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TUTORIAL 1

This is the first of two tutorials covering vector calculus.

Specifically, there are 'introductory exercises and 'components' to expect in main exam questions.

The 'introductory section' includes:

scalar product, $\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$

vector product, $\underset{\sim}{a}\times\underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \underset{\sim}{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

where $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$

and

vector differentiation, $\frac{d}{dt}\underset{\sim}{a} = \frac{da_1}{dt}\underset{\sim}{i} + \frac{da_2}{dt}\underset{\sim}{j} + \frac{da_3}{dt}\underset{\sim}{k}$.

We then examine

$$\text{grad}\,\phi = \underset{\sim}{\nabla}\phi = \underset{\sim}{i}\frac{\partial\phi}{\partial x} + \underset{\sim}{j}\frac{\partial\phi}{\partial y} + \underset{\sim}{k}\frac{\partial\phi}{\partial z}$$

Note that $|\text{grad}\,\phi|$ gives the (magnitude) of the maximum rate of change of ϕ,

while $\frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$ gives the direction (i.e. the unit vector pointing in the direction) of this maximum rate of change.

In any other direction, the direction derivative gives the (magnitude of) the rate of change of ϕ,

i.e. $\underset{\sim}{\nabla}\phi\cdot\hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}}$ is the unit vector pointing in the specified direction.

We conclude by covering key aspects regarding conservative fields

If $\underset{\sim}{F}$ is a conservative field

then $\int_A^B \underset{\sim}{F}.d\underset{\sim}{r}$ is path independent,

where $\int (F_x \underset{\sim}{i} + F_y \underset{\sim}{j}).(dx\underset{\sim}{i} + dy\underset{\sim}{j})$

$$= \int F_x\,dx + \int F_y\,dy.$$

This can only give an indication that a field may be conservative.

A "sure-fire" test is

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{0} \iff \underset{\sim}{F} \text{ is conservative},$$

where $\underset{\sim}{\nabla}\times\underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$.

Introductory Exercises [Section A type]

1. Determine the value of a so that $\underset{\sim}{F}$ and $\underset{\sim}{r}$ are perpendicular when:

(a) $\underset{\sim}{F} = 2\underset{\sim}{i} + a\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{r} = 4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$

(b) $\underset{\sim}{F} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{r} = a\underset{\sim}{i} - 7\underset{\sim}{j} - 18\underset{\sim}{k}$

(c) $\underset{\sim}{F} = a\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{r} = 3\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}$

(d) $\underset{\sim}{F} = 13(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$, $\underset{\sim}{r} = a\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

2. Evaluate $\underset{\sim}{v} = \underset{\sim}{w} \times \underset{\sim}{r}$ when:

(a) $\underset{\sim}{w} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{r} = 2\underset{\sim}{i} - 6\underset{\sim}{j} - 3\underset{\sim}{k}$

(b) $\underset{\sim}{w} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{r} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

(c) $\underset{\sim}{w} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

3. If $\underset{\sim}{A} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{B} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, use the properties of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ to show that

$\underset{\sim}{A}.\underset{\sim}{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$

4. Prove that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = 2\,\mathbf{b}\times\mathbf{a}$.

5. If the position of a particle is given by the time-dependent position vector $\mathbf{r}(t)$, derive an expression for its velocity $\mathbf{v}(t)$ when:

(a) $\mathbf{r}(t) = e^{-t}\,\mathbf{i} + 2\cos 3t\,\mathbf{j} + 2\sin 3t\,\mathbf{k}$

(b) $\mathbf{r}(t) = (2t+3)\,\mathbf{i} + (t^2+3t)\,\mathbf{j} + (t^3+2t^2)\,\mathbf{k}$

(c) $\mathbf{r}(t) = 3t\,\mathbf{i} + 2t^2\,\mathbf{j} + (t^2+t)\,\mathbf{k}$

Components of Section B type questions

$\boxed{\text{GRAD}}$

6. Find grad ϕ for the following:

(a) $\phi = x+y+z$

(b) $\phi = x^2+y^2-z^2$

(c) $\phi = 2xz^4 - x^2y$

(d) $\phi = 2z - x^3y$

(e) $\phi = x^2yz^3$

7. Find the maximum rate of change of the following scalar fields and the direction in which this occurs:

(a) temperature $T(x,y,z) = T_0(x^2-y^2+xyz)$ at the point $(1,1,1)$,

(b) $\phi(x,y,z) = 4x^2y^2 - 3xz^2 - 2y^2z + 4$ at the point $(2,-1,2)$,

(c) $\phi(x,y,z) = 2xy^2 + y^2z + x^2z - 11$ at the point $(-2,1,3)$.

8. Find the rate of change of the following scalar fields in the given direction:

(a) pressure $p(x,y,z) = p_0\,xyz^2$ in the direction $2\mathbf{i}+\mathbf{j}-\mathbf{k}$ at the point $(2,3,1)$

(b) $\phi = xe^y + yz^2 + xyz$ at point $(2,0,3)$ in direction $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

(c) $\phi = (x+2y+z)^2 - (x-y-z)^2$ at point $(2,1,-1)$ in direction $\mathbf{A} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

9. Show that: (a) $\nabla \phi = -\dfrac{\hat{\mathbf{r}}}{r^2}$, when $\phi = \dfrac{1}{r}$

(b) $\nabla \phi = n r^{n-1} \hat{\mathbf{r}}$, when $\phi = r^n$

(c) $\nabla^2 \phi = 0$, when $\phi = \dfrac{1}{r}$

CONSERVATIVE FIELDS

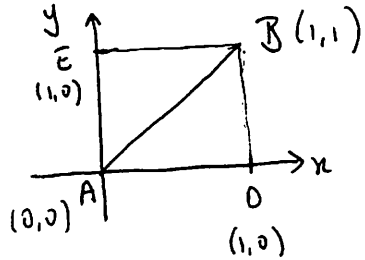
10.(a) A space-dependent force defines a vector field $\mathbf{F}(x,y) = (2x+y)\,\mathbf{i} + (x-3y)\,\mathbf{j}$.

