HANDOUT 4 [without gaps]

— VECTOR CALCULUS (continued)

Flux calculations (surface integrals like $\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$)

Divergence

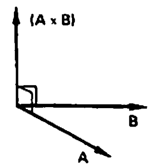
Curl
- definition
- physical examples

Multiple operations
- grad div , div grad , curl curl
- curl grad , div curl , div grad (revisited, Laplacian, physical examples)

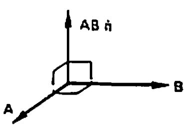
Revision summary (so far)

VECTOR PRODUCT → VECTOR AREA → SURFACE INTEGRALS

- The vector product of two vectors **A** and **B** is defined as

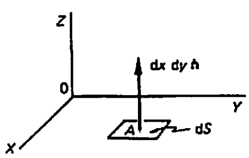


$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$, at right angles to the plane of **A** and **B** to form a right-handed set.



If $\theta = \frac{\pi}{2}$, then $|\mathbf{A} \times \mathbf{B}| = AB$ in the direction of the normal. Therefore, if $\hat{\mathbf{n}}$ is a unit normal then

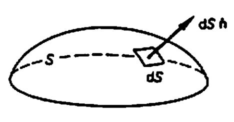
$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\hat{\mathbf{n}} = AB\,\hat{\mathbf{n}}$$



- If $P(x, y)$ is a point in the xy-plane, the element of area $d\underset{\sim}{\mathbf{S}}$ can be written

$$d\underset{\sim}{\mathbf{S}} = (\mathbf{i}\,dx) \times (\mathbf{j}\,dy) = dx\,dy\,\hat{\mathbf{n}}$$

i.e. a vector of magnitude $dx\,dy$ acting in the direction of $\hat{\mathbf{n}}$ and referred to as the *vector area*.



- For a general surface S in space, each element of surface dS has a *vector area* $d\mathbf{S}$ such that $d\mathbf{S} = dS\,\hat{\mathbf{n}}$.

You will also remember that we established previously that the unit normal $\hat{\mathbf{n}}$ to a surface S is given by

$$\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{\nabla} S}{|\underset{\sim}{\nabla} S|} .$$

Let's work out some surface integrals of the form

$$\int_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

to illustrate the technique.

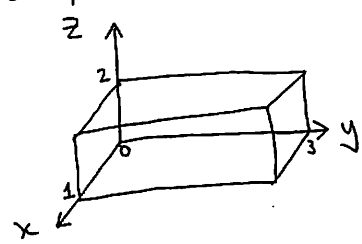
This is a long example, but we will use the result again when we get to the Divergence Theorem.

Ex. Consider a vector field $\underset{\sim}{F}(x,y,z) = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$ and a surface S with flat sides that is bounded by the planes $x=0,\ y=0,\ z=0$; $x=1,\ y=3,\ z=2$.

What is the total flux of $\underset{\sim}{F}$ over S?

In other words, what is $\oint_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$?

Ans. The surface S is a "box" in x, y, z ...

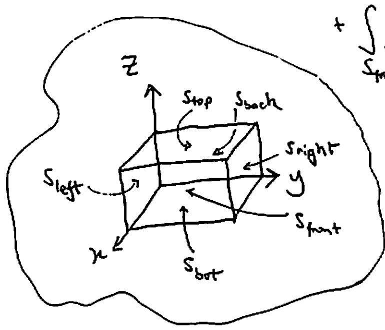


(95)

To work out the flux of $\mathbf{F}$ over the whole surface, consider each side of the box in turn.

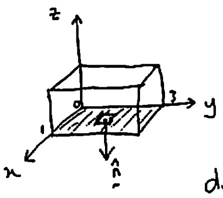
$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \int_{S_{bot}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{top}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{right}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{left}} \mathbf{F}\cdot d\mathbf{S}$$

$$+ \int_{S_{front}} \mathbf{F}\cdot d\mathbf{S} + \int_{S_{back}} \mathbf{F}\cdot d\mathbf{S}$$



And recall that, for this closed surface, each $d\mathbf{S}$ will point OUTWARDS from the enclosed volume.

(i) The base of the box, S_{bot}



$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

but here $z = 0$, so

$\mathbf{F} = x^2\mathbf{i} + y\mathbf{k}$.

dS is in the xy plane, i.e. $dS = dx\,dy$

while $\hat{\mathbf{n}} = -\mathbf{k}$ (pointing outwards and therefore downwards).

(96)

So we have $\quad d\mathbf{S} = \hat{\mathbf{n}}\,dS = -\mathbf{k}\,dS$

and
$$\int_{S_{bot}} \mathbf{F}\cdot d\mathbf{S} = \int_{S_{bot}} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$
$$= \int_{S_{bot}} (x^2\mathbf{i} + y\mathbf{k})\cdot(-\mathbf{k})\,dS$$
$$= \int_{S_{bot}} \left[x^2(-\mathbf{k}\cdot\mathbf{i}) + y(-\mathbf{k})\cdot\mathbf{k}\right] dS$$
$$= \int_{S_{bot}} (0 - y)\,dS$$
$$= \int_{S_{bot}} (-y)\,dS$$

On this surface, $dS = dx\,dy$ and x varies from 0 to 1 and y varies from 0 to 3

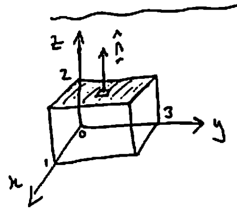
$$\therefore \int_{S_{bot}} \mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{x=1}\int_{y=0}^{y=3} (-y)\,dx\,dy$$
$$= \int_{x=0}^{x=1} \left[-\frac{y^2}{2}\right]_0^3 dx$$

(97)

i.e. $\displaystyle\int_{S_{bot}} \mathbf{F}\cdot d\mathbf{S} = \int_0^1 \left(-\frac{9}{2}\right) dx = \left[-\frac{9}{2}x\right]_0^1$

$$= -\frac{9}{2}.$$

(ii) The top of the box, S_{top}



$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

but here $z = 2$, so

$\mathbf{F} = x^2\mathbf{i} + 2\mathbf{j} + y\mathbf{k}$.

