HANDOUT 5 [without gaps]

- VECTOR CALCULUS (continued)

curl curl → Laplacian
Divergence Theorem
- interpretation
- electrostatics
- magnetism
- hydrodynamics
- proof
- worked example

Summary of integrals

HANDOUT 6 [without gaps]

- VECTOR CALCULUS (concluded)

Stoke's Theorem
- proof
- applications

Conservative Fields - Revisited
- the five equivalent conditions
- examples of conservative fields

Examples of solenoidal fields (zero divergence everywhere)

Alternative space coordinate systems (reference material)

Siméon Denis Poisson

Born: 21 June 1781 in Pithiviers, France
Died: 25 April 1840 in Sceaux (near Paris), France

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

A quotation by Poisson

Life is good for only two things, discovering mathematics and teaching mathematics.
Mathematics Magazine, v. 64, no. 1, Feb. 1991.

Libri said of him:

His only passion has been science: he lived and is dead for it.

Fellow of the Royal Society	Elected 1818
Royal Society Copley Medal	Awarded 1832
Lunar features	Crater Poisson
Paris street names	Rue Denis Poisson (17th Arrondissement)
Commemorated on the Eiffel Tower	

Originally forced to study medicine, Siméon Poisson began to study mathematics in 1798 at the Ecole Polytechnique. His teachers Laplace and Lagrange were to become friends for life. A memoir on finite differences, written when Poisson was 18, attracted the attention of Legendre.

Poisson taught at Ecole Polytechnique from 1802 until 1808 when he became an astronomer at Bureau des Longitudes. In 1809 he was appointed to the chair of pure mathematics in the newly opened Faculté des Sciences.

His most important works were a series of papers on definite integrals and his advances in Fourier series. This work was the foundation of later work in this area by Dirichlet and Riemann.

In *Recherchés sur la probabilité des jugements...*, an important work on probability published in 1837, the Poisson distribution first appeared. The Poisson distribution describes the probability that a random event will occur in a time or space interval under the conditions that the probability of the event occurring is very small, but the number of trials is very large so that the event actually occurs a few times.

He published between 300 and 400 mathematical works including applications to electricity and magnetism, and astronomy. His *Traité de mécanique* published in 1811 and again in 1833 was the standard work on mechanics for many years.

His name is attached to a wide area of ideas, for example:- Poisson's integral, Poisson's equation in potential theory, Poisson brackets in differential equations, Poisson's ratio in elasticity, and Poisson's constant in electricity.

"curl curl → Laplacian"

A further important identity, which is used for example to derive wave equations in electromagnetism, is...

$$\boxed{\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{V}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{V}) - \nabla^2\underset{\sim}{V}},$$

or in words ...

$$\boxed{\text{curl curl } \underset{\sim}{V} = \text{grad div } \underset{\sim}{V} - \text{Laplacian } \underset{\sim}{V}}$$

Then, for example, if $\text{div } \underset{\sim}{V} = 0$ then this reduces to

$$\boxed{\text{curl curl } \underset{\sim}{V} = -\text{Laplacian } \underset{\sim}{V}}.$$

PROOF

If $\mathbf{V} = (V_1, V_2, V_3)$ then

$$\operatorname{curl} \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

$$\therefore \operatorname{curl}\operatorname{curl} \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right]\mathbf{i}$$

$$- \left[\frac{\partial}{\partial x}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\right]\mathbf{k}$$

i.e. $\operatorname{curl}\operatorname{curl} \mathbf{V} =$

$$\left[\frac{\partial^2 V_2}{\partial y \partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_3}{\partial z \partial x}\right]\mathbf{i}$$

$$- \left[\frac{\partial^2 V_2}{\partial x^2} - \frac{\partial^2 V_1}{\partial x \partial y} - \frac{\partial^2 V_3}{\partial z \partial y} + \frac{\partial^2 V_2}{\partial z^2}\right]\mathbf{j}$$

$$+ \left[\frac{\partial^2 V_1}{\partial x \partial z} - \frac{\partial^2 V_3}{\partial x^2} - \frac{\partial^2 V_3}{\partial y^2} + \frac{\partial^2 V_2}{\partial y \partial z}\right]\mathbf{k}$$

$$= -\left[\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right]\mathbf{i} + \frac{\partial}{\partial x}\left[\frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right]\mathbf{i}$$

$$- \left[\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial z^2}\right]\mathbf{j} + \frac{\partial}{\partial y}\left[\frac{\partial V_1}{\partial x} + \frac{\partial V_3}{\partial z}\right]\mathbf{j}$$

$$- \left[\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial^2 V_3}{\partial y^2}\right]\mathbf{k} + \frac{\partial}{\partial z}\left[\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}\right]\mathbf{k}$$

$$= -\left[\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right]\mathbf{i} + \frac{\partial}{\partial x}\left[\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right]\mathbf{i}$$

$$- \left[\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_2}{\partial z^2}\right]\mathbf{j} + \frac{\partial}{\partial y}\left[\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right]\mathbf{j}$$

$$- \left[\frac{\partial^2 V_3}{\partial x^2} + \frac{\partial^2 V_3}{\partial y^2} + \frac{\partial^2 V_3}{\partial z^2}\right]\mathbf{k} + \frac{\partial}{\partial z}\left[\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right]\mathbf{k}$$

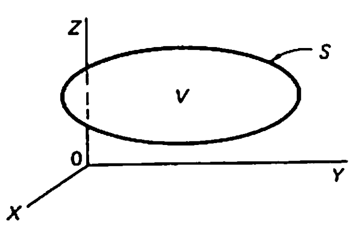
i.e. $\operatorname{curl}\operatorname{curl} \mathbf{V} =$

$$-\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$+ \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right)$$

$$\therefore \nabla \times (\nabla \times \mathbf{V}) = -\nabla^2 \mathbf{V} + \nabla(\nabla \cdot \mathbf{V})$$

Divergence Theorem (Gauss' theorem)



For a closed surface S, enclosing a region V in a vector field $\mathbf{F}$,

$$\int_V \operatorname{div} \mathbf{F}\, dV = \oint_S \mathbf{F} . d\mathbf{S}$$

The Divergence Theorem

$$\int_V \operatorname{div} \mathbf{F}\, dV = \oint_S \mathbf{F} . d\mathbf{S}$$

relates

- the volume integral (triple integral) on the left hand side to the surface integral (double integral) on the right hand side, and
- the divergence of vector field $\mathbf{F}$ within volume V to the total flux of $\mathbf{F}$ across the closed surface S around V.

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Consider, for example, electrostatic charges contained within volume V.

The total charge within V is

$$\int_V \rho \, dV$$

where ρ is the (VOLUME) CHARGE DENSITY,

i.e. the charge per unit volume.

So, in the expression $\int_V \operatorname{div} \underset{\sim}{F} \, dV$,

$\operatorname{div} \underset{\sim}{F}$ can be seen to be representing some kind of <u>volume density</u>.

But, the volume density of what ?

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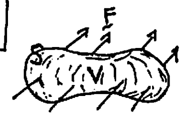
The answer to this question is given by the right hand side of the divergence theorem:

$$\int_V \operatorname{div} \underset{\sim}{F} \, dV = \oint_S \underset{\sim}{F} . d\underset{\sim}{S}$$

The volume integral of this density equates to the

$$\boxed{\text{NET FLUX OF } \underset{\sim}{F} \text{ ACROSS THE ENCLOSING SURFACE.}}$$



So, why shouldn't the flux of $\hat{F}$ into the volume equal the flux of $\hat{F}$ out of the volume ?

→ If there are effectively "sources" or "sinks" of flux within the volume V

i.e. $\boxed{\operatorname{div} \underset{\sim}{F} \equiv \text{volume density of sources and sinks of flux}}$

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In other words,

if $\int_V \operatorname{div} \underset{\sim}{F} \, dV$ is..	then within volume V there is/are ...
POSITIVE	SOURCES "CREATING" FLUX OF $\underset{\sim}{F}$
NEGATIVE	SINKS "EATING" FLUX OF $\underset{\sim}{F}$
ZERO	SOURCE/SINK – FREE ZONE OR THE SOURCES AND SINKS CANCEL OUT

Let's look at a physical example ...

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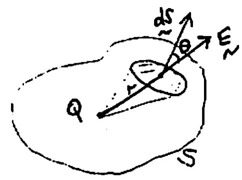
Electrostatics

Consider a point charge Q that is enclosed by <u>any</u> surface S.



Define the origin at Q, so that the electric field is

$$\underset{\sim}{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r}$$

where r is the distance to surface element dS

$\hat{r}$ is the radial unit vector pointing away from Q.