Consider the action of this force along three different paths between points $A(0,0)$ and $B(1,1)$

(i) AD then DB

(ii) AE then EB

(iii) straight line AB



Calculate the work done $\displaystyle\int_A^B \mathbf{F} \cdot d\mathbf{r}$ for each case.

(b) Repeat the above for $\mathbf{F}(x,y) = (1+x^2y)\,\mathbf{i} + 2xy\,\mathbf{j}$.

(c) What can be said regarding whether each of the above fields is conservative?

11. Verify that the following vector fields are conservative

(a) $\mathbf{F} = (3x^2 - 3yz + 2xz)\,\mathbf{i} + (3y^2 - 3xz + z^2)\,\mathbf{j} + (3z^2 - 3xy + x^2 + 2yz)\,\mathbf{k}$

(b) $\mathbf{F} = (y^2 \cos x + z^3)\,\mathbf{i} + (2y \sin x + 4)\,\mathbf{j} + (3xz^2 + 2)\,\mathbf{k}$

12. Determine which of the following vector fields are conservative:

(a) $\mathbf{F} = (2xy + z)\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + (x + y^2)\,\mathbf{k}$

(b) $\mathbf{F} = (yz + 2y)\,\mathbf{i} + (xz + 2x)\,\mathbf{j} + (xy + 3)\,\mathbf{k}$

(c) $\mathbf{F} = (yz^2 + 3)\,\mathbf{i} + (xz^2 + 2)\,\mathbf{j} + (2xyz + 4)\,\mathbf{k}$

BS1

Brief solutions

(full solutions will follow)

1. $\underset{\sim}{F} \cdot \underset{\sim}{r} = 0$ (a) 3, (b) −2, (c) 8, (d) −4.

2. (a) $-15\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\sim}{k}$

   (b) $-5\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

   (c) $-13\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}$

3. Use the facts that $\underset{\sim}{i} \cdot \underset{\sim}{i} = 1$, $\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$, etc.

4. Recognise that $\underset{\sim}{A} \times \underset{\sim}{A} = \underset{\sim}{B} \times \underset{\sim}{B} = \underset{\sim}{0}$ and $\underset{\sim}{B} \times \underset{\sim}{A} = -\underset{\sim}{A} \times \underset{\sim}{B}$

5. $\underset{\sim}{v}(t) = \frac{d}{dt}\underset{\sim}{r}$ (a) $-e^{-t}\underset{\sim}{i} - 6\sin 3t\,\underset{\sim}{j} + 6\cos 3t\,\underset{\sim}{k}$,

   (b) $2\underset{\sim}{i} + (2t+3)\underset{\sim}{j} + (3t^2+4t)\underset{\sim}{k}$,

   (c) $3\underset{\sim}{i} + 4t\underset{\sim}{j} + (2t+1)\underset{\sim}{k}$.

6. (a) $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

   (b) $2(x\underset{\sim}{i} + y\underset{\sim}{j} - z\underset{\sim}{k})$

   (c) $2(z^4 - xy)\underset{\sim}{i} - x^2\underset{\sim}{j} + 8xz^3\underset{\sim}{k}$

   (d) $-3x^2y\,\underset{\sim}{i} - x^3\underset{\sim}{j} + 2\underset{\sim}{k}$

   (e) $2xyz^3\underset{\sim}{i} + x^2z^3\underset{\sim}{j} + 3x^2yz^2\underset{\sim}{k}$

BS2

7. (a) $\underset{\sim}{\nabla} T$ at $(1,1,1)$ is $T_0(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$

   $|\underset{\sim}{\nabla} T| = T_0\sqrt{11}$

   Direction given by unit vector $\hat{\underset{\sim}{n}} = \dfrac{\underset{\sim}{\nabla} T}{|\underset{\sim}{\nabla} T|} = \dfrac{3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}}{\sqrt{11}}$.

   (b) $\hat{\underset{\sim}{n}} = \dfrac{1}{5\sqrt{21}}(2\underset{\sim}{i} - 20\underset{\sim}{j} + 11\underset{\sim}{k})$

   (c) $\hat{\underset{\sim}{n}} = -\dfrac{1}{\sqrt{129}}(10\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k})$

8. We need the direction derivative $\underset{\sim}{\nabla} p \cdot \hat{\underset{\sim}{a}}$

   where $\hat{\underset{\sim}{a}}$ is the unit vector in the specified direction

   (a) $\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$, $\underset{\sim}{\nabla} p$ at $(2,3,1)$

   is $p_0(3\underset{\sim}{i} + 2\underset{\sim}{j} + 12\underset{\sim}{k})$

   $\therefore\ \hat{\underset{\sim}{a}} \cdot \underset{\sim}{\nabla} p = \dfrac{p_0}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 2\underset{\sim}{j} + 12\underset{\sim}{k})$

   $= -\dfrac{4p_0}{\sqrt{6}}$

8. (b) -8.285 , (c) -9.165

9. (a) $\underset{\sim}{\nabla}\left(\frac{1}{r}\right) = \underset{\sim}{\nabla}\left(\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right) = \dfrac{-x\underset{\sim}{i}-y\underset{\sim}{j}-z\underset{\sim}{k}}{(x^2+y^2+z^2)^{3/2}}$

$$= \frac{-\underset{\sim}{r}}{r^3} = \frac{-r\,\underset{\sim}{\hat{r}}}{r^3} = -\frac{\underset{\sim}{\hat{r}}}{r^2}$$

(b) similar steps to part (a)

(c) Can either do this in the two steps $\underset{\sim}{\nabla}\cdot\left(\underset{\sim}{\nabla}\left(\frac{1}{r}\right)\right)$

or use $\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right)$

$$\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right] = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right] = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2}{\partial y^2}\left[\quad\right] = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \quad \frac{\partial^2}{\partial z^2}\left[\quad\right] = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Then, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left[\quad\right] = 0$.