S_{top} is composed of elements dS that, again, can be written in terms of dx and dy i.e. $dS = dx\,dy$

while $\hat{\mathbf{n}} = +\mathbf{k}$ (pointing outwards and therefore upwards).

So we have $\quad d\mathbf{S} = \hat{\mathbf{n}}\,dS = +\mathbf{k}\,dx\,dy$

and $\displaystyle\int_{S_{top}} \mathbf{F}\cdot d\mathbf{S} = \int_{S_{top}} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_{S_{top}} (x^2\mathbf{i} + 2\mathbf{j} + y\mathbf{k})\cdot\mathbf{k}\,dx\,dy$

(98)

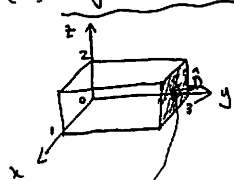
Since $(x^2\mathbf{i} + 2\mathbf{j} + y\mathbf{k})\cdot\mathbf{k} = y$,

$$\int_{S_{top}} \mathbf{F}\cdot d\mathbf{S} = \int_{S_{top}} y\,dS.$$

On this surface, $dS = dx\,dy$ and x varies from 0 to 1 and y varies from 0 to 3

$$\therefore \int_{S_{top}} \mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{x=1}\int_{y=0}^{y=3} y\,dx\,dy = \int_{x=0}^{x=1} \left[\frac{y^2}{2}\right]_0^3 dx$$
$$= \left[\frac{9}{2}x\right]_0^1 = +\frac{9}{2}.$$

(iii) Right hand side of the box, S_{right}



NB $dS = dx\,dz$.

Here, $y = 3$ and

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + 3\mathbf{k}$

$\hat{\mathbf{n}} = \mathbf{j}$ (outwards along positive y)

$$d\mathbf{S}_{right} = \hat{\mathbf{n}}\,dS = \hat{\mathbf{n}}\,dx\,dz = \mathbf{j}\,dx\,dz$$

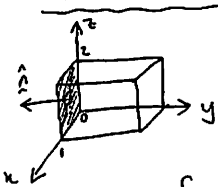
$$\therefore \int_{S_{right}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{S_{right}} (x^2\underset{\sim}{i} + z\underset{\sim}{j} + 3\underset{\sim}{k})\cdot \underset{\sim}{j}\; dxdz$$

$$= \iint_{S_{right}} z\; dxdz = \int_{x=0}^{x=1}\int_{z=0}^{z=2} z\; dxdz$$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^2 dx = \int_0^1 2\,dx$$

$$= \left[2x\right]_0^1 = 2 .$$

(iv) Left hand side of the box, S_{left}



Here, $y=0$ and $\underset{\sim}{F} = x^2\underset{\sim}{i} + z\underset{\sim}{j}$.

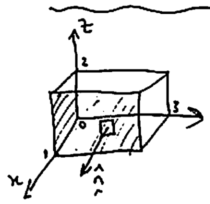
$$d\underset{\sim}{S}_{left} = -\underset{\sim}{j}\; dxdz$$

and
$$\int_{S_{left}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{x=0}^{x=1}\int_{z=0}^{z=2} (-z)\; dzdx$$

$$= \int_{x=0}^{x=1} \left[-\frac{z^2}{2}\right]_0^2 dx$$

$$= \int_0^1 (-2)\,dx = -2 .$$

(v) Front side of the box, S_{front}

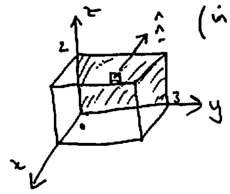


Here, $x=1$ and $\underset{\sim}{F} = \underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$.

$$d\underset{\sim}{S}_{front} = \underset{\sim}{i}\; dydz$$

and
$$\int_{S_{front}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{y=0}^{y=3}\int_{z=0}^{z=2} (1)\; dy\,dz = 6 .$$

(vi) Back side of the box, S_{back}



(in negative x direction)

Here, $x=0$ and $\underset{\sim}{F} = z\underset{\sim}{j} + y\underset{\sim}{k}$

$$d\underset{\sim}{S}_{back} = -\underset{\sim}{i}\; dydz$$

and
$$\int_{S_{back}} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_{S_{back}} (z\underset{\sim}{j} + y\underset{\sim}{k})\cdot(-\underset{\sim}{i})\; dS$$

$$= \int_{S_{back}} (0)\; dS = 0 .$$

Finally, the flux of $\underset{\sim}{F} = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$ over the whole (closed) surface S is

$$\oint_S \underset{\sim}{F}\cdot d\underset{\sim}{S} = \text{" sum of the integrals over the sides"}$$

$$= -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6 .$$

DIV (THE DIVERGENCE OF A VECTOR FUNCTION)

The div operator is the second differential operator that we will define, but first lets clarify what is meant by an "operator" and an "operation".