The flux of $\hat{E}$ over dS is

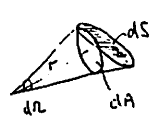
$$\hat{E} . d\underset{\sim}{S} = |E||d\hat{S}| \cos\theta$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2} dS \cos\theta .$$

Now recall that for an element of solid angle $d\Omega$ we have

$$d\Omega = \frac{dA}{r^2}$$

$$= \frac{dS \cos\theta}{r^2}$$



$\therefore$ Flux of $\underset{\sim}{E}$ over dS is

$$\underset{\sim}{E}\cdot d\underset{\sim}{S} = \frac{Q}{4\pi\varepsilon_0}\,\frac{dS\cos\theta}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\,d\Omega$$

$\therefore$ Flux of $\underset{\sim}{E}$ over closed surface S is

$$\oint_S \underset{\sim}{E}\cdot d\underset{\sim}{S} = \oint \frac{Q}{4\pi\varepsilon_0}\,d\Omega$$

$$= \frac{Q}{4\pi\varepsilon_0}\oint d\Omega$$

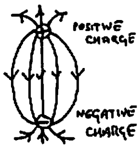
$$= \frac{Q}{4\pi\varepsilon_0}\cdot 4\pi$$

(recall that the solid angle subtended by a sphere is 4π steradians)

i.e. $$\boxed{\oint_S \underset{\sim}{E}\cdot d\underset{\sim}{S} = \frac{Q}{\varepsilon_0}}$$ Gauss's flux law in integral form

i.e. the flux of $\underset{\sim}{E}$ over S is proportional to the (net) charge Q enclosed.

i.e. positive (negative) charges are the sources (sinks) of flux of the $\underset{\sim}{E}$ field.



Also, we can express the total charge Q enclosed by S in terms of a volume charge density ρ, whereby

$$\frac{Q}{\varepsilon_0} = \oint_S \underset{\sim}{E}\cdot d\underset{\sim}{S}$$

becomes

$$\int_V \frac{\rho}{\varepsilon_0}\,dV = \oint_S \underset{\sim}{E}\cdot d\underset{\sim}{S}$$ (V = volume enclosed by surface S)

but the divergence theorem gives

$$\int_V \operatorname{div}\underset{\sim}{E}\,dV = \oint_S \underset{\sim}{E}\cdot d\underset{\sim}{S}.$$

$\therefore$ $$\boxed{\operatorname{div}\underset{\sim}{E} = \frac{\rho}{\varepsilon_0}.}$$ Gauss's flux law in differential form (i.e. applying at a point and not expressed in terms of an extended region of space).

i.e. $\operatorname{div}\underset{\sim}{E} \equiv$ volume density of sources and sinks of $\underset{\sim}{E}$-flux.

Johann Carl Friedrich Gauss

Born: 30 April 1777 in Brunswick, Duchy of Brunswick (now Germany)
Died: 23 Feb 1855 in Göttingen, Hanover (now Germany)

At the age of seven, Carl Friedrich Gauss started elementary school, and his potential was noticed almost immediately. His teacher, Büttner, and his assistant, Martin Bartels, were amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101

In 1788 Gauss began his education at the Gymnasium with the help of Büttner and Bartels, where he learnt High German and Latin. After receiving a stipend from the Duke of Brunswick-Wolfenbüttel, Gauss entered Brunswick Collegium Carolinum in 1792. At the academy Gauss independently discovered Bode's law, the binomial theorem and the arithmetic- geometric mean, as well as the law of quadratic reciprocity and the prime number theorem.

In 1795 Gauss left Brunswick to study at Göttingen University. Gauss's teacher there was Kaestner, whom Gauss often ridiculed. His only known friend amongst the students was Farkas Bolyai. They met in 1799 and corresponded with each other for many years.

Gauss returned to Brunswick where he received a degree in 1799. After the Duke of Brunswick had agreed to continue Gauss's stipend, he requested that Gauss submit a doctoral dissertation to the University of Helmstedt. He already knew Pfaff, who was chosen to be his advisor. Gauss's dissertation was a discussion of the fundamental theorem of algebra.

With his stipend to support him, Gauss did not need to find a job so devoted himself to research. He published the book *Disquisitiones Arithmeticae* in the summer of 1801. There were seven sections, all but the last section, referred to above, being devoted to number theory.

Gauss's work never seemed to suffer from his personal tragedy. He published his second book, *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium*, in 1809, a major two volume treatise on the motion of celestial bodies. In the first volume he discussed differential equations, conic sections and elliptic orbits, while in the second volume, the main part of the work, he showed how to estimate and then to refine the estimation of a planet's orbit. Gauss's contributions to theoretical astronomy stopped after 1817, although he went on making observations until the age of 70.

Much of Gauss's time was spent on a new observatory, completed in 1816, but he still found the time to work on other subjects. His publications during this time include *Disquisitiones generales circa seriem infinitam*, a rigorous treatment of series and an introduction of the hypergeometric function, *Methodus nova integralium valores per approximationem inveniendi*, a practical essay on approximate integration, *Bestimmung der Genauigkeit der Beobachtungen*, a discussion of statistical estimators, and *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodus nova tractata*. The latter work was inspired by geodesic problems and was principally concerned with potential theory. In fact, Gauss found himself more and more interested in geodesy in the 1820's.

From the early 1800's Gauss had an interest in the question of the possible existence of a non-Euclidean geometry. He discussed this topic at length with Farkas Bolyai and in his correspondence with Gerling and Schumacher. In a book review in 1816 he discussed proofs which deduced the axiom of parallels from the other Euclidean axioms, suggesting that he believed in the existence of non-Euclidean geometry, although he was rather vague. Gauss confided in Schumacher, telling him that he believed his reputation would suffer if he admitted in public that he believed in the existence of such a geometry.

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In 1832, Gauss and Weber began investigating the theory of terrestrial magnetism after Alexander von Humboldt attempted to obtain Gauss's assistance in making a grid of magnetic observation points around the Earth. Gauss was excited by this prospect and by 1840 he had written three important papers on the subject: *Intensitas vis magneticae terrestris ad mensuram absolutam revocata* (1832), *Allgemeine Theorie des Erdmagnetismus* (1839) and *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungskräfte* (1840). These papers all dealt with the current theories on terrestrial magnetism, including Poisson's ideas, absolute measure for magnetic force and an empirical definition of terrestrial magnetism. Dirichlet's principle was mentioned without proof.

Gauss and Weber achieved much in their six years together. They discovered Kirchhoff's laws, as well as building a primitive telegraph device which could send messages over a distance of 5000 ft. However, this was just an enjoyable pastime for Gauss. He was more interested in the task of establishing a world-wide net of magnetic observation points. This occupation produced many concrete results. The *Magnetischer Verein* and its journal were founded, and the atlas of geomagnetism was published, while Gauss and Weber's own journal in which their results were published ran from 1836 to 1841.

Gauss spent the years from 1845 to 1851 updating the Göttingen University widow's fund. This work gave him practical experience in financial matters, and he went on to make his fortune through shrewd investments in bonds issued by private companies.

From 1850 onwards Gauss's work was again of nearly all of a practical nature although he did approve Riemann's doctoral thesis and heard his probationary lecture. His last known scientific exchange was with Gerling. He discussed a modified Foucault pendulum in 1854. He was also able to attend the opening of the new railway link between Hanover and Göttingen, but this proved to be his last outing. His health deteriorated slowly, and Gauss died in his sleep early in the morning of 23 February, 1855.

Application of the divergence theorem in magnetism... (148)

$\underset{\sim}{B}$ = magnetic induction

$\equiv$ magnetic flux density

- Here, we cannot isolate a magnetic pole inside a surface S.
  They always occur as north-south pairs

$\therefore$ No sources or sinks of magnetic flux at any point

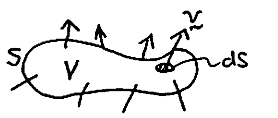
$\Rightarrow$ $\underset{\sim}{B}$ flux into a surface S = $\underset{\sim}{B}$ flux out of S

i.e. $\boxed{\oint_S \underset{\sim}{B}.d\underset{\sim}{S} = 0}$

But, divergence theorem states $\int_V \text{div}\,\underset{\sim}{B}\,dV = \oint_S \underset{\sim}{B}.d\underset{\sim}{S}$

$\therefore$ $\boxed{\text{div}\,\underset{\sim}{B} = 0}$ at every point
i.e. it is a "solenoidal" vector field.

