$; $-4\dfrac{\partial^2 u}{\partial y^2} = -4Cxe^{2x+y}$

$\therefore$ $4Ce^{2x+y} = e^{2x+y}$ and $C = \frac{1}{4}$.

$\therefore$ General solution is $u = F(2x+y) + G(2x-y) + \frac{1}{4}xe^{2x+y}$.

Separation of variables

Here we assume that the solution can be expressed as a product of unknown functions of each of the independent variables e.g. $u(x,y) = X(x)Y(y)$.

How do we know that the solution is of this form?

Generally, the solution we seek is not of this form!

But we can combine separable solutions together to get the desired solution.

Let's start with some simple examples...

Ex Solve the boundary-value problem

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \quad u(0,y) = 8e^{-3y}$$

Ans Set $u = X(x)Y(y)$ and substitute to find...

$X_x Y = 4XY_y$, where the subscript denotes partial derivative.

$\therefore \dfrac{X_x}{4X} = \dfrac{Y_y}{Y}$.

Left-hand side only depends on x but is true for all x.

Right-hand side only depends on y but is true for all y.

This implies that each side of the equation equals a constant since x and y are independent.

i.e. $\dfrac{X_x}{4X} = c = \dfrac{Y_y}{Y}$, $c =$ "separation constant"

We can now write this as two ordinary differential equations

i.e. $\dfrac{dX}{dx} = c4X$ and $\dfrac{dY}{dy} = cY$

with solutions $X = Ae^{4cx}$ and $Y = Be^{cy}$

$\therefore\ u = XY = Ke^{c(4x+y)}$, $(K = AB)$.

Now apply the boundary condition $u(0,y) = 8e^{-3y}$

i.e. $Ke^{c(4x+y)} \xrightarrow{x=0} Ke^{cy} = 8e^{-3y}$

i.e. $K = 8$, $c = -3$.

Required solution is

$$u = 8e^{-3(4x+y)}$$

Note This is a solution that is separable.

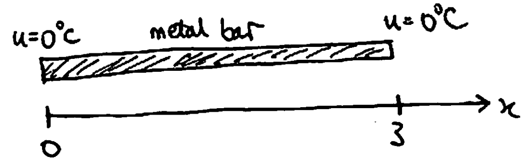
The following example results in a final solution that is not separable ...

Ex Solve the heat flow equation $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$

for $0 < x < 3$, $t > 0$, given that $u(0,t) = u(3,t) = 0$

and $u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$.

Ans If $u =$ temperature then we could be describing the following ...



i.e. a bar of length 3 units whose temperature is kept at 0°C at each end.

Initially, at $t=0$, the distribution of temperature along the bar is given by $u(x,0)$.

We wish to know how the temperature evolves with time i.e. $u(x,t)$.

Set $u = X(x)T(t)$ in the pde : $XT_t = 2X_{xx}T$

i.e. $\dfrac{X_{xx}}{X} = \dfrac{T_t}{2T} = -\lambda^2$ (separation constant)

We use $-\lambda^2$ to avoid unphysical solutions that result if $+\lambda^2$ is taken.

This gives two ode's

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + 2\lambda^2 T = 0$$

Simple harmonic oscillator

$$X = A_1 \cos\lambda x + B_1 \sin\lambda x \quad ; \quad T = c_1 e^{-2\lambda^2 t}$$

i.e. a solution is $u = XT = c_1 e^{-2\lambda^2 t}(A_1 \cos\lambda x + B_1 \sin\lambda x)$

or simply, $\boxed{u = e^{-2\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)}$

Apply boundary conditions at $x = 0, 3$

$x=0$, $u = 0 = e^{-2\lambda^2 t}(A + 0)$ i.e. $A = 0$

then $u = e^{-2\lambda^2 t} B\sin\lambda x$

$x=3$, $u = 0 = e^{-2\lambda^2 t} B\sin\lambda 3$ i.e. $3\lambda = m\pi$, $m = 0, \pm1, \pm2, \ldots$

i.e. $\lambda = \frac{m\pi}{3}$

Solution is now $\boxed{u = e^{-\frac{2m^2\pi^2}{9}t}\left(B\sin\frac{m\pi x}{3}\right)}$

Superposition principle: sum of such solutions is also a solution

e.g. $$u = e^{-\frac{2m_1^2\pi^2 t}{9}} B_1 \sin\left(\frac{m_1\pi x}{3}\right) + e^{-\frac{2m_2^2\pi^2 t}{9}} B_2 \sin\left(\frac{m_2\pi x}{3}\right) + e^{-\frac{2m_3^2\pi^2 t}{9}} B_3 \sin\left(\frac{m_3\pi x}{3}\right)$$

Final boundary condition is

$$u(x,0) = 5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x)$$

where $u(x,0) = B_1 \sin\left(\frac{m_1\pi x}{3}\right) + B_2 \sin\left(\frac{m_2\pi x}{3}\right) + B_3 \sin\left(\frac{m_3\pi x}{3}\right)$

i.e. $B_1 = 5$, $B_2 = -3$, $B_3 = 2$ and $m_1 = 12$, $m_2 = 24$, $m_3 = 30$

$\therefore$ Required solution is

$$u(x,t) = 5e^{-32\pi^2 t}\sin(4\pi x) - 3e^{-128\pi^2 t}\sin(8\pi x) + 2e^{-200\pi^2 t}\sin(10\pi x).$$