The scalar product of two vectors is an operation between two vectors that yields a scalar result.

This can also be thought of in terms of an operator that acts on a vector ....

....

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \text{'a scalar'}$$

But we may consider

$$\underset{\sim}{a}\cdot \;=\; \text{'an operator'}$$

that acts on $\underset{\sim}{b}$.

An operator is like a function

e.g. $f(x) = x^2$ is function f acting upon x,

where f is the "square it" operator.

Similarly, if $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$

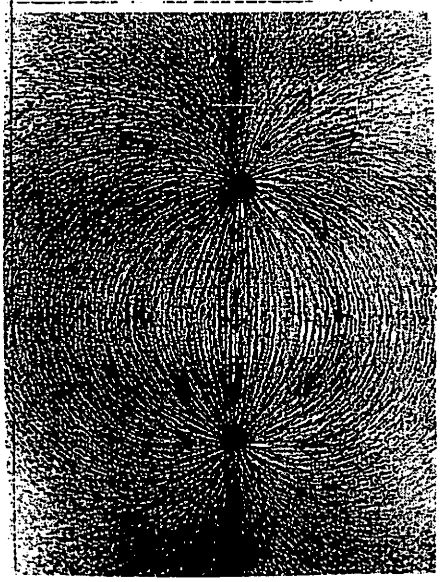
then $\underset{\sim}{a}\cdot$ is an operator acting upon $\underset{\sim}{b}$

that gives the result $a_1b_1 + a_2b_2 + a_3b_3$.

To define the differential operator div,

we replace $\underset{\sim}{a}$ with $\underset{\sim}{\nabla}$

div



SOURCE

SINK

If $\underset{\sim}{V} = V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}$ then

$$\text{div}\,\underset{\sim}{V} \equiv \underset{\sim}{\nabla}.\underset{\sim}{V} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}\right)$$

$$\Rightarrow \boxed{\underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

→ net outflow per unit volume (at a point)

$$\boxed{\text{div (vector)} = \text{scalar}}$$

In other words,

- the div operator $\underset{\sim}{\nabla}.$ acts on a vector and gives a scalar
- $\underset{\sim}{\nabla}.\underset{\sim}{V}$ has the physical interpretation of the net outflow per unit volume (at a point) of the vector field $\underset{\sim}{V}$. This can be deduced from the "Divergence Theorem" that is covered later.
- The "outflow" of a vector field can be related to the presence of "sources" and "sinks" within the vector field.

Ex If $\underset{\sim}{V} = x^2y\,\underset{\sim}{i} - xyz\,\underset{\sim}{j} + yz^2\underset{\sim}{k}$ then work out $\text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}.\underset{\sim}{V}$.

Ans

$$\underset{\sim}{V} = (V_x, V_y, V_z)$$
$$= (x^2y, -xyz, yz^2)$$

So,

$$\underset{\sim}{\nabla}.\underset{\sim}{V} = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)\cdot\left(V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}\right)$$
$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z},$$

where

$$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}(x^2y) = 2xy$$
$$\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}(-xyz) = -xz$$
$$\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}(yz^2) = 2yz.$$

$$\therefore\ \text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}.\underset{\sim}{V} = 2xy + (-xz) + 2yz.$$

This is the divergence of vector field $\underset{\sim}{V}(x,y,z)$. At any point (x,y,z), we can work out the divergence of $\underset{\sim}{V}$ but substituting the values of x, y and z in the right hand side of the above.

Ex If vector field $\underset{\sim}{A} = 2x^2y\,\underset{\sim}{i} - 2(xy^2 + y^3z)\underset{\sim}{j} + 3y^2z^2\underset{\sim}{k}$

then determine $\underset{\sim}{\nabla}.\underset{\sim}{A}$.

Ans $\underset{\sim}{A} = (A_x, A_y, A_z)$ where

$$A_x = 2x^2y$$
$$A_y = -2(xy^2 + y^3z)$$
$$A_z = 3y^2z^2$$

$$\therefore\ \underset{\sim}{\nabla}.\underset{\sim}{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= 4xy - 2(2xy + 3y^2z) + 6y^2z$$
$$= 4xy - 4xy - 6y^2z + 6y^2z = 0.$$

- Any such vector field $\underset{\sim}{A}$ for which $\underset{\sim}{\nabla}.\underset{\sim}{A} = 0$ at <u>all</u> points (x,y,z), as in the above example, is termed SOLENOIDAL.

Let's define a third differential operator called curl ...

CURL (THE CURL OF A VECTOR FUNCTION) (107)

The cross product of two vectors is an operation between two vectors that yields a vector result.

This can also be thought of in terms of an operator that acts on a vector ....

$$\underset{\sim}{a} \times \underset{\sim}{b} = \text{'a vector'}$$

and we may consider

$$\underset{\sim}{a}\times \; = \text{'an operator' that acts on } \underset{\sim}{b} \text{ and gives a vector result.}$$

To define the <u>differential</u> operator curl,

we replace $\underset{\sim}{a}$ with $\underset{\sim}{\nabla}$.

curl is most concisely defined in terms of a 3x3 determinant but can also be written in terms of the expansion of this determinant ...