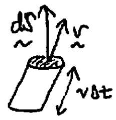
Application of the divergence theorem in hydrodynamics... (149)



$\underset{\sim}{v}$ = velocity of fluid particles

$\rho(x,y,z)$ = density of fluid $\left(\frac{\text{mass}}{\text{vol.}}\right)$

$\frac{\partial \rho}{\partial t}$ = rate of <u>increase</u> of density



- mass of fluid through dS in time Δt,
  dm = density x vol. = $\rho\, d\underset{\sim}{S}.(\underset{\sim}{v}\Delta t)$
- mass flowing out of whole surface S,

$$M = \Delta t \oint_S \rho \underset{\sim}{v}.d\underset{\sim}{S} = \underline{\text{decrease}} \text{ in mass in } V$$
$$= -\Delta t \int_V \frac{\partial \rho}{\partial t}\, dV$$

i.e. $\oint_S \rho \underset{\sim}{v}.d\underset{\sim}{S} = -\int_V \frac{\partial \rho}{\partial t}\, dV$

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We can now use the divergence theorem to relate the surface integral, on the left hand side, to a volume integral

i.e. $\int_V \text{div}(\rho \underset{\sim}{v})\, dV = \oint_S \rho\, \underset{\sim}{v}.d\underset{\sim}{S}$

giving $\int_V \text{div}(\rho \underset{\sim}{v})\, dV = -\int_V \frac{\partial \rho}{\partial t}\, dV$

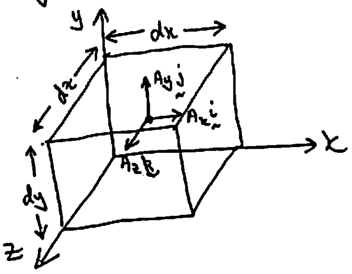
and $\boxed{\text{div}(\rho \underset{\sim}{v}) = -\frac{\partial \rho}{\partial t}}$ <u>Equation of continuity</u> <u>(conservation of matter)</u>

If there are no sources or sinks of fluid within V AND
if the fluid is also incompressible (ρ = constant and thus $\frac{\partial \rho}{\partial t} = 0$)

THEN $\text{div}\,\underset{\sim}{v} = 0$.

A PROOF OF THE DIVERGENCE THEOREM

Consider any vector field $\underset{\sim}{A}$ and let dx, dy and dz be the side lengths of an elemental volume $dV = dx\,dy\,dz$.

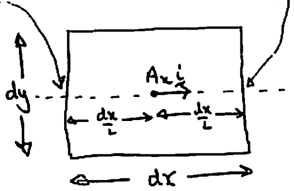


At the centre of this volume, let $\underset{\sim}{A}$ have the value (A_x, A_y, A_z).

- Move along x only and the x component of $\underset{\sim}{A}$ changes to

$$A_x - \frac{\partial A_x}{\partial x}\frac{dx}{2} \ \begin{pmatrix}\text{at the}\\ \text{left face}\end{pmatrix} \quad \text{and} \quad A_x + \frac{\partial A_x}{\partial x}\frac{dx}{2} \ \begin{pmatrix}\text{at the}\\ \text{right face}\end{pmatrix}$$

ie looking at a cross-section



Flux of $\underset{\sim}{A}$ = $\underset{\sim}{A}.d\underset{\sim}{S}$ (small surface element)

The x-component of $\underset{\sim}{A}$ is perpendicular to the left and right hand faces and these faces have area $dy\,dz$

$\therefore$ Flux over right face $= \left(A_x + \frac{\partial A_x}{\partial x}\frac{dx}{2}\right) dy\,dz$

" " left " $= -\left(A_x - \frac{\partial A_x}{\partial x}\frac{dx}{2}\right) dy\,dz$ (A_x in direction $\underset{\sim}{i}$)

- Contribution to flux over whole surface = left + right

$$= \left(\frac{\partial A_x}{\partial x} dx\right) dy\,dz$$

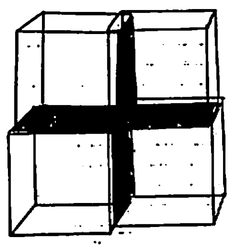
$$= \frac{\partial A_x}{\partial x}\, dV$$

Similarly, flux over top & bottom faces $= \frac{\partial A_y}{\partial y}\, dV$

and " " front & back " $= \frac{\partial A_z}{\partial z}\, dV$

- $\therefore$ total flux over surface $= \oint \underset{\sim}{A}.d\underset{\sim}{S} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dV$

$$= \operatorname{div}(\underset{\sim}{A})\, dV$$



- Fill a large volume by stacking up these little "sugar cubes"
- Flux over pairs of internal surfaces cancel out – leaving the external surface S of the large volume V

i.e.

$$\boxed{\oint_S \underset{\sim}{A}.d\underset{\sim}{S} = \int_V \operatorname{div} \underset{\sim}{A}\, dV}$$

Worked example verifying the divergence ..

Return to the example considered on page 94 in which the flux of the vector field $\underset{\sim}{F}(x,y,z) = x^2\underset{\sim}{i} + z\underset{\sim}{j} + y\underset{\sim}{k}$ over a surface S bounded by the planes $x=0$, $x=2$, $y=0$, $y=3$, and $z=0$, $z=4$, was calculated.

It was shown that $\oint_S \underset{\sim}{F}.d\underset{\sim}{S} = 6$.

To verify the divergence theorem for this case, we wish to show that

$$\int_V \operatorname{div} \underset{\sim}{F}\, dV = \oint_S \underset{\sim}{F}.d\underset{\sim}{S}$$

Considering the volume integral, we firstly want to calculate $\operatorname{div} \underset{\sim}{F}$.

$$\operatorname{div} \underset{\sim}{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \quad \text{where } \underset{\sim}{F} = (F_x, F_y, F_z)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y)$$

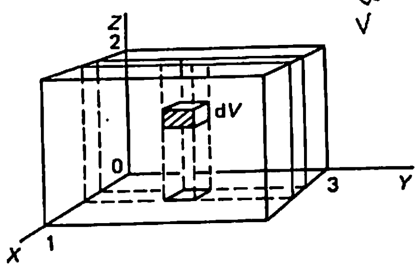
$$= 2x + 0 + 0$$

$$= 2x \,.$$

Then,

$$\int_V \operatorname{div} \underset{\sim}{F}\, dV = \int_V 2x\, dV$$

$$= \iiint_V 2x\, dx\,dy\,dz$$



i.e.

$$\int_V \operatorname{div} \underset{\sim}{F}\, dV = \int_{x=0}^{x=1}\int_{y=0}^{y=3}\int_{z=0}^{z=2} 2x\, dx\,dy\,dz$$

$$= \int_0^3\int_0^2 \left[\frac{2x^2}{2}\right]_0^1 dy\,dz$$

$$= \int_0^3\int_0^2 dy\,dz = \int_0^3 2.dz$$

$$= \left[2z\right]_0^3 = 6 \,.$$

INTEGRALS INVOLVING

SCALAR $\phi(x,y,z)$ AND VECTOR $\underset{\sim}{V}(x,y,z)$ FIELDS

LINE INTEGRALS: $\displaystyle\int_C \underset{\sim}{V}\cdot d\underset{\sim}{r}$, $\displaystyle\int_C \underset{\sim}{V}\, dr$, $\displaystyle\int_C \phi\, d\underset{\sim}{r}$

(see p130)

SURFACE INTEGRALS: $\displaystyle\int_S \underset{\sim}{V}\cdot d\underset{\sim}{S}$, $\displaystyle\int_S \underset{\sim}{V}\, dS$, $\displaystyle\int_S \phi\, d\underset{\sim}{S}$

VOLUME INTEGRALS: $\displaystyle\int_V \underset{\sim}{V}\, dV$, $\displaystyle\int_V \phi\, dV$

[where, for example, $\phi = \text{div}(\underset{\sim}{V})$]

$\longrightarrow$ Let's briefly look at the integrals of scalar fields with respect to $d\underset{\sim}{r}$ and $d\underset{\sim}{S}$.

For a scalar field $\phi(x,y,z)$:

I
$$\boxed{\begin{aligned}\int_C \phi\, d\underset{\sim}{r} &= \int_C \phi\,(\underset{\sim}{i}\,dx + \underset{\sim}{j}\,dy + \underset{\sim}{k}\,dz) \\ &= \underset{\sim}{i}\int_C \phi\,dx + \underset{\sim}{j}\int_C \phi\,dy + \underset{\sim}{k}\int_C \phi\,dz\end{aligned}}$$

- The answer is a vector.
- In each case, the particular curve C gives the form of ϕ (just like when we looked at line integrals for the work done).

Example with a parametric representation of curve C ....