10. (a) (i) $\displaystyle\int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^D \underset{\sim}{F}\cdot d\underset{\sim}{r} + \int_D^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$

$y=0$, $d\underset{\sim}{r} = dx\,\underset{\sim}{i}$

$\underset{\sim}{F} = 2x\underset{\sim}{i} + x\underset{\sim}{j}$

$\underset{\sim}{F}\cdot d\underset{\sim}{r} = 2x\,dx$

$x=1$, $d\underset{\sim}{r} = dy\,\underset{\sim}{j}$

$\underset{\sim}{F} = (2+y)\underset{\sim}{i} + (1-3y)\underset{\sim}{j}$

$\underset{\sim}{F}\cdot d\underset{\sim}{r} = (1-3y)\,dy$

integral $= 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$

Similarly, (ii) and (iii) give $\frac{1}{2}$

(b) (i) 2 , (ii) $\frac{4}{3}$, (iii) $\frac{23}{12}$

(c) 1st case : field may be conservative but, since we would need to check all paths to ensure path independence of $\int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$, it is inconclusive.

2nd case : not path independent
$\Rightarrow$ non-conservative.

11. In each case, show that $\underset{\sim}{\nabla}\times\underset{\sim}{F} = \underset{\sim}{0}$

12. All conservative i.e. $\underset{\sim}{\nabla}\times\underset{\sim}{F} = \underset{\sim}{0}$.

TUTORIAL 1 – FULL WORKED SOLUTIONS

1. $\underset{\sim}{F}.\underset{\sim}{r} = |\underset{\sim}{F}||\underset{\sim}{r}|\cos\theta$, where θ is the angle between $\underset{\sim}{F}$ and r

$\underset{\sim}{F}$ perpendicular to $\underset{\sim}{r}$ when $\theta = 90°$ i.e. $\cos\theta = 0$

i.e. $\underset{\sim}{F}.\underset{\sim}{r} = 0$

i.e. $F_1 r_1 + F_2 r_2 + F_3 r_3 = 0$

(if $\underset{\sim}{F} = F_1 \underset{\sim}{i} + F_2 \underset{\sim}{j} + F_3 \underset{\sim}{k}$ and $\underset{\sim}{r} = r_1 \underset{\sim}{i} + r_2 \underset{\sim}{j} + r_3 \underset{\sim}{k}$)

(a) $\left.\begin{array}{l}\underset{\sim}{F} = 2\underset{\sim}{i} + a\underset{\sim}{j} + \underset{\sim}{k} \\ \underset{\sim}{r} = 4\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}\end{array}\right\}$ we need to find "a" that gives $\underset{\sim}{F}.\underset{\sim}{r} = 0$

i.e. $(2)(4) + (a)(-2) + (1)(-2) = 0$

i.e. $8 - 2a - 2 = 0$

i.e. $6 = 2a$

i.e. $a = 3$.

(b) $\left.\begin{array}{l}\underset{\sim}{F} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} \\ \underset{\sim}{r} = a\underset{\sim}{i} - 7\underset{\sim}{j} - 18\underset{\sim}{k}\end{array}\right\}$ $\underset{\sim}{F}.\underset{\sim}{r} = 2a + (2)(-7) + (-1)(-18) = 0$

i.e. $2a - 14 + 18 = 0$

i.e. $2a = -4$

i.e. $a = -2$.

1. (c) $\left.\begin{array}{l} \mathbf{F} = a\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \\ \mathbf{r} = 3\mathbf{i} - 6\mathbf{j} + 4\mathbf{k} \end{array}\right\}$ $\mathbf{F}\cdot\mathbf{r} = 3a - 12 - 12 = 0$

i.e. $3a = 24$, i.e. $a = 8$.

(d) $\left.\begin{array}{l} \mathbf{F} = 13(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\ \mathbf{r} = a\mathbf{i} + \mathbf{j} + 2\mathbf{k} \end{array}\right\}$ $\mathbf{F}\cdot\mathbf{r} = 13(\mathbf{i} + 2\mathbf{j} + \mathbf{k})\cdot(a\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 0$

i.e. $(\mathbf{i} + 2\mathbf{j} + \mathbf{k})\cdot(a\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 0$

i.e. $a + 2 + 2 = 0$, i.e. $a = -4$

2. If $\mathbf{w} = (w_1, w_2, w_3)$ and $\mathbf{r} = (r_1, r_2, r_3)$

then $\mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} w_2 & w_3 \\ r_2 & r_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} w_1 & w_3 \\ r_1 & r_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} w_1 & w_2 \\ r_1 & r_2 \end{vmatrix}$,

where, for each 2x2 determinant,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(a) $\left.\begin{array}{l} \mathbf{w} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k} \\ \mathbf{r} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k} \end{array}\right\}$ $\mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix}$

i.e. $\mathbf{w}\times\mathbf{r} = \mathbf{i}\begin{vmatrix} 3 & -1 \\ -6 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & 3 \\ 2 & -6 \end{vmatrix}$

$= \mathbf{i}\left[(3)(-3) - (-1)(-6)\right] - \mathbf{j}\left[(4)(-3) - (-1)(2)\right] + \mathbf{k}\left[(4)(-6) - (3)(2)\right]$.

$= \mathbf{i}\left[-9 - 6\right] - \mathbf{j}\left[-12 + 2\right] + \mathbf{k}\left[-24 - 6\right]$

$= -15\mathbf{i} + 10\mathbf{j} - 30\mathbf{k}$.

(b) $\left.\begin{array}{l} \mathbf{w} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \\ \mathbf{r} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k} \end{array}\right\}$ $\mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 2 & -1 \end{vmatrix}$

$= \mathbf{i}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$

$= \mathbf{i}\left[(1)(-1) - (2)(2)\right] - \mathbf{j}\left[(1)(-1) - (2)(-1)\right] + \mathbf{k}\left[(1)(2) - (1)(-1)\right]$

i.e. $\hat{w} \times \underset{\sim}{r} = \underset{\sim}{i}\left[-1-4\right] - \hat{j}\left[-1+2\right] + \hat{k}\left[2+1\right]$

$= -5\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}.$

(c) $\hat{w} \times \hat{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = \hat{i}\begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}$

$= \hat{i}\left[(-3)(4)-(1)(1)\right] - \hat{j}\left[(2)(4)-(1)(1)\right]$

$+ \hat{k}\left[(2)(1)-(-3)(1)\right]$

$= \underset{\sim}{i}\left[-12-1\right] - \underset{\sim}{j}\left[8-1\right] + \hat{k}\left[2+3\right]$