Recall that (108)

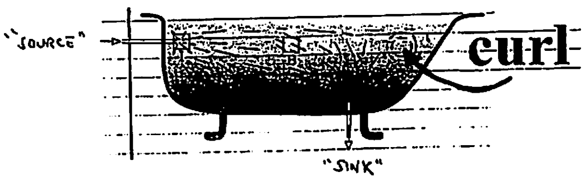
$$\underset{\sim}{a}\times\underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and $\underset{\sim}{\nabla} = \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$.

curl $\underset{\sim}{V}$ (also called rot $\underset{\sim}{V}$) of a vector field $\underset{\sim}{V}(x,y,z) = (V_x, V_y, V_z)$ is given by

$$\underset{\sim}{\nabla}\times\underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

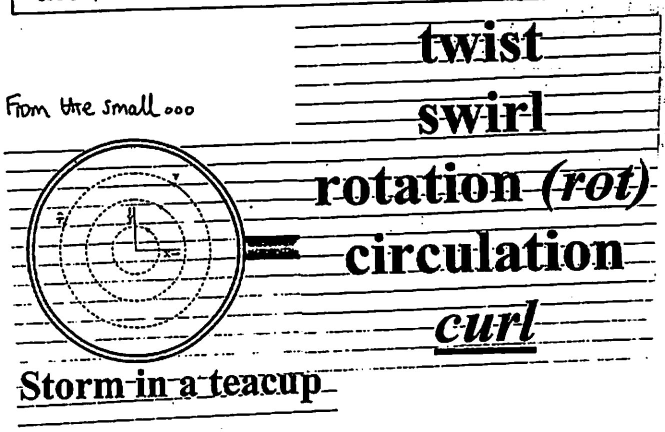
i.e.
$$\underset{\sim}{\nabla}\times\underset{\sim}{V} = \underset{\sim}{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$



Since grad is a vector <u>operator</u>, div and curl yield <u>properties</u> of a vector field. (109)

div ⇒ the flux of the field originating from a point
curl ⇒ the "circulation" there is about a point

From the small ooo



ooo to the large

Hurricane Mitch

And be clear that while (110)

$\boxed{\text{div (vector) = scalar}}$ (scalar product),

$\boxed{\text{curl (vector) = vector}}$ (vector product).

Ex If vector field $\underset{\sim}{V} = (y^4 - x^2z^2)\underset{\sim}{i} + (x^2+y^2)\underset{\sim}{j} - x^2yz\underset{\sim}{k}$

then determine curl $\underset{\sim}{V} = \underset{\sim}{\nabla}\times\underset{\sim}{V}$.

Ans ... $, V_z)$ where $V_x = y^4 - x^2z^2$
$V_y = x^2 + y^2$
$V_z = -x^2yz$

and ... $+ \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}$

(111)

$$\text{curl}\,\mathbf{V} = \nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-x^2yz) - \frac{\partial}{\partial z}(x^2+y^2)\right]$$
$$-\mathbf{j}\left[\frac{\partial}{\partial x}(-x^2yz) - \frac{\partial}{\partial z}(y^4-x^2z^2)\right]$$
$$+\mathbf{k}\left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(y^4-x^2z^2)\right]$$

$$= \mathbf{i}\left[-x^2z+0\right] - \mathbf{j}\left[-2xyz+2x^2z\right]$$
$$+\mathbf{k}\left[2x-4y^3\right]$$

(112)

Ex Determine curl $\mathbf{F}$ at the point $(2,0,3)$ where $\mathbf{F} = ze^{2xy}\,\mathbf{i} + 2xz\cos y\,\mathbf{j} + (x+2y)\,\mathbf{k}$.

Ans

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{2xy} & 2xz\cos y & x+2y \end{vmatrix}$$

$$= \mathbf{i}\left[2-2x\cos y\right] - \mathbf{j}\left[1-e^{2xy}\right] + \mathbf{k}\left[2z\cos y - 2xze^{2xy}\right].$$

At point $(2,0,3)$, $\nabla\times\mathbf{F} = \mathbf{i}\left[2-4\cos 0\right] - \mathbf{j}\left[1-e^0\right] + \mathbf{k}\left[6\cos 0 - 12e^0\right]$

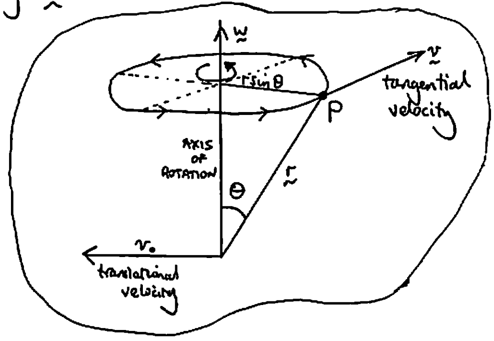
$$= -2\mathbf{i} + 0 - 6\mathbf{k}$$
$$= -2(\mathbf{i}+3\mathbf{k}).$$

(113)

Let's look at two physical examples to see if some justification can be found that 'curl expresses rotation'.

An example from mechanics ...

Ex Consider a rotating body with constant angular velocity $\mathbf{w}$ and that is also moving with a translational velocity $\mathbf{v}_0$



TOTAL VELOCITY AT ANY POINT P ON THE TRANSLATING AND ROTATING BODY,

$$\mathbf{V} = \mathbf{v}_0 + \mathbf{v}$$
$$= \mathbf{v}_0 + \mathbf{w}\times\mathbf{r}$$

$$\text{curl}\,\mathbf{V} = \nabla\times\mathbf{V} = (\nabla\times\mathbf{v}_0) + (\nabla\times\mathbf{w}\times\mathbf{r})$$
$$= \nabla\times\mathbf{w}\times\mathbf{r}$$

, for a constant translational velocity i.e. not space dependent.