Ex Evaluate $\displaystyle\int_C \phi\, d\underset{\sim}{r}$ from $A(0,0,0)$ to $B(3,2,1)$ when $\phi = xy^2z$ and curve C has parametric equations:

$x = 3u,\ y = 2u^2,\ z = u^3$

Ans Here, we write everything in terms of the parameter u, i.e. dx, dy, dz, ϕ, and the start and end points (A, B)

We have:

$$\int_C \phi\, d\underset{\sim}{r} = \underset{\sim}{i}\int_C \phi\,dx + \underset{\sim}{j}\int_C \phi\,dy + \underset{\sim}{k}\int_C \phi\,dz$$

where $\phi = xy^2z = (3u)(2u^2)^2u^3 = 12u^8$

and
$$\begin{cases} \dfrac{dx}{du} = 3, & \text{i.e. } dx = 3\,du \\ \dfrac{dy}{du} = 4u, & \text{i.e. } dy = 4u\,du \\ \dfrac{dz}{du} = 3u^2, & \text{i.e. } dz = 3u^2\,du \end{cases}$$

and $A(0,0,0)$ corresponds to $u=0$ (then $x=0, y=0, z=0$)
$B(3,2,1)$ corresponds to $u=1$ (then $x=3.1, y=2.1, z=1^3$)

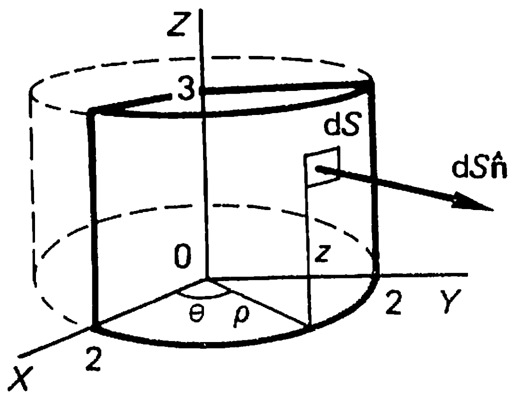
$$\therefore \int_C \phi\, d\underset{\sim}{r} = \underset{\sim}{i}\int_0^1 12u^8 . 3du + \underset{\sim}{j}\int_0^1 12u^8 . 4u\,du + \underset{\sim}{k}\int_0^1 12u^8 . 3u^2\,du$$

$$= \underset{\sim}{i}\int_0^1 36u^8\,du + \underset{\sim}{j}\int_0^1 48u^9\,du + \underset{\sim}{k}\int_0^1 36u^{10}\,du$$

$$= \underset{\sim}{i}\left[\frac{36u^9}{9}\right]_0^1 + \underset{\sim}{j}\left[\frac{48u^{10}}{10}\right]_0^1 + \underset{\sim}{k}\left[\frac{36}{11}u^{11}\right]_0^1 = 4\underset{\sim}{i} + \frac{24}{5}\underset{\sim}{j} + \frac{36}{11}\underset{\sim}{k}.$$

II $\displaystyle\boxed{\int_S \phi\, d\underset{\sim}{S} = \int_S \phi\, \hat{\underset{\sim}{n}}\, dS, \quad \text{where } \hat{\underset{\sim}{n}} = \frac{\nabla S}{|\nabla S|}, \text{ i.e. unit normal to the surface.}}$

Ex



Evaluate $\int_S \phi\, d\underset{\sim}{S}$ over curved surface S defined by $x^2+y^2=4$ between the planes $z=0$ and $z=3$ in the first octant, where $\phi = xyz$.

(154e)

Ans $\displaystyle\int_S \phi\, d\underset{\sim}{S} = \int_S \phi\, \hat{\underset{\sim}{n}}\, dS,$ where $\hat{\underset{\sim}{n}} = \frac{\nabla S}{|\nabla S|}$.

(154f)

- We need to work out an expression for the unit normal $\hat{\underset{\sim}{n}}$.

S is given by $x^2+y^2-4=0$ (between $z=0$ and $z=3$)

$$\therefore \nabla S = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)(x^2+y^2-4)$$

i.e. $\nabla S = 2x\underset{\sim}{i} + 2y\underset{\sim}{j} + 0\underset{\sim}{k}$

and $|\nabla S| = \sqrt{(2x)^2+(2y)^2} = 2\sqrt{x^2+y^2}$
$= 2\sqrt{4} = 4$ (using the equation of the surface S)

$$\therefore \hat{\underset{\sim}{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\underset{\sim}{i}+2y\underset{\sim}{j}}{4} = \frac{x\underset{\sim}{i}+y\underset{\sim}{j}}{2}.$$

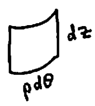
- We now need to specify the surface and a suitable form of dS in the integral:

$$\int_S \phi\, d\underset{\sim}{S} = \int_S \phi\,\hat{\underset{\sim}{n}}\, dS = \int_S (xyz)\left(\frac{x\underset{\sim}{i}+y\underset{\sim}{j}}{2}\right) dS$$
$$= \frac{1}{2}\int_S x^2yz\,\underset{\sim}{i} + xy^2z\,\underset{\sim}{j}\; dS$$

To evaluate the integral, consider the cylindrical coordinates (ρ, θ, z) shown in the original diagram....

(155)

For dS we have



i.e. $dS = \rho\, d\theta . dz$

i.e. $dS = 2\, d\theta\, dz$

(since $\rho = 2$)

For x, y we have

$x = 2\cos\theta$
$y = 2\sin\theta$

Putting all this together...

$$\int_S \phi\, d\underset{\sim}{S} = \frac{1}{2}\int_S x^2yz\,\underset{\sim}{i} + xy^2z\,\underset{\sim}{j}\; dS$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^3 \left[(2\cos\theta)^2(2\sin\theta)z\,\underset{\sim}{i} + (2\cos\theta)(2\sin\theta)^2 z\,\underset{\sim}{j}\right] 2\, d\theta\, dz$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^3 \left[8\cos^2\theta.\sin\theta.z\,\underset{\sim}{i} + 8\cos\theta.\sin^2\theta.z\,\underset{\sim}{j}\right] 2\, d\theta\, dz$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{8\cos^2\theta\sin\theta\,\underset{\sim}{i} + 8\cos\theta\sin^2\theta\,\underset{\sim}{j}\right\}.\left[\frac{z^2}{2}\right]_0^3 2\, d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} \left\{\cos^2\theta\sin\theta\,\underset{\sim}{i} + \cos\theta\sin^2\theta\,\underset{\sim}{j}\right\}.9\, d\theta$$
$$= 36\left[-\frac{\cos^3\theta}{3}\underset{\sim}{i} + \frac{\sin^3\theta}{3}\underset{\sim}{j}\right]_0^{\frac{\pi}{2}} = 12(\underset{\sim}{i}+\underset{\sim}{j}).$$

THEORETICAL PHYSICS I   DR GRAHAM McDONALD

HANDOUT 6

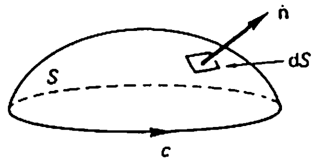
— VECTOR CALCULUS (concluded)

Stoke's Theorem
- proof
- applications

Conservative Fields – Revisited
- the five equivalent conditions
- examples of conservative fields

Examples of solenoidal fields (zero divergence everywhere)

Alternative space coordinate systems (reference material)



If **F** is a vector field existing over an open surface *S* and around its boundary closed curve c, then

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$$

where $\oint \underset{\sim}{F} \cdot \underset{\sim}{dr}$ is called THE CIRCULATION OF $\underset{\sim}{F}$ AROUND THE CURVE C.

Stoke's theorem expresses the relationship between $\underset{\sim}{\nabla} \times \underset{\sim}{F}$ and the 'circulation'.

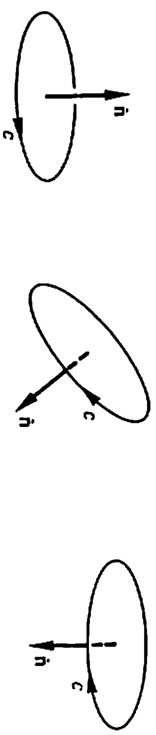
For example, recall the paddle wheel on page 116. One can quantify the rotational character of the field by working out the circulation around the paddle wheel.

An important convention ...

Direction of unit normal vectors to a surface S

When we were dealing with the divergence theorem, the normal vectors were drawn in a direction outward from the enclosed region.

With an open surface, as we now have, there is, in fact, no inward or outward direction. With any general surface, a normal vector can be drawn in either of two opposite directions. To avoid confusion, a convention must therefore be agreed upon and the established rule is as follows.

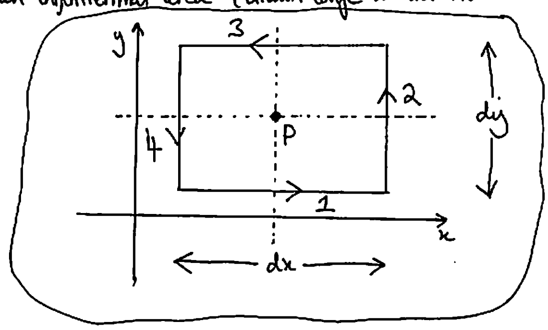


A unit normal n̂ is drawn perpendicular to the surface *S* at any point in the direction indicated by applying a right-handed screw sense to the direction of integration round the boundary c.

Proof of Stoke's Theorem

Consider a vector field $\underset{\sim}{V}$ and an area element $dS = dx\,dy$ which, for simplicity, lies in the xy plane.

Consider an infinitesimal area (drawn large to aid visualisation!)

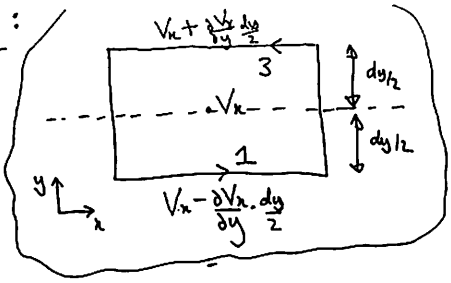


Convention: The circulation is defined around the curve such that the area enclosed is kept to the LEFT (as above). This corresponds to a CLOCKWISE navigation around a normal to this area coming OUT of the page.