$= -13\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}.$

3. $\underset{\sim}{A}\cdot\underset{\sim}{B} = (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k})\cdot(b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k})$

$= a_1b_1\,\underset{\sim}{i}\cdot\underset{\sim}{i} + a_1b_2\,\underset{\sim}{i}\cdot\underset{\sim}{j} + a_1b_3\,\underset{\sim}{i}\cdot\underset{\sim}{k}$

$+ a_2b_1\,\underset{\sim}{j}\cdot\underset{\sim}{i} + a_2b_2\,\underset{\sim}{j}\cdot\underset{\sim}{j} + a_2b_3\,\underset{\sim}{j}\cdot\underset{\sim}{k}$

$+ a_3b_1\,\underset{\sim}{k}\cdot\underset{\sim}{i} + a_3b_2\,\underset{\sim}{k}\cdot\underset{\sim}{j} + a_3b_3\,\underset{\sim}{k}\cdot\underset{\sim}{k}$

$= a_1b_1 + a_2b_2 + a_3b_3$, since $\underset{\sim}{i}\cdot\underset{\sim}{i} = \underset{\sim}{j}\cdot\underset{\sim}{j} = \underset{\sim}{k}\cdot\underset{\sim}{k} = 1$

and $\underset{\sim}{i}\cdot\underset{\sim}{j} = \underset{\sim}{j}\cdot\underset{\sim}{i} = \underset{\sim}{i}\cdot\underset{\sim}{k} = \underset{\sim}{k}\cdot\underset{\sim}{i}$

$= \underset{\sim}{j}\cdot\underset{\sim}{k} = \underset{\sim}{k}\cdot\underset{\sim}{j} = 0.$

4. $(\underset{\sim}{a} + \underset{\sim}{b}) \times (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{a} \times (\underset{\sim}{a} - \underset{\sim}{b}) + \underset{\sim}{b} \times (\underset{\sim}{a} - \underset{\sim}{b})$

$= \underset{\sim}{a}\times\underset{\sim}{a} - \underset{\sim}{a}\times\underset{\sim}{b} + \underset{\sim}{b}\times\underset{\sim}{a} - \underset{\sim}{b}\times\underset{\sim}{b}$

$= -\underset{\sim}{a}\times\underset{\sim}{b} + \underset{\sim}{b}\times\underset{\sim}{a}$ (since any vector is parallel to itself i.e. $\theta = 0$, $\sin\theta = 0$ and $|\underset{\sim}{a}\times\underset{\sim}{a}| = |\underset{\sim}{a}||\underset{\sim}{a}|\sin\theta$, $|\underset{\sim}{b}\times\underset{\sim}{b}| = |\underset{\sim}{b}||\underset{\sim}{b}|\sin\theta$)

$= 2\,\underset{\sim}{b}\times\underset{\sim}{a}.$

(since $\underset{\sim}{b}\times\underset{\sim}{a} = -\underset{\sim}{a}\times\underset{\sim}{b}$)

5. $\underset{\sim}{v}(t) = \frac{d}{dt}\underset{\sim}{r}(t) = \frac{d}{dt}r_1\,\underset{\sim}{i} + \frac{d}{dt}r_2\,\underset{\sim}{j} + \frac{d}{dt}r_3\,\underset{\sim}{k}$

(if $\underset{\sim}{r} = (r_1, r_2, r_3)$)

(a) $\underset{\sim}{v}(t) = \frac{d}{dt}(e^{-t})\,\underset{\sim}{i} + \frac{d}{dt}(2\cos 3t)\,\underset{\sim}{j} + \frac{d}{dt}(2\sin 3t)\,\underset{\sim}{k}$

$= -e^{-t}\,\underset{\sim}{i} - 6\sin 3t\,\underset{\sim}{j} + 6\cos 3t\,\underset{\sim}{k}.$

(b) $\underset{\sim}{v}(t) = \frac{d}{dt}(2t+3)\,\underset{\sim}{i} + \frac{d}{dt}(t^2+3t)\,\underset{\sim}{j} + \frac{d}{dt}(t^3+2t^2)\,\underset{\sim}{k}$

$= 2\underset{\sim}{i} + (2t+3)\underset{\sim}{j} + (3t^2+4t)\underset{\sim}{k}.$

(c) $\underset{\sim}{v}(t) = \frac{d}{dt}(3t)\,\underset{\sim}{i} + \frac{d}{dt}(2t^2)\,\underset{\sim}{j} + \frac{d}{dt}(t^2+t)\,\underset{\sim}{k}$

$= 3\underset{\sim}{i} + 4t\,\underset{\sim}{j} + (2t+1)\,\underset{\sim}{k}.$

6. $$\operatorname{grad}\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

i.e. $$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

(a) $$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(x+y+z) + \mathbf{j}\frac{\partial}{\partial y}(x+y+z) + \mathbf{k}\frac{\partial}{\partial z}(x+y+z)$$

$$= \mathbf{i}(1) + \mathbf{j}(1) + \mathbf{k}(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

(b) $$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2-z^2) + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2-z^2) + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2-z^2)$$

$$= \mathbf{i}(2x) + \mathbf{j}(2y) + \mathbf{k}(-2z)$$

$$= 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} = 2(x\mathbf{i} + y\mathbf{j} - z\mathbf{k}).$$

(c) $$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(2xz^4-x^2y) + \mathbf{j}\frac{\partial}{\partial y}(2xz^4-x^2y) + \mathbf{k}\frac{\partial}{\partial z}(2xz^4-x^2y)$$

$$= \mathbf{i}(2z^4 - 2xy) + \mathbf{j}(0 - x^2) + \mathbf{k}(2x.4z^3 + 0)$$

$$= \mathbf{i}\,2(z^4 - xy) - x^2\mathbf{j} + 8xz^3\mathbf{k}.$$

(d) $$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(2z-x^3y) + \mathbf{j}\frac{\partial}{\partial y}(2z-x^3y) + \mathbf{k}\frac{\partial}{\partial z}(2z-x^3y)$$

$$= \mathbf{i}(0 - 3x^2y) + \mathbf{j}(0 - x^3) + \mathbf{k}(2 - 0)$$

$$= -3x^2y\,\mathbf{i} - x^3\mathbf{j} + 2\mathbf{k}.$$

(e) $$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(x^2yz^3) + \mathbf{j}\frac{\partial}{\partial y}(x^2yz^3) + \mathbf{k}\frac{\partial}{\partial z}(x^2yz^3)$$

$$= \mathbf{i}(2xyz^3) + \mathbf{j}(x^2z^3) + \mathbf{k}(3x^2yz^2)$$

$$= xz^2\left[2yz\,\mathbf{i} + xz\,\mathbf{j} + 3xy\,\mathbf{k}\right].$$

7. (a) ∇T gives the maximum rate of change of T as a vector.