(114)

- Evaluate $\mathbf{w}\times\mathbf{r}$.

$$\mathbf{w} = w_x\,\mathbf{i} + w_y\,\mathbf{j} + w_z\,\mathbf{k}$$
$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\therefore\ \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix}$$

$$= \mathbf{i}(w_yz - w_zy) - \mathbf{j}(w_xz - w_zx) + \mathbf{k}(w_xy - w_yx)$$

- Evaluate $\nabla\times(\mathbf{w}\times\mathbf{r})$, noting $\mathbf{w}$ not a function of x, y or z.

$$\nabla\times(\mathbf{w}\times\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_yz - w_zy & -(w_xz - w_zx) & w_xy - w_yx \end{vmatrix}$$

$$= \mathbf{i}(w_x + w_x) + \mathbf{j}(w_y + w_y) + \mathbf{k}(w_z + w_z)$$
$$= 2\mathbf{w}$$

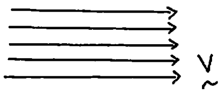
$$\therefore\ \boxed{\text{curl}\,\mathbf{V} = 2\mathbf{w}}$$

i.e. curl expresses both the direction and magnitude of the rotational property of the total velocity vector.

(115)

Examples from fluid dynamics ...

Ex (a)



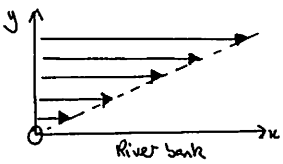
UNIFORM FLOW OF FLUID (space independent)

say,

and $\underset{\sim}{V} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ (i.e. an example of a constant vector)

$$\Rightarrow \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & 2 & -4 \end{vmatrix} = \underset{\sim}{0} \quad \Rightarrow \quad \underset{\sim}{v} \text{ field is IRROTATIONAL.}$$

(b) Flow near a river bank (at $y=0$)



$\underset{\sim}{V} = \alpha y \underset{\sim}{i}$, for example.

i.e. x-component depends on y and α = constant

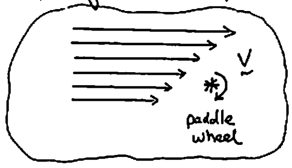
$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha y & 0 & 0 \end{vmatrix} = \underset{\sim}{k}\left(0 - \frac{\partial}{\partial y}(\alpha y)\right) = -\alpha \underset{\sim}{k}$$

(INTO the paper)

(116)

In this case, $\underset{\sim}{\nabla} \times \underset{\sim}{V} \neq \underset{\sim}{0}$: the field is ROTATIONAL and has some "circulation".

To see this, imagine a small paddle wheel in the fluid flow ...



If the fluid velocity is not uniform across the side of the wheel then the wheel will turn i.e. the paddle measures $\underset{\sim}{\nabla} \times \underset{\sim}{V}$ (or the circulation) of the flow at that point.

(c) But not all non-uniform flows have such circulation.

Consider $\underset{\sim}{V} = V_x(x)\underset{\sim}{i} + V_y(y)\underset{\sim}{j} + V_z(z)\underset{\sim}{k}$

i.e. x, y, z components are functions of x, y, z, respectively.

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x(x) & V_y(y) & V_z(z) \end{vmatrix} = \underset{\sim}{i}\left[\frac{\partial V_z(z)}{\partial y} - \frac{\partial}{\partial z}V_y(y)\right] - \underset{\sim}{j}\left[\frac{\partial V_z(z)}{\partial x} - \frac{\partial}{\partial z}V_x(x)\right] + \underset{\sim}{k}\left[\frac{\partial V_y(y)}{\partial x} - \frac{\partial}{\partial y}V_x(x)\right]$$

$= \underset{\sim}{0}$, since each component is only a function of stated coord.

(117)

Summary of grad, div and curl

(a) *Grad* operator ∇ acts on a *scalar* field to give a *vector* field
(b) *Div* operator $\nabla.$ acts on a *vector* field to give a *scalar* field
(c) *Curl* operator $\nabla\times$ acts on a *vector* field to give a *vector* field.
(d) With a *scalar function* $\phi(x, y, z)$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a *vector function* $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

(i) $\text{div}\,\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

(ii) $\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$

Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

Ex $\boxed{\text{grad div } \underset{\sim}{A} = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}.\underset{\sim}{A})}$

If $\underset{\sim}{A} = x^2y\,\underset{\sim}{i} + yz^3\,\underset{\sim}{j} - zx^3\,\underset{\sim}{k} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j} + a_z\underset{\sim}{k}$

then $\text{div}\,\underset{\sim}{A} = \underset{\sim}{\nabla}.\underset{\sim}{A} = \frac{\partial a_x}{\partial x} + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z$

$= 2xy + z^3 - x^3$

$= \phi(x,y,z)$, a scalar field.