Let $\underset{\sim}{V} = (V_x, V_y)$ at P in the centre of area element.

The circulation $\oint \underset{\sim}{V} \cdot \underset{\sim}{dr} = \int_{\text{along } 1} \underset{\sim}{V} \cdot \underset{\sim}{dr} + \int_{\text{along } 2} \underset{\sim}{V} \cdot \underset{\sim}{dr} + \int_{\text{along } 3} \underset{\sim}{V} \cdot \underset{\sim}{dr} + \int_{\text{along } 4} \underset{\sim}{V} \cdot \underset{\sim}{dr}$

Along sides 1 and 3 :



We have $V_x \mp \frac{\partial V_x}{\partial y}\frac{dy}{2}$ and $dr \to dx$.

$$\therefore \int_{\substack{\text{along}\\1}} \underline{V}\cdot d\underline{r} = \left(V_x - \frac{\partial V_x}{\partial y}\frac{dy}{2}\right)dx \,, \quad \int_{\substack{\text{along}\\3}} \underline{V}\cdot d\underline{r} = -\left(V_x + \frac{\partial V_x}{\partial y}\frac{dy}{2}\right)dx$$

(we are going in the negative x-direction this time)

Along sides 2 and 4 :

$$\int_{\substack{\text{along}\\2}} \underline{V}\cdot d\underline{r} = \left(V_y + \frac{\partial V_y}{\partial x}\frac{dx}{2}\right)dy \,, \quad \int_{\substack{\text{along}\\4}} \underline{V}\cdot d\underline{r} = -\left(V_y - \frac{\partial V_y}{\partial x}\frac{dx}{2}\right)dy$$

$$\therefore \boxed{\oint \underline{V}\cdot d\underline{r} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy}$$

z-component of $\nabla\times\underline{V}$ since $\nabla\times\underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$

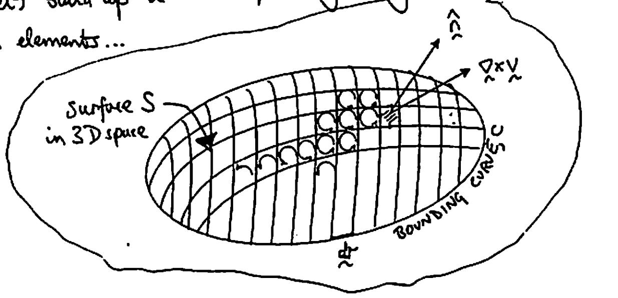
Denote the element area as $dS = dx\,dy$ and the unit normal to this area as $\hat{\underline{n}}$.

Then the z-component of curl $\underline{V}$ is $(\text{curl}\,\underline{V})\cdot\hat{\underline{n}}$

i.e. $$\boxed{\begin{aligned} &\oint \underline{V}\cdot d\underline{r} = (\nabla\times\underline{V})\cdot\hat{\underline{n}}\,dS \\ \text{or,}\quad &\oint \underline{V}\cdot d\underline{r} = (\nabla\times\underline{V})\cdot d\underline{S} \end{aligned}}$$

FOR AN ELEMENT OF AREA $d\underline{S}$

- The above is true also for an element in 3D space by allowing the normal to this element to point in any appropriate direction
- Let's build up a 3D surface by tiling it with lots of such elements...



- Adjacent line integrals cancel out, leaving just the line integral around the bounding curve.
- Adding up all the surface elements turns the right-hand side of $\oint \underline{V}\cdot d\underline{r}$ into a surface integral.

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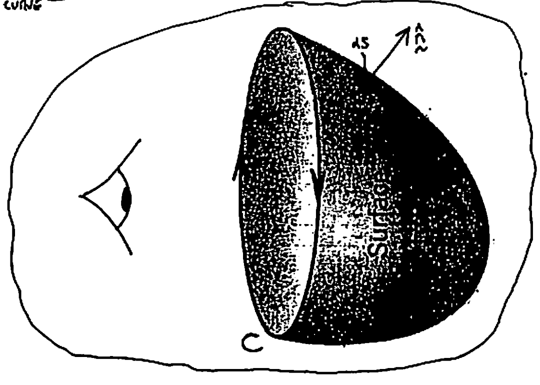
This gives **STOKE'S THEOREM**

$$\boxed{\oint_C \underline{V}\cdot d\underline{r} = \int_S (\nabla\times\underline{V})\cdot d\underline{S}}$$

LINE INTEGRAL AROUND THE BOUNDING CURVE

SURFACE INTEGRAL OF $\nabla\times\underline{V}$

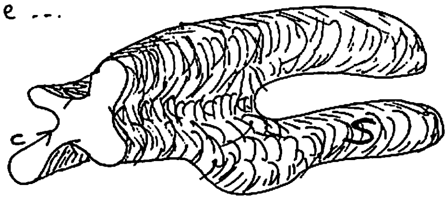


- Applies to an OPEN SURFACE S having a BOUNDING CURVE C.
- All the $\hat{\underline{n}}$'s point outwards and this gives the CLOCKWISE directional sense around C.
- It's like a fishing net where the surface is composed of all the elemental loops (the net itself) and the bounding curve is the rim of the net.

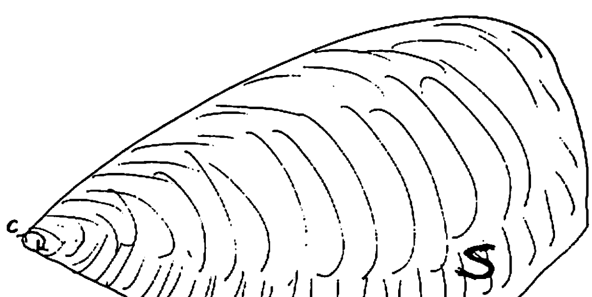
AND the bounding curve doesn't need to lie in a plane

AND we haven't specified what the ~~surface~~ looks like

We could have ...



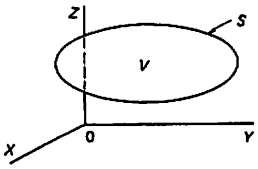
or ...



and we still have $$\oint_C \underline{V}\cdot d\underline{r} = \int_S (\nabla\times\underline{V})\cdot d\underline{S} \;!$$

A comparison ...

- *Divergence theorem* (Gauss' theorem)

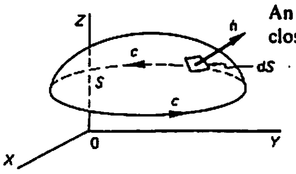


Closed surface S enclosing a region V in a vector field $\mathbf{F}$.

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

Quantifies the net flux through S and relates this to $\underset{\sim}{\nabla}\cdot\underset{\sim}{F}$ (the volume density of sources and sinks of flux).

- *Stokes theorem*



An open surface S bounded by a simple closed curve c, then

$$\int_S \operatorname{curl} \mathbf{F}\cdot dS = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Quantifies the circulation (twist/swirl/rotation/vorticity) of the field around curve C and relates this to $(\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot\hat{\underset{\sim}{n}}$ (the surface density of the circulation).

George Gabriel Stokes

Born: 13 Aug 1819 in Skreen, County Sligo, Ireland
Died: 1 Feb 1903 in Cambridge, Cambridgeshire, England

George Stokes' father, Gabriel Stokes, was the Protestant minister of the parish of Skreen in County Sligo. His mother was the daughter of a minister of the church so George Stokes's upbringing was a very religious one. He was the youngest of six children and every one of his three older brothers went on to become a priest.

In 1835, at the age of 16, George Stokes moved to England and entered Bristol College in Bristol. The two years which Stokes spent in Bristol at this College were important ones in preparing him for his studies at Cambridge.

In 1841 Stokes graduated as Senior Wrangler (the top First Class degree) in the Mathematical Tripos and he was the first Smith's prizeman. Pembroke College immediately gave him a Fellowship. He wrote [3]:-

> *After taking my degree I continued to reside in College and took private pupils. I thought I would try my hand at original research....*

It was William Hopkins who advised Stokes to undertake research into hydrodynamics and indeed this was the area in which Stokes began to work. In addition to Hopkins' advice, Stokes

was also inspired to enter this field by the recent work by George Green. Stokes published papers on the motion of incompressible fluids in 1842 and 1843, in particular *On the steady motion of incompressible fluids* in 1842. After completing the research Stokes discovered that Duhamel had already obtained similar results but, since Duhamel had been working on the distribution of heat in solids, Stokes decided that his results were obtained in a sufficiently different situation to justify him publishing.

Perhaps the most important event in the recognition of Stokes as a leading mathematician was his *Report on recent researches in hydrodynamics* presented to the British Association for the Advancement of Science in 1846. But a study of fluids was certainly not the only area in which he was making major contributions at this time. In 1845 Stokes had published an important work on the aberration of light, the first of a number of important works on this topic. He also used his work on the motion of pendulums in fluids to consider the variation of gravity at different points on the earth, publishing a work on geodesy of major importance *On the variation of gravity at the surface of the earth* in 1849.