The magnitude of the maximum rate of change is $|\nabla T|$
The direction of the maximum rate of change is the unit vector in the direction of ∇T. The unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$, for example.

The unit vector associated with ∇T is $$\frac{\nabla T}{|\nabla T|}$$

$$T(x,y,z) = T_0\,(x^2 - y^2 + xyz)$$

$$\underset{\sim}{\nabla} T = \underset{\sim}{i}\frac{\partial}{\partial x}T + \underset{\sim}{j}\frac{\partial}{\partial y}T + \underset{\sim}{k}\frac{\partial}{\partial z}T$$

i.e. $\underset{\sim}{\nabla} T = \underset{\sim}{i}\,T_0\,(2x + yz) + \underset{\sim}{j}\,T_0(-2y + xz) + \underset{\sim}{k}\,(xy)T_0$

At the point $(1,1,1)$

$$\underset{\sim}{\nabla} T = \underset{\sim}{i}\,T_0\,(2+1) + \underset{\sim}{j}\,T_0(-2+1) + \underset{\sim}{k}\,(1)T_0$$

(setting $x=y=z=1$)

$$= 3T_0\,\underset{\sim}{i} - T_0\,\underset{\sim}{j} + T_0\,\underset{\sim}{k}$$

$$= T_0\,(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$$

Max. rate of change of $\underset{\sim}{\nabla}T$ is $|\underset{\sim}{\nabla}T| = T_0\,(3^2 + (-1)^2 + (1)^2)^{\frac{1}{2}}$

$$= T_0\,(9+1+1)^{\frac{1}{2}}$$

$\underset{\sim}{\text{NB}}$ This is the same as

$$\sqrt{(3T_0)^2 + (-T_0)^2 + T_0^2}$$

$$= \sqrt{T_0^2\,(3^2 + (-1)^2 + 1^2)}$$

$$= T_0\sqrt{3^2 + (-1)^2 + 1^2}$$

i.e. $|\underset{\sim}{\nabla}T| = \sqrt{11}\,T_0$

direction (of unit vector) is

$$\frac{\underset{\sim}{\nabla}T}{|\underset{\sim}{\nabla}T|} = \frac{T_0\,(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})}{\sqrt{11}\,T_0}$$

$$= \frac{1}{\sqrt{11}}\,(3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}).$$

(b) $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\,(4.2\,xy^2 - 3z^2) + \underset{\sim}{j}\,(4x^2.2y - 2.2y\,z)$

$$+\underset{\sim}{k}\,(-3x.2z - 2y^2)$$

i.e. $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\,(8xy^2 - 3z^2) + \underset{\sim}{j}\,(8x^2y - 4yz) - \underset{\sim}{k}\,(6xz + 2y^2)$

At point $(2,-1,2)$ $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\,(8.2.(-1)^2 - 3.2^2) + \underset{\sim}{j}\,(8.2^2.(-1) - 4(-1).2)$

$$-\underset{\sim}{k}\,(6.2.2 + 2.(-1)^2)$$

$$= \underset{\sim}{i}\,(16-12) + \underset{\sim}{j}\,(-32+8) - \underset{\sim}{k}\,(24+2)$$

i.e. $\underset{\sim}{\nabla}\phi = 4\underset{\sim}{i} - 24\underset{\sim}{j} - 26\underset{\sim}{k}$.

Max. rate of change of $\underset{\sim}{\nabla}\phi$ is $|\underset{\sim}{\nabla}\phi| = |2(2\underset{\sim}{i} - 12\underset{\sim}{j} - 13\underset{\sim}{k})|$

$$= 2\sqrt{4 + 144 + 169} = 2\sqrt{317}$$

$$\approx 35.6$$

Direction (of unit vector) is

$$\frac{\underset{\sim}{\nabla}\phi}{|\underset{\sim}{\nabla}\phi|} = \frac{1}{\sqrt{317}}\,(2\underset{\sim}{i} - 12\underset{\sim}{j} - 13\underset{\sim}{k}).$$

→ mistake in the original brief solution distributed ←

(c) $\phi = 2xy^2 + y^2 z + x^2 z - 11$ @ point $(-2,1,3)$

$$\underset{\sim}{\nabla}\phi = \hat{i}\left(2y^2 + 2xz\right) + \hat{j}\left(4xy + 2yz\right) + \hat{k}\left(y^2 + x^2\right)$$

i.e. $\hat{\nabla}\phi = 2\hat{i}\left(y^2 + xz\right) + 2y\,\hat{j}\left(2x + z\right) + \hat{k}\left(x^2 + y^2\right)$

At point $(-2,1,3)$ $\hat{\nabla}\phi = 2\hat{i}\left(1-6\right) + 2\underset{\sim}{j}\left(-4+3\right) + \hat{k}\left(4+1\right)$

i.e. $\hat{\nabla}\phi = -10\hat{i} - 2\hat{j} + 5\underset{\sim}{k}$

Max. rate of change of $\underset{\sim}{\nabla}\phi$ is $|\nabla\phi| = \sqrt{10^2 + 2^2 + 5^2}$

i.e. $|\hat{\nabla}\phi| = \sqrt{100 + 4 + 25} = \sqrt{129}$

Direction (of unit vector) is

$$\frac{\hat{\nabla}\phi}{|\underset{\sim}{\nabla}\phi|} = \frac{1}{\sqrt{129}}\left(-10\hat{i} - 2\underset{\sim}{j} + 5\hat{k}\right) = \frac{-1}{\sqrt{129}}\left(10\hat{i} + 2\hat{j} - 5\underset{\sim}{k}\right)$$

8. (a) Rate of change of scalar field $p(x,y,z)$ in direction of a unit vector $\hat{a}$ is $\underset{\sim}{\nabla} p \cdot \hat{a}$

where $p(x,y,z) = p_0\, xyz^2$. $\hat{\nabla} p = p_0\left[\hat{i}\left(yz^2\right) + \hat{j}\left(xz^2\right) + \hat{k}\left(2xyz\right)\right]$

At point $(2,3,1)$ $\hat{\nabla} p = p_0\left[\hat{i}\left(3\right) + \underset{\sim}{j}\left(2\right) + \hat{k}\left(12\right)\right]$

ie $\hat{\nabla} p = p_0\left(3\hat{i} + 2\underset{\sim}{j} + 12\hat{k}\right)$.