(118)

Now, div and curl both act on vector fields but grad acts on a scalar field.

e.g. $\text{grad}(\text{div}\,\underset{\sim}{A}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}.\underset{\sim}{A})$

$= \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$

$= (2y - 3x^2)\underset{\sim}{i} + 2x\underset{\sim}{j} + 3z^2\underset{\sim}{k}$

Ex $\boxed{\text{div grad}\,\phi = \underset{\sim}{\nabla}.(\underset{\sim}{\nabla}\phi)}$

If scalar field $\phi = xyz - 2y^2z + x^2z^2$

then $\text{grad}\,\phi = \underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$

$= (yz + 2xz^2)\underset{\sim}{i} + (xz - 4yz)\underset{\sim}{j} + (xy - 2y^2 + 2x^2z)\underset{\sim}{k}$

i.e. a vector field

(119)

and

$$\operatorname{div}\operatorname{grad}\phi = \nabla.(\nabla\phi)$$

$$= \frac{\partial}{\partial x}\left(yz+2xz^2\right) + \frac{\partial}{\partial y}\left(xz-4yz\right) + \frac{\partial}{\partial z}\left(xy-2y^2+2x^2z\right)$$

$$= 2z^2 - 4z + 2x^2.$$

Ex

$$\boxed{\operatorname{curl}\operatorname{curl}\mathbf{A} = \nabla\times(\nabla\times\mathbf{A})}$$

If vector field $\mathbf{A} = x^2yz\,\mathbf{i} + xyz^2\,\mathbf{j} + y^2z\,\mathbf{k}$

then

$$\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix}$$

$$= \mathbf{i}\left(2yz-2xyz\right) - \mathbf{j}\left(-x^2y\right) + \mathbf{k}\left(yz^2-x^2z\right).$$

(120)

The result of $\nabla\times\mathbf{A}$ is another vector field so we can take the curl of this new field...

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = \nabla\times(\nabla\times\mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz-2xyz & x^2y & yz^2-x^2z \end{vmatrix}$$

$$= \mathbf{i}\,z^2 - \mathbf{j}\left(-2xz-2y+2xy\right) + \mathbf{k}\left(2xy-2z+2xz\right).$$

The following three multiple operations lead to general results (often called 'vector identities').

(a) $\boxed{\operatorname{curl}\operatorname{grad}\phi \text{, where } \phi \text{ is scalar field}}$

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(121)

Then,

$$\operatorname{curl}\operatorname{grad}\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \mathbf{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right] + \mathbf{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= \mathbf{0}$$

i.e. $\operatorname{curl}\operatorname{grad}\phi = \nabla\times(\nabla\phi) = \mathbf{0}$

TRUE FOR ANY SCALAR FIELD ϕ

(122)

(b) $\boxed{\operatorname{div}\operatorname{curl}\mathbf{A} = \nabla.(\nabla\times\mathbf{A})}$

Let $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, then

$$\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

and

$$\operatorname{div}\operatorname{curl}\mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = \frac{\partial}{\partial x}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$= \frac{\partial^2 a_z}{\partial x\partial y} - \frac{\partial^2 a_y}{\partial x\partial z} - \frac{\partial^2 a_z}{\partial y\partial x} + \frac{\partial^2 a_x}{\partial y\partial z} + \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_x}{\partial z\partial y}$$

$$= 0.$$

i.e. $\operatorname{div}\operatorname{curl}\mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = 0$

TRUE FOR ANY VECTOR FIELD $\mathbf{A}$

(c) $\boxed{\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi)}$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \underset{\sim}{i} + \frac{\partial \phi}{\partial y} \underset{\sim}{j} + \frac{\partial \phi}{\partial z} \underset{\sim}{k}$$

$$\therefore \text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

i.e. $\boxed{\text{div grad } \phi = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}}$

In physics, we commonly write the divgrad operator as

$$\boxed{\nabla^2 = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}.$$

It is an important operator in its own right and it is usually called

THE LAPLACIAN.

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Note that the Laplacian is written without an underscore; it is a scalar differential operator.

This means that either $\nabla^2 \phi$ or $\nabla^2 \underset{\sim}{V}$ are possible, where ϕ is a scalar field and $\underset{\sim}{V}$ is a vector field.

The Laplacian appears in numerous important equations such as...

$\nabla^2 \phi = 0$: LAPLACE'S EQUATION

$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$: THE WAVE EQUATION

$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial \phi}{\partial t}$: DIFFUSION or HEAT CONDUCTION EQUATION

∇^2 arises in heat, hydrodynamics, electricity, magnetism, aerodynamics, elasticity, optics, quantum mechanics, and more!!

Particular examples of the Laplacian in electrostatics ...

"Gauss's Law" $\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho / \varepsilon_0}$

where $\underset{\sim}{E}$ = electric field

ρ = charge density

Then, since $\underset{\sim}{E} = -\underset{\sim}{\nabla} V$

where V = (scalar) potential function,

$$\underset{\sim}{\nabla} \cdot (-\underset{\sim}{\nabla} V) = \rho / \varepsilon_0$$

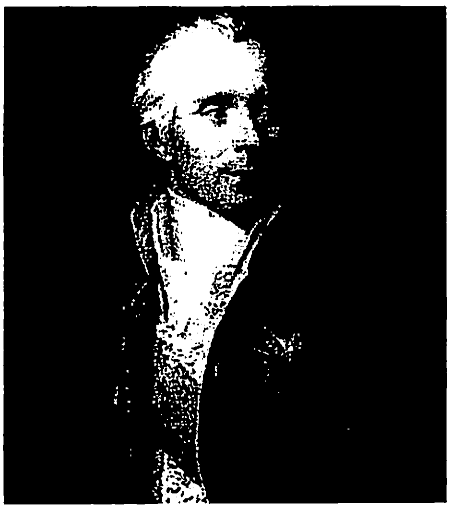
giving "POISSON'S EQUATION" $\boxed{\nabla^2 V = -\rho / \varepsilon_0}$

If there is no charge i.e. $\rho = 0$, we get $\boxed{\nabla^2 V = 0}$.