Stokes's work on the motion of pendulums in fluids led to a fundamental paper on hydrodynamics in 1851 when he published his law of viscosity, describing the velocity of a small sphere through a viscous fluid. In addition to several important investigations concerning the wave theory of light, such as a paper on diffraction in 1849. This paper is discussed in detail in [10] in which the authors write:-

> *...the results of Stokes are related to the elastic theory of light, and supplement and expand a number of questions, previously studied for the most part in the works of A Cauchy. Stokes's methods for solving diffraction problems, differing considerably from the methods employed by Cauchy, form the basis of the further studies of the mathematical theory of the phenomenon of diffraction.*

Stokes named and explained the phenomenon of fluorescence in 1852. Stokes's interpretation of this phenomenon, which results from absorption of ultraviolet light and emission of blue light, is based on an elastic aether which vibrates as a consequence of the illuminated molecules.

Stokes's influence is summed up well by Parkinson in [1]:-

> *... Stokes was a very important formative influence on subsequent generations of Cambridge men, including Maxwell. With Green, who in turn had influenced him, Stokes followed the work of the French, especially Lagrange, Laplace, Fourier, Poisson and Cauchy. This is seen most clearly in his theoretical studies in optics and hydrodynamics; but it should also be noted that Stokes, even as an undergraduate, experimented incessantly. Yet his interests and investigations extended beyond physics, for his knowledge of chemistry and botany was extensive, and often his work in optics drew him into those fields.*

Applications of Stoke's theorem

Ex 1 Induction currents.

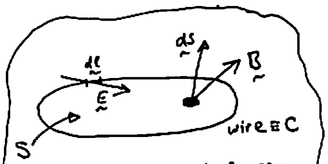
Consider a loop of wire . If we wave this through a magnetic field $\underset{\sim}{B}$ then currents are induced in the wire.

More precisely, the change in the magnetic flux Φ gives rise to an electric field $\underset{\sim}{E}$ that, in turn, gives rise to the current.

The circuital law,

$$\oint_C \underset{\sim}{E}\cdot d\underset{\sim}{l} = -\frac{\partial \Phi}{\partial t}$$

$$= -\frac{\partial}{\partial t}\int_S \underset{\sim}{B}\cdot d\underset{\sim}{S}$$



dS is area element of open surface bounded by curve C

$$\therefore \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{E})\cdot d\underset{\sim}{S} = -\frac{\partial}{\partial t}\int_S \underset{\sim}{B}\cdot d\underset{\sim}{S} \quad \text{(using Stoke's theorem)}$$

i.e.

$$\boxed{\underset{\sim}{\nabla}\times\underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}}$$

This is the differential form of one of Maxwell's equations.

Ex 2 There is an interplay between the magnetic field and moving charges such that moving charges give rise to twists (circulation) in the magnetic field.

Ampère's law in differential form : $\boxed{\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}}$

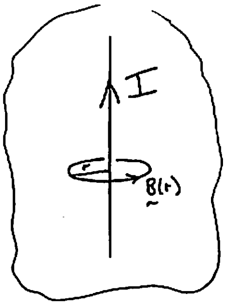
where $\underset{\sim}{J}$ = electric current density (A/m^2)

Stoke's theorem $\Rightarrow$ $\int_S (\nabla \times \underset{\sim}{B}) . d\underset{\sim}{S} = \oint_C \underset{\sim}{B} . d\underset{\sim}{r}$

while current, $I = \int_S \underset{\sim}{J} . d\underset{\sim}{S}$

i.e. $\oint_C \underset{\sim}{B} . d\underset{\sim}{r} = \mu_0 \int_S \underset{\sim}{J} . d\underset{\sim}{S} = \mu_0 I$

... Ampère's law in integral form

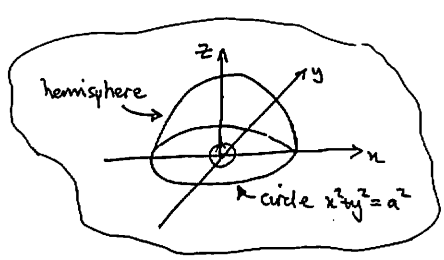


$B \times 2\pi r = \mu_0 I$

$\Rightarrow \boxed{B = \dfrac{\mu_0 I}{2\pi r}}$

Ex 3 Given a vector field $\underset{\sim}{V} = 4y\underset{\sim}{i} + x\underset{\sim}{j} + 2z\underset{\sim}{k}$, find $\int (\nabla \times \underset{\sim}{V}) . \hat{\underset{\sim}{n}}\, dS$ over the hemisphere $x^2+y^2+z^2 = a^2$, $z \geq 0$

i.e.



Ans Looks like this integral might be a bit difficult.

BUT, Stoke's theorem implies that the integral is the same over any surface bounded by the circle at $z=0$ i.e. the bounding curve given by $x^2+y^2=a^2$.

So, let's use the plane area inside the circle for this surface integral.

Recall that all the unit normal vectors $\hat{\underset{\sim}{n}}$ of the hemisphere point outwards. Imagine the hemisphere "deflating" onto the circle with the $\hat{\underset{\sim}{n}}$ vectors still pointing outwards.

Then, for the circle let's choose $\hat{\underset{\sim}{n}} = \underset{\sim}{k}$ (pointing upwards). This would then define which direction one would calculate the circulation around the circle.

On the xy plane, we have $z=0$ and $\underset{\sim}{V} = 4y\underset{\sim}{i} + x\underset{\sim}{j}$.

Then, $\nabla \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & x & 0 \end{vmatrix}$

$= \underset{\sim}{i}\left[\frac{\partial(0)}{\partial y} - \frac{\partial(x)}{\partial z}\right] - \underset{\sim}{j}\left[\frac{\partial(0)}{\partial x} - \frac{\partial(4y)}{\partial z}\right]$

$+ \underset{\sim}{k}\left[\frac{\partial(x)}{\partial x} - \frac{\partial(4y)}{\partial y}\right]$

i.e. $\nabla \times \underset{\sim}{V} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + (1-4)\underset{\sim}{k} = -3\underset{\sim}{k}$

We want to calculate $\int (\nabla \times \underset{\sim}{V}) . d\underset{\sim}{S}$

$= \int (\nabla \times \underset{\sim}{V}) . \hat{\underset{\sim}{n}}\, dS$

where $\hat{\underset{\sim}{n}} = \underset{\sim}{k}$ across the whole circle.

$(\nabla \times \underset{\sim}{V}) . \hat{\underset{\sim}{n}} = -3\underset{\sim}{k} . \hat{\underset{\sim}{n}} = -3\underset{\sim}{k} . \underset{\sim}{k} = -3$.

$\therefore \int_{\substack{\text{circular}\\\text{disk}}} (\nabla \times \underset{\sim}{V}) . \hat{\underset{\sim}{n}}\, dS = \int_{\substack{\text{circular}\\\text{disk}}} (-3)\, dS = -3 \int_{\substack{\text{circular}\\\text{disk}}} dS$

$= -3\pi a^2$.

Conservative Fields – Revisited

(172)

Earlier, we obtained three equivalent conditions for a vector field $\underset{\sim}{V}$ to be conservative.

These were ...

(I) • the existence of a scalar potential $\phi(x,y,t)$ such that $\int_A^B \underset{\sim}{V}.d\underset{\sim}{r} = \int_A^B d\phi = \phi_B - \phi_A$ [path independence]

(II) • for $\underset{\sim}{V}.d\underset{\sim}{r} = d\phi = V_x dx + V_y dy + V_z dz$ [$\underset{\sim}{V}.d\underset{\sim}{r} = d\phi$, an exact differential]

(III) • the reciprocity relations : $\frac{\partial V_x}{\partial y} = \frac{\partial V_y}{\partial x}$; $\frac{\partial V_x}{\partial z} = \frac{\partial V_z}{\partial x}$ and $\frac{\partial V_y}{\partial z} = \frac{\partial V_z}{\partial y}$

(173)

With the help of the vector algebra that has been developed, we can re-cast these three conditions in terms of five equivalent conditions.

Firstly,

note that if $\underset{\sim}{V} = (V_x, V_y, V_z)$ then

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \underset{\sim}{j}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \underset{\sim}{k}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right]$

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}(0) - \underset{\sim}{j}(0) + \underset{\sim}{k}(0)$, using the reciprocity relations (III).

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$

$\therefore$ $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ if $\underset{\sim}{V}$ is conservative.

(174)

Secondly, note condition (II) requiring $d\phi$ to be an exact differential implies that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz .$$

But this is just the dot product of $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial \phi}{\partial x} + \underset{\sim}{j}\frac{\partial \phi}{\partial y} + \underset{\sim}{k}\frac{\partial \phi}{\partial z}$ and $d\underset{\sim}{r} = \underset{\sim}{i}dx + \underset{\sim}{j}dy + \underset{\sim}{k}dz$.