The required unit vector is in the direction $2\hat{i} + \hat{j} - \hat{k}$

i.e. $\hat{a}$ with unit magnitude and with components in the ratio 2:1:1

$$\therefore \hat{a} = \frac{2\hat{i} + \hat{j} - \underset{\sim}{k}}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{6}}\left(2\hat{i} + \hat{j} - \hat{k}\right)$$

Projection of $\hat{\nabla} p$ in this direction at point $(2,3,1)$ is

$$\underset{\sim}{\nabla} p \cdot \hat{a} = p_0\left(3\hat{i} + 2\hat{j} + 12\hat{k}\right) \cdot \frac{1}{\sqrt{6}}\left(2\hat{i} + \hat{j} - \hat{k}\right)$$

$$= \frac{p_0}{\sqrt{6}}\left(6 + 2 - 12\right) = \frac{-4p_0}{\sqrt{6}}.$$

(b) $\phi = xe^y + yz^2 + xyz$; $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\left(e^y + yz\right) + \hat{j}\left(xe^y + z^2 + xz\right) + \underset{\sim}{k}\left(2yz + xy\right)$

i.e. $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\left(e^y + yz\right) + \underset{\sim}{j}\left(xe^y + z^2 + xz\right) + \hat{k}\, y\left(2z + x\right)$.

At point $(2,0,3)$ $\underset{\sim}{\nabla}\phi = \underset{\sim}{i}\left(e^0 + 0\right) + \hat{j}\left(2 + 9 + 6\right) + \hat{k}\left(0\right)$

i.e $\hat{\nabla}\phi = \hat{i} + 17\hat{j}$; $\hat{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\hat{a} = \frac{\hat{A}}{|\hat{A}|}$

i.e. $\hat{a} = \frac{1}{\sqrt{3^2 + 2^2 + 1^2}}\left(3\hat{i} - 2\underset{\sim}{j} + \hat{k}\right) = \frac{1}{\sqrt{14}}\left(3\hat{i} - 2\hat{j} + \hat{k}\right)$

$$\therefore \underset{\sim}{\nabla}\phi \cdot \hat{a} = \frac{1}{\sqrt{14}}\left(\hat{i} + 17\underset{\sim}{j}\right) \cdot \left(3\hat{i} - 2\hat{j} + \hat{k}\right) = \frac{1}{\sqrt{14}}\left(3 - 34 + 0\right) = -\frac{31}{\sqrt{14}}$$

$$\simeq -8.285.$$

(12)

(c) $\phi = (x+2y+z)^2 - (x-y-z)^2$ at point $(2,1,-1)$ in direction $\underset{\sim}{A} = \underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$$\nabla\phi = \underset{\sim}{i}\left[2(x+2y+z) - 2(x-y-z)\right] + \underset{\sim}{j}\left[4(x+2y+z) + 2(x-y-z)\right] + \underset{\sim}{k}\left[2(x+2y+z) + 2(x-y-z)\right]$$

$$= \underset{\sim}{i}\,(2x+4y+2z-2x+2y+2z) + \underset{\sim}{j}\,(4x+8y+4z+2x-2y-2z) + \underset{\sim}{k}\,(2x+4y+2z+2x-2y-2z)$$

$$= \underset{\sim}{i}\,(6y+4z) + \underset{\sim}{j}\,(6x+6y+2z) + \underset{\sim}{k}\,(4x+2y)$$

$$= 2\left[\underset{\sim}{i}\,(3y+2z) + \underset{\sim}{j}\,(3x+3y+z) + \underset{\sim}{k}\,(2x+y)\right].$$

At point $(2,1,-1)$ $\quad \nabla\phi = 2\left[\underset{\sim}{i}\,(3-2) + \underset{\sim}{j}\,(6+3-1) + \underset{\sim}{k}\,(4+1)\right]$

i.e. $\nabla\phi = 2\left[\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k}\right]$

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}}{\sqrt{1^2+4^2+2^2}} = \frac{\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}}{\sqrt{21}}$$

$$\therefore \nabla\phi \cdot \hat{\underset{\sim}{a}} = \frac{2}{\sqrt{21}}(\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k})\cdot(\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}) = \frac{2}{\sqrt{21}}(1-32+10)$$

$$= \frac{2}{\sqrt{21}}(-21) = \frac{-42}{\sqrt{21}} \approx -9.165$$

(13)

9. (a) $\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right)$; $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

where
$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -x(x^2+y^2+z^2)^{-3/2}$$
$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2y) = -y(x^2+y^2+z^2)^{-3/2}$$
$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2z) = -z(x^2+y^2+z^2)^{-3/2}$$

$$\therefore \nabla\left(\frac{1}{r}\right) = \underset{\sim}{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \underset{\sim}{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \underset{\sim}{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= -(x^2+y^2+z^2)^{-3/2}\left[\underset{\sim}{i}x + \underset{\sim}{j}y + \underset{\sim}{k}z\right]$$

$$= \frac{-(\underset{\sim}{i}x + \underset{\sim}{j}y + \underset{\sim}{k}z)}{(x^2+y^2+z^2)^{3/2}} = \frac{-\underset{\sim}{r}}{|\underset{\sim}{r}|^3} = \frac{-\underset{\sim}{r}}{r^3}$$

$$= \frac{-|\underset{\sim}{r}|\hat{\underset{\sim}{r}}}{r^3} = -\frac{r}{r^3}\hat{\underset{\sim}{r}} = -\frac{\hat{\underset{\sim}{r}}}{r^2}$$

NB Any vector can be written as the product of its modulus and a unit vector in the same direction e.g. $\underset{\sim}{A} = |\underset{\sim}{A}|\hat{\underset{\sim}{a}}$, $\underset{\sim}{r} = |\underset{\sim}{r}|\hat{\underset{\sim}{r}} = r\hat{\underset{\sim}{r}}$

(b) Show that $\underset{\sim}{\nabla}(r^n) = n r^{n-1}\hat{r}$

Ans $\underset{\sim}{\nabla} r^n = \underset{\sim}{\nabla}\left(\sqrt{x^2+y^2+z^2}\right)^n$, since $\underset{\sim}{r} = x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$.

$$= \underset{\sim}{\nabla}\,(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{n}{2}} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x) = nx(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

AND $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{n}{2}} = ny(x^2+y^2+z^2)^{\frac{n}{2}-1}$; $\frac{\partial}{\partial z}(x^2+y^2+z^2) = nz(x^2+y^2+z^2)^{\frac{n}{2}-1}$

$$\therefore \underset{\sim}{\nabla} r^n = n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k})$$

$$= n(r^2)^{\frac{n}{2}-1}\underset{\sim}{r} = n r^{\frac{2n}{2}-2}\underset{\sim}{r}$$

i.e. $\underset{\sim}{\nabla} r^n = n r^{n-2}\underset{\sim}{r} = n r^{n-2}|\underset{\sim}{r}|\hat{r}$

i.e. $\underset{\sim}{\nabla} r^n = n r^{n-2} r\hat{r} = n r^{n-1}\hat{r}$

(c) Show that $\nabla^2\phi = 0$ when $\phi = \frac{1}{r}$

Ans $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ i.e. the Laplacian

where $\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -y(x^2+y^2+z^2)^{-3/2}$; $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -z(x^2+y^2+z^2)^{-3/2}$

Then, $\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right)$

$$= \frac{\partial}{\partial x}\left(-x(x^2+y^2+z^2)^{-3/2}\right)$$

$$= -(x^2+y^2+z^2)^{-3/2} - \left(-\frac{3}{2}x(x^2+y^2+z^2)^{-5/2}(2x)\right)$$

[product rule]

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$= -\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

i.e. $\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$.

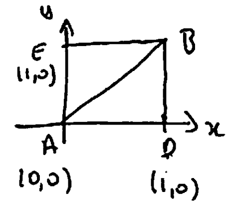
The symmetry in x, y, z implies that $\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$

and $\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$

Whereby, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right) = \frac{(2x^2-y^2-z^2)+(2y^2-x^2-z^2)+(2z^2-x^2-y^2)}{(x^2+y^2+z^2)^{5/2}}$

$$= \frac{0}{(x^2+y^2+z^2)^{5/2}} = 0$$

10. $\underset{\sim}{F} = (2x+y)\underset{\sim}{i} + (x-3y)\underset{\sim}{j}$



(i) Work done $= \int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$

AD then DB   i.e. $(0,0)$ to $(1,0)$ to $(1,1)$

$$\int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^D \underset{\sim}{F}\cdot d\underset{\sim}{r} + \int_D^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$$

Along AD : $\left\{\begin{array}{l} y=0 \\ dy=0 \end{array}\right\}$ and $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} = dx\,\underset{\sim}{i}$

$$\underset{\sim}{F} = 2x\,\underset{\sim}{i} + x\,\underset{\sim}{j}$$

$$\therefore \int_A^D \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^D 2x\,(\underset{\sim}{i}+\underset{\sim}{j})\cdot dx\,\underset{\sim}{i} = \int_A^D 2x\,dx$$

$$= \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1$$

Along DB : $\left\{\begin{array}{l} x=1 \\ dx=0 \end{array}\right\}$ and $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} = dy\,\underset{\sim}{j}$

$$\underset{\sim}{F} = (2+y)\,\underset{\sim}{i} + (1-3y)\,\underset{\sim}{j}$$

$$\therefore \int_D^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_D^B \left[(2+y)\,\underset{\sim}{i} + (1-3y)\,\underset{\sim}{j}\right]\cdot \underset{\sim}{j}\,dy$$

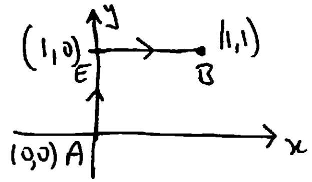
$$= \int_D^B (1-3y)\,dy = \int_0^1 (1-3y)\,dy$$

$$= \left[y - \tfrac{3}{2}y^2\right]_0^1 = 1 - \tfrac{3}{2} = -\tfrac{1}{2}$$

$$\therefore \int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^D \underset{\sim}{F}\cdot d\underset{\sim}{r} + \int_D^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$$

$$= 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

(ii) AE then EB



$$\int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^E \underset{\sim}{F}\cdot d\underset{\sim}{r} + \int_E^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$$

Along AE : $\left\{\begin{array}{l} x=0 \\ dx=0 \end{array}\right\}$ $d\underset{\sim}{r} = dy\,\underset{\sim}{j}$, $\underset{\sim}{F} = (2x+y)\,\underset{\sim}{i} + (x-3y)\,\underset{\sim}{j}$

i.e. $\underset{\sim}{F} = y\,\underset{\sim}{i} - 3y\,\underset{\sim}{j}$

i.e. $\underset{\sim}{F} = y\,(\underset{\sim}{i} - 3\underset{\sim}{j})$

$$\therefore \int_A^E \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^E y\,(\underset{\sim}{i}-3\underset{\sim}{j})\cdot dy\,\underset{\sim}{j} = \int_A^E (-3y)\,dy = \int_0^1 (-3y)\,dy$$

$$= \left[-\tfrac{3}{2}y^2\right]_0^1 = -\tfrac{3}{2}$$

Along EB : $\left\{\begin{array}{l} y=1 \\ dy=0 \end{array}\right\}$ $d\underset{\sim}{r} = dx\,\underset{\sim}{i}$, $\underset{\sim}{F} = (2x+1)\,\underset{\sim}{i} + (x-3)\,\underset{\sim}{j}$

$$\therefore \int_E^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_E^B \left[(2x+1)\,\underset{\sim}{i} + (x-3)\,\underset{\sim}{j}\right]\cdot dx\,\underset{\sim}{i}$$

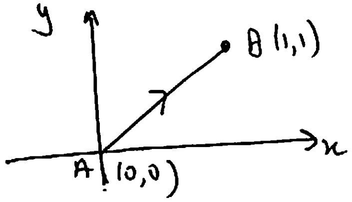
$$= \int_E^B (2x+1)\,dx = \int_0^1 (2x+1)\,dx = \left[x^2+x\right]_0^1$$

$$= 2$$

$$\therefore \int_A^B \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_A^E \underset{\sim}{F}\cdot d\underset{\sim}{r} + \int_E^B \underset{\sim}{F}\cdot d\underset{\sim}{r}$$

$$= -\tfrac{3}{2} + 2 = \tfrac{1}{2}$$

(iii) along straight line AB



Straight line connects (0,0), (1,1)
i.e. it is the $y = x$

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B \left[(2x+y)\,\mathbf{i} + (x-3y)\,\mathbf{j}\right]\cdot\left[dx\,\mathbf{i} + dy\,\mathbf{j}\right]$$

$$= \int_A^B (2x+y)\,dx + \int_A^B (x-3y)\,dy$$

$$= \int_A^B (2x+x)\,dx + \int_A^B (y-3y)\,dy \qquad \{\text{using } y = x\}$$

$$= \int_A^B 3x\,dx + \int_A^B (-2y)\,dy$$

$$= \int_0^1 3x\,dx + \int_0^1 (-2y)\,dy$$

$$= \left[\tfrac{3}{2}x^2\right]_0^1 + \left[-y^2\right]_0^1 = \tfrac{3}{2} - 1 = \tfrac{1}{2}.$$

In all three cases (i), (ii), (iii) the work done $= \frac{1}{2}$.

(b) $\mathbf{F}(x,y) = (1+x^2y)\,\mathbf{i} + (2xy)\,\mathbf{j}$

(i) Along AD then DB   Along AD : $\left\{\begin{array}{l} y=0 \\ dy=0 \end{array}\right\}$

and $d\mathbf{r} = dx\,\mathbf{i}$, $\mathbf{F} = \mathbf{i}$

$$\therefore \int_A^D \mathbf{F}\cdot d\mathbf{r} = \int_A^D \mathbf{i}\cdot\mathbf{i}\,dx = \int_A^D dx = [x]_0^1 = 1.$$

Along DB $\left(\begin{array}{l} x=1 \\ dx=0 \end{array}\right)$ $d\mathbf{r} = \mathbf{j}\,dy$, $\mathbf{F} = (1+y)\,\mathbf{i} + 2y\,\mathbf{j}$

$$\int_D^B \mathbf{F}\cdot d\mathbf{r} = \int_D^B 2y\,dy = \left[y^2\right]_0^1 = 1.$$

$$\therefore \int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^D \mathbf{F}\cdot d\mathbf{r} + \int_D^B \mathbf{F}\cdot d\mathbf{r} = 1 + 1 = 2.$$

(ii) along AE then EB

Along AE : $\left\{\begin{array}{l} x=0 \\ dx=0 \end{array}\right\}$ $d\mathbf{r} = dy\,\mathbf{j}$, $\mathbf{F} = \mathbf{i}$

$$\int_A^E \mathbf{F}\cdot d\mathbf{r} = \int_A^E \mathbf{i}\cdot\mathbf{j}\,dy = 0.$$

Along EB : $\left\{\begin{array}{l} y=1 \\ dy=0 \end{array}\right\}$ $d\mathbf{r} = dx\,\mathbf{i}$, $\mathbf{F} = (1+x^2)\,\mathbf{i} + 2x\,\mathbf{j}$

$$\int_E^B \mathbf{F}\cdot d\mathbf{r} = \int_E^B (1+x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

$$\therefore \int_A^B \mathbf{F}\cdot d\mathbf{r} = 0 + \frac{4}{3} = \frac{4}{3}.$$

(iii) Along $y=x$

$$\int_A^B \underset{\sim}{F}.d\underset{\sim}{r} = \int_A^B \left[(1+x^2y)\underset{\sim}{i} + (2xy)\underset{\sim}{j}\right].\left[dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j}\right]$$

$$= \int_A^B (1+x^2y)\,dx + \int_A^B 2xy\,dy$$

$$= \int_A^B (1+x^3)\,dx + \int_A^B 2y^2\,dy \qquad \{\text{using } y=x\}$$

$$= \left[x + \frac{x^4}{4}\right]_0^1 + \left[\frac{2}{3}y^3\right]_0^1$$

$$= 1 + \frac{1}{4} + \frac{2}{3} = \frac{12}{12} + \frac{3}{12} + \frac{8}{12} = \frac{23}{12}.$$

(c) 1st field may be conservative but, as we would need to check all possible paths to verify this our results showing path independence of $\int_A^B \underset{\sim}{F}.d\underset{\sim}{r}$ for just 3 paths are inconclusive.

2nd field gives three different values for $\int_A^B \underset{\sim}{F}.d\underset{\sim}{r}$ along different paths i.e. work done not path independent and 2nd field is definitely non-conservative.

11. (a) $\underset{\sim}{F} = (3x^2 - 3yz + 2xz)\underset{\sim}{i} + (3y^2 - 3xz + z^2)\underset{\sim}{j} + (3z^2 - 3xy + x^2 + 2yz)\underset{\sim}{k}$

$\boxed{\underset{\sim}{F} \text{ conservative if and only if } \underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{0}}$

if $\underset{\sim}{F} = F_x(x,y,z)\,\underset{\sim}{i} + F_y(x,y,z)\,\underset{\sim}{j} + F_z(x,y,z)\,\underset{\sim}{k}$.

then
$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \underset{\sim}{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \underset{\sim}{i}\left[(-3x+2z) - (-3x+2z)\right] - \underset{\sim}{j}\left[(-3y+2x) - (-3y+2x)\right] + \underset{\sim}{k}\left[(-3z) - (-3z)\right]$$

$$= \underset{\sim}{