"LAPLACE'S EQUATION"

Pierre-Simon Laplace

Born: 23 March 1749 in Beaumont-en-Auge, Normandy, France
Died: 5 March 1827 in Paris, France



Laplace attended a Benedictine priory school in Beaumont-en-Auge, as a day pupil, between the ages of 7 and 16. His father expected him to make a career in the Church and indeed either the Church or the army were the usual destinations of pupils at the priory school. At the age of 16 Laplace entered Caen University. As he was still intending to enter the Church, he enrolled to study theology. However, during his two years at the University of Caen, Laplace discovered his mathematical talents and his love of the subject. Credit for this must go largely to two teachers of mathematics at Caen, C Gadbled and P Le Canu of whom little is known except that they realised Laplace's great mathematical potential.

Once he knew that mathematics was to be his subject, Laplace left Caen without taking his degree, and went to Paris. He took with him a letter of introduction to d'Alembert from Le Canu, his teacher at Caen. Although Laplace was only 19 years old when he arrived in Paris he quickly impressed d'Alembert. Not only did d'Alembert begin to direct Laplace's mathematical studies, he also tried to find him a position to earn enough money to support himself in Paris. Finding a position for such a talented young man did not prove hard, and Laplace was soon appointed as professor of mathematics at the Ecole Militaire. Gillespie writes in [1]:-

> *Imparting geometry, trigonometry, elementary analysis, and statics to adolescent cadets of good family, average attainment, and no commitment to the subjects*

He began producing a steady stream of remarkable mathematical papers, the first presented to the Académie des Sciences in Paris on 28 March 1770. This first paper, read to the Society but not published, was on maxima and minima of curves where he improved on methods given by Lagrange. His next paper for the Academy followed soon afterwards, and on 18 July 1770 he read a paper on difference equations.

Not only had he made major contributions to difference equations and differential equations but he had examined applications to mathematical astronomy and to the theory of probability, two major topics which he would work on throughout his life. His work on mathematical astronomy before his election to the Academy included work on the inclination of planetary orbits, a study of how planets were perturbed by their moons, and in a paper read to the Academy on 27 November 1771 he made a study of the motions of the planets which would be the first step towards his later masterpiece on the stability of the solar system.

The 1780s were the period in which Laplace produced the depth of results which have made him one of the most important and influential scientists that the world has seen. It was not achieved, however, with good relationships with his colleagues. Although d'Alembert had been proud to have considered Laplace as his protégé, he certainly began to feel that Laplace was rapidly making much of his own life's work obsolete and this did nothing to improve relations. Laplace tried to ease the pain for d'Alembert by stressing the importance of d'Alembert's work since he undoubtedly felt well disposed towards d'Alembert for the help and support he had given.

In 1784 Laplace was appointed as examiner at the Royal Artillery Corps, and in this role in 1785, he examined and passed the 16 year old Napoleon Bonaparte. In fact this position gave Laplace much work in writing reports on the cadets that he examined but the rewards were that he became well known to the ministers of the government and others in positions of power in France.

Laplace was made a member of the committee of the Académie des Sciences to standardise weights and measures in May 1790. This committee worked on the metric system and advocated a decimal base. In 1793 the Reign of Terror commenced and the Académie des Sciences, along with the other learned societies, was suppressed on 8 August. The weights and measures commission was the only one allowed to continue but soon Laplace, together with Lavoisier, Borda, Coulomb, Brisson and Delambre were thrown off the commission since all those on the committee had to be worthy:-

... by their Republican virtues and hatred of kings.

Before the 1793 Reign of Terror Laplace together with his wife and two children left Paris and lived 50 km southeast of Paris. He did not return to Paris until after July 1794. Although Laplace managed to avoid the fate of some of his colleagues during the Revolution, such as Lavoisier who was guillotined in May 1794 while Laplace was out of Paris, he did have some difficult times. He was consulted, together with Lagrange and Laland, over the new calendar for the Revolution. Laplace knew well that the proposed scheme did not really work because the length of the proposed year did not fit with the astronomical data. However he was wise enough not to try to overrule political dogma with scientific facts. He also conformed, perhaps more happily, to the decisions regarding the metric division of angles into 100 subdivisions.

Exposition du systeme du monde was written as a non-mathematical introduction to Laplace's most important work *Traité du Mécanique Céleste* whose first volume appeared three years later. Laplace had already discovered the invariability of planetary mean motions. In 1786 he had proved that the eccentricities and inclinations of planetary orbits to each other always remain small, constant, and self-correcting. These and many other of his earlier results formed the basis for his great work the *Traité du Mécanique Céleste* published in 5 volumes, the first two in 1799.