$\therefore$ Condition (II) implies that

$$d\phi = \underset{\sim}{\nabla}\phi . d\underset{\sim}{r} .$$

while condition (I) requires that $\underset{\sim}{V}.d\underset{\sim}{r} = d\phi$

$\Rightarrow$ If we can write a vector field $\underset{\sim}{V}$ as $\underset{\sim}{V} = \underset{\sim}{\nabla}\phi$, where ϕ is a scalar field, then $\underset{\sim}{V}$ is a conservative field.

(175)

Let's also note that

when $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ (implying $\underset{\sim}{V}$ conservative)

Stoke's theorem gives

$$\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{V}) . d\underset{\sim}{s} = \oint_C \underset{\sim}{V} . d\underset{\sim}{r} = 0 .$$

So that $\oint_C \underset{\sim}{V}.d\underset{\sim}{r} = 0$ around any closed curve when $\underset{\sim}{V}$ conservative.

This is consistent with condition (I) which gives

$$\int_A^A \underset{\sim}{V}.d\underset{\sim}{r} = \int_A^A d\phi = \phi_A - \phi_A = 0 .$$

Let's re-state the five equivalent conditions for $\underset{\sim}{V}$ to be a conservative field ...

(i) $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$

(ii) $\oint \underset{\sim}{V} \cdot d\underset{\sim}{r} = 0$ around every simple closed curve

(iii) $\int_A^B \underset{\sim}{V} \cdot d\underset{\sim}{r}$ is path-independent

(iv) $\underset{\sim}{V} \cdot d\underset{\sim}{r} = d\phi$ = an exact differential

(v) $\underset{\sim}{V} = \underset{\sim}{\nabla}\phi$, ϕ a (single-valued) scalar field.

→ equivalent conditions for $\underset{\sim}{V}$ conservative ←

Notes (a) A "simple" closed curve does not cross itself and thus a single circulation direction of the curve and directions for the surface normals are possible.

(b) Any scalar or vector field is defined in a region of space. Thus, (i) to (v) apply to a region of space. This region needs to be "simply connected" ⇒ any simple closed curve C can be shrunk down to a point within the region.

e.g.

Ex It was shown earlier that any vector field obeying a radial inverse square law is conservative.

Let's now test a 'central field' of the form $\underset{\sim}{V} = r^n \hat{\underset{\sim}{r}}$.

Note that inverse square $(n=-2)$ is a special case.

Ans Recall that $\hat{\underset{\sim}{r}} = \dfrac{\underset{\sim}{r}}{|\underset{\sim}{r}|} = \dfrac{\underset{\sim}{r}}{r}$

where $\underset{\sim}{r} = (x, y, z) = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

$$\therefore \underset{\sim}{V} = r^n \hat{\underset{\sim}{r}} = \frac{r^n \underset{\sim}{r}}{r} = r^{n-1}(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})$$

i.e. $\underset{\sim}{V} = (V_x, V_y, V_z)$ where
$$V_x = r^{n-1}x$$
$$V_y = r^{n-1}y$$
$$V_z = r^{n-1}z.$$

$$\therefore \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\text{i.e. } \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^{n-1}x & r^{n-1}y & r^{n-1}z \end{vmatrix}$$

$$\text{i.e. } \underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}y)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}x)\right] + \underset{\sim}{k}\left[\frac{\partial}{\partial x}(r^{n-1}y) - \frac{\partial}{\partial y}(r^{n-1}x)\right]$$

Now note that implicit partial differentiation of $r^2 = x^2 + y^2 + z^2$ gives

$$\left\{\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \\ 2r\frac{\partial r}{\partial y} &= 2y \\ 2r\frac{\partial r}{\partial z} &= 2z \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} \\ \frac{\partial r}{\partial y} &= \frac{y}{r} \\ \frac{\partial r}{\partial z} &= \frac{z}{r} \end{aligned}\right\}$$

Then, for example, the $\underset{\sim}{i}$ component of $\underset{\sim}{\nabla} \times \underset{\sim}{V}$ is

$$\underset{\sim}{i}\left[\frac{\partial}{\partial y}(r^{n-1}z) - \frac{\partial}{\partial z}(r^{n-1}y)\right]$$

$$= \underset{\sim}{i}\left[(n-1)r^{n-2}\frac{\partial r}{\partial y}.z - (n-1)r^{n-2}\frac{\partial r}{\partial z}.y\right]$$

$$= \underset{\sim}{i}\left[(n-1)r^{n-2}.\frac{y}{r}.z - (n-1)r^{n-2}.\frac{z}{r}.y\right] = \underset{\sim}{0}.$$

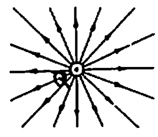
And a similar result is obtained for both the $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components of $\underset{\sim}{\nabla} \times \underset{\sim}{V}$.

So, since $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$,

the central field $\underset{\sim}{V} = r^n \hat{\underset{\sim}{r}}$ is conservative.

Examples of conservative fields

(a) $\underset{\sim}{E}$-field from a static point charge



(b) Any radial inverse square field

i.e. $\underset{\sim}{V}(\underset{\sim}{r}) = \eta \dfrac{\hat{\underset{\sim}{r}}}{r^2}$

(where η defines the particular constants of the physical system).

(c) Any central field of the form $\underset{\sim}{V}(\underset{\sim}{r}) = \eta r^n \hat{\underset{\sim}{r}}$.

But note that $\underset{\sim}{V}$ does not need to be radial to be conservative.

One can envisage, for example, non-dissipative mechanical force fields that are not radial.

However,

if $\underset{\sim}{V}$ is a conservative vector field

then $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ everywhere

$\Rightarrow$ the field is "IRROTATIONAL"

i.e. it has no vortices / circulation / swirl / etc.

Also note that if $\underset{\sim}{V}$ is conservative

then $\underset{\sim}{V} = \underset{\sim}{\nabla} \phi$ (for some scalar field).

$$\Rightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \phi)$$

$$= \nabla^2 \phi \Leftarrow \text{THIS DOES NOT NEED TO BE ZERO.}$$

$\therefore$ A conservative field can have sources and sinks of flux but no vortices.

Ex 2 Imagine walking into an electromagnetic theory revision class dealing with the $\underset{\sim}{E}$-field of static charges and a scalar (potential difference) field $V = -\phi$. What's on the board regarding the conservative nature of $\underset{\sim}{E}$?

Ans Something like this ...

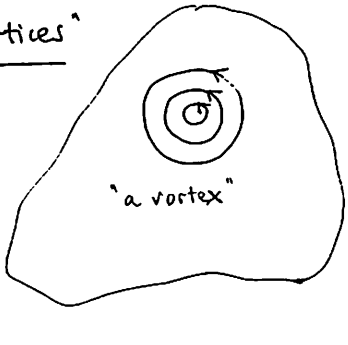
$$\underset{\sim}{E} = -\underset{\sim}{\nabla} V \Leftrightarrow \underset{\sim}{\nabla} \times \underset{\sim}{E} = \underset{\sim}{0} \Leftrightarrow \oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$$

$$\Leftrightarrow \int_A^B \underset{\sim}{E} \cdot d\underset{\sim}{l} \text{ path independent.}$$

Recall that there can be sources (+ve charges) and sinks (-ve charges) of the flux of $\underset{\sim}{E}$ over a closed surface (the net charge is inside the surface).

However, $\oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = 0$ means that there cannot be circulation of the field. This is often said as the field having "no vortices"

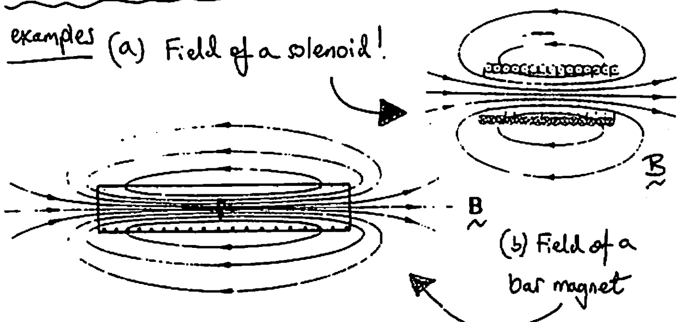
— This isn't the full story but it gives you an idea of the type of thing to expect.



Another special type of vector field:

ZERO DIVERGENCE $\Rightarrow$ "SOLENOIDAL"

examples (a) Field of a solenoid!