The first volume of the *Mécanique Céleste* is divided into two books, the first on general laws of equilibrium and motion of solids and also fluids, while the second book is on the law of universal gravitation and the motions of the centres of gravity of the bodies in the solar system. The main mathematical approach here is the setting up of differential equations and solving them to describe the resulting motions. The second volume deals with mechanics applied to a study of the planets. In it Laplace included a study of the shape of the Earth which included a discussion of data obtained from several different expeditions, and Laplace applied his theory of errors to the results. Another topic studied here by Laplace was the theory of the tides but Airy, giving his own results nearly 50 years later, wrote:-

It would be useless to offer this theory in the same shape in which Laplace has given it; for that part of the Mécanique Céleste which contains the theory of tides is perhaps on the whole more obscure than any other part...

In the *Mécanique Céleste* Laplace's equation appears but although we now name this equation after Laplace, it was in fact known before the time of Laplace. The Legendre functions also appear here and were known for many years as the Laplace coefficients. The *Mécanique Céleste* does not attribute many of the ideas to the work of others but Laplace was heavily influenced by Lagrange and by Legendre and used methods which they had developed with few references to the originators of the ideas.

After the publication of the fourth volume of the *Mécanique Céleste*, Laplace continued to apply his ideas of physics to other problems such as capillary action (1806-07), double refraction (1809), the velocity of sound (1816), the theory of heat, in particular the shape and rotation of the cooling Earth (1817-1820), and elastic fluids (1821). However during this period his dominant position in French science came to an end and others with different physical theories began to grow in importance.

The Société d'Arcueil, after a few years of high activity, began to become less active with the meetings becoming less regular around 1812. The meetings ended completely the following year. Arago, who had been a staunch member of the Society, began to favour the wave theory of light as proposed by Fresnel around 1815 which was directly opposed to the corpuscular theory which Laplace supported and developed. Many of Laplace's other physical theories were attacked, for instance his caloric theory of heat was at odds with the work of Petit and of Fourier. However, Laplace did not concede that his physical theories were wrong and kept his belief in fluids of heat and light, writing papers on these topics when over 70 years of age.

Revision Summary

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; $\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$; then we have the following relationships.

1. *Scalar product* (dot product) $\mathbf{A}.\mathbf{B} = AB\cos\theta$

$$\mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A} \quad \text{and} \quad \mathbf{A}.(\mathbf{B}+\mathbf{C}) = \mathbf{A}.\mathbf{B} + \mathbf{A}.\mathbf{C}$$

2. *Vector product* (cross product) $\mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{N}$

$\mathbf{N}$ = unit normal vector where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{N}$ form a right-handed set.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A}) \text{ and } \mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

3. *Unit vectors*

(a) $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

$\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0.$

(b) $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \underline{0}$

$|\mathbf{i}\times\mathbf{j}| = |\mathbf{j}\times\mathbf{k}| = |\mathbf{k}\times\mathbf{i}| = 1.$

4. *Scalar triple product* $\mathbf{A}.(\mathbf{B}\times\mathbf{C})$

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \mathbf{B}.(\mathbf{C}\times\mathbf{A}) = \mathbf{C}.(\mathbf{A}\times\mathbf{B})$$

Unchanged by cyclic change of vectors.
Sign reversed by non-cyclic change of vectors.

5. *Coplanar vectors* $\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = 0.$

6. *Vector triple product* $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}$$

and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}.\mathbf{A})\mathbf{B} - (\mathbf{C}.\mathbf{B})\mathbf{A}.$

7. *Differentiation of vectors*

If $\mathbf{A}$, a_x, a_y, a_z are functions of u,

$$\frac{d\mathbf{A}}{du} = \frac{da_x}{du}\mathbf{i} + \frac{da_y}{du}\mathbf{j} + \frac{da_z}{du}\mathbf{k}$$

8. *Unit tangent vector* $\mathbf{T}$

$$\mathbf{T} = \frac{\dfrac{d\mathbf{A}}{du}}{\left|\dfrac{d\mathbf{A}}{du}\right|}$$

Not covered but follows the result on page 70, when 3 dimensions are considered.

9. *Integration of vectors*

$$\int_a^b \mathbf{A}\,du = \mathbf{i}\int_a^b a_x\,du + \mathbf{j}\int_a^b a_y\,du + \mathbf{k}\int_a^b a_z\,du$$

10. *Grad* (gradient of a scalar function ϕ)

$$\text{grad }\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\text{'del'} = \text{operator } \nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

(a) *Directional derivative* $\dfrac{d\phi}{ds} = \hat{\mathbf{a}}.\text{grad }\phi = \hat{\mathbf{a}}.\nabla\phi$ where $\hat{\mathbf{a}}$ is a unit vector in a stated direction. Grad ϕ gives the direction for maximum rate of change of ϕ.

(b) *Unit normal vector* $\mathbf{N}$ to surface $\phi(x, y, z) = \text{constant}$.

$$\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|}$$

11. *Div* (divergence of a vector function $\mathbf{A}$)

$$\text{div }\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

If $\nabla.\mathbf{A} = 0$ for all points, $\mathbf{A}$ is a *solenoid* vector.

12. *Curl* (curl of a vector function $\mathbf{A}$)

$$\text{curl }\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

13. *Operators*

grad (∇) acts on a *scalar* and gives a *vector*
div ($\nabla.$) acts on a *vector* and gives a *scalar*
curl ($\nabla\times$) acts on a *vector* and gives a *vector.*

14. *Multiple operations*

(a) curl grad $\phi = \nabla\times(\nabla\phi) = 0$

(b) div curl $\mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = 0$

(c) div grad $\phi = \nabla.(\nabla\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}.$