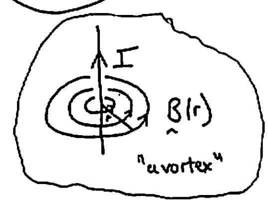


(b) Field of a bar magnet

(c) Field from a current-carrying wire:

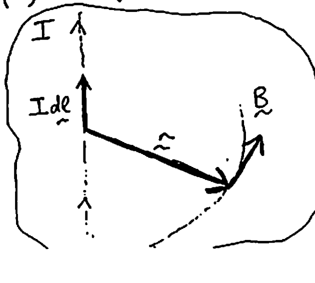
: $\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$

can be non-zero, can have vortices



(d) : $\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$, but no sources or sinks (no magnetic "monopoles")

(e) Field from a small current element $I d\underset{\sim}{l}$:



$$\underset{\sim}{B} \propto \frac{I d\underset{\sim}{l} \times \hat{\underset{\sim}{r}}}{r^2}$$

- Inverse-square but not radial
- not conservative
- it is solenoidal

[REFERENCE MATERIAL]

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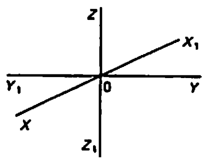
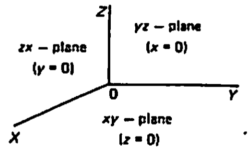
Space coordinate systems

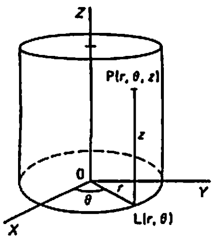
1. *Cartesian coordinates* (x, y, z)—referred to three coordinate axes OX, OY, OZ at right angles to each other. These are arranged in a *right-handed* manner, i.e. turning from OX to OY gives a right-handed screw action in the positive direction of OZ.





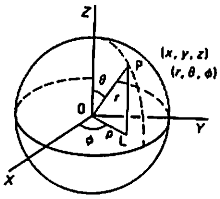
2. *Cylindrical coordinates* (r, θ, z) are useful where an axis of symmetry occurs.

$$x = r\cos\theta; \quad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta; \quad \theta = \arctan(y/x)$$
$$z = z; \quad z = z$$



Any point P is considered as having a position on a cylinder. If L is the projection of P on the xy-plane, then (r, θ) are the usual polar coordinates of L. The cylindrical coordinates of P then merely require the addition of the z-coordinate.

3. *Spherical coordinates* (r, θ, ϕ) are appropriate where a centre of symmetry occurs. The position of a point is considered as being a point on a sphere.



$$x = r\sin\theta\cos\phi \quad r = \sqrt{x^2 + y^2 + z^2}$$
$$y = r\sin\theta\sin\phi \quad \theta = \arccos(z/r)$$
$$z = r\cos\theta \quad \phi = \arctan(y/x)$$

(NB $\rho = r\sin\theta$)

r is the distance of P from the origin and is always taken as positive.

L is the projection of P on the xy-plane;
θ is the angle between OP and the positive OZ axis;
ϕ is the angle between OL and the OX axis.

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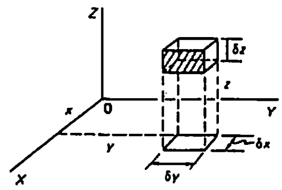
Element of volume in space in the three coordinate systems

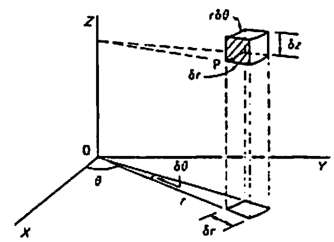
1. *Cartesian coordinates*

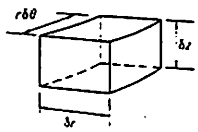


We have already used this many times.

$$\delta v = \delta x\,\delta y\,\delta z$$

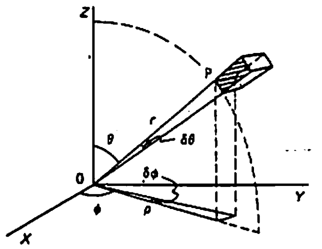
2. *Cylindrical coordinates*

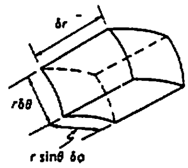




$$\delta v = r\delta\theta\,\delta r\,\delta z$$
$$\therefore\ \delta v = r\,\delta r\,\delta\theta\,\delta z$$

3. *Spherical coordinates*





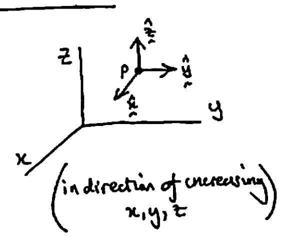
$$\delta v = \delta r\,r\delta\theta\,r\sin\theta\,\delta\phi$$
$$\therefore\ \delta v = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

UNIT BASIS VECTORS

(186)

Cartesian

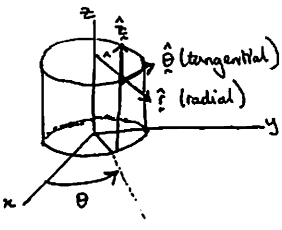
$\underset{\sim}{e}_x \equiv \underset{\sim}{i} \equiv \hat{\underset{\sim}{x}}$
$\underset{\sim}{e}_y \equiv \underset{\sim}{j} \equiv \hat{\underset{\sim}{y}}$
$\underset{\sim}{e}_z \equiv \underset{\sim}{k} \equiv \hat{\underset{\sim}{z}}$



(in direction of increasing x, y, z)

Cylindrical

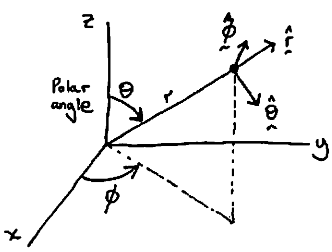
$\underset{\sim}{e}_r \equiv \hat{\underset{\sim}{r}}$
$\underset{\sim}{e}_\theta \equiv \hat{\underset{\sim}{\theta}}$
$\underset{\sim}{e}_z \equiv \hat{\underset{\sim}{z}}$



(in direction of increasing r, θ, z)

Spherical

$\underset{\sim}{e}_r \equiv \hat{\underset{\sim}{r}}$
$\underset{\sim}{e}_\theta \equiv \hat{\underset{\sim}{\theta}}$
$\underset{\sim}{e}_\phi \equiv \hat{\underset{\sim}{\phi}}$



(in direction of increasing r, θ, φ)

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SUMMARY – DIFFERENTIAL OPERATORS IN OTHER ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

Below we list the vector differential operators in cylindrical and spherical coordinates. For reference, the corresponding expressions in Cartesian coordinates are also given. f and $\mathbf{A}$ are arbitrary differentiable scalar and vector fields respectively.

- Cartesian coordinates (x, y, z)

$$\mathbf{e}_x \equiv \mathbf{i}, \quad \mathbf{e}_y \equiv \mathbf{j}, \quad \mathbf{e}_z \equiv \mathbf{k}$$
$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

$$(\mathrm{grad}\, f)_1 = \frac{\partial f}{\partial x}$$
$$(\mathrm{grad}\, f)_2 = \frac{\partial f}{\partial y}$$
$$(\mathrm{grad}\, f)_3 = \frac{\partial f}{\partial z}$$
$$\mathrm{div}\,\mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$
$$(\mathrm{curl}\,\mathbf{A})_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}$$
$$(\mathrm{curl}\,\mathbf{A})_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}$$
$$(\mathrm{curl}\,\mathbf{A})_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$$
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- cylindrical polar coordinates (r, θ, z)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \quad \mathbf{e}_z = \mathbf{k}$$
$$\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$$

$$(\mathrm{grad}\, f)_r = \frac{\partial f}{\partial r}$$
$$(\mathrm{grad}\, f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$(\text{grad } f)_z = \frac{\partial f}{\partial z}$$
$$\text{div } \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$
$$(\text{curl } \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$
$$(\text{curl } \mathbf{A})_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$
$$(\text{curl } \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

- spherical polar coordinates (r, θ, ϕ)

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta$$

$$\mathbf{e}_r = \cos\phi\sin\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k}$$
$$\mathbf{e}_\theta = \cos\phi\cos\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j} - \sin\theta\,\mathbf{k}$$
$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$
$$\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi$$

$$(\text{grad } f)_r = \frac{\partial f}{\partial r}$$
$$(\text{grad } f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$
$$(\text{grad } f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$
$$\text{div } \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$
$$(\text{curl } \mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \phi}$$
$$(\text{curl } \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$
$$(\text{curl } \mathbf{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}$$
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

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The power of vectors is that physical laws such as the divergence theorem and Stoke's theorem do not change in different coordinate systems ...

... but one must substitute the appropriate expressions for quantities such as dS and dV. Using the tables for components of vector operations, one finds ...

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CYLINDRICAL COORDINATES

$$\nabla V \equiv \text{grad } V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$
$$\nabla \cdot \mathbf{A} \equiv \text{div } \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$
$$\nabla \times \mathbf{A} \equiv \text{curl } \mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)$$
$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES

$$\nabla V \equiv \text{grad } V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$
$$\nabla \cdot \mathbf{A} \equiv \text{div } \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$
$$\nabla \times \mathbf{A} \equiv \text{curl } \mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)$$
$$+ \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\boldsymbol{\phi}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)$$
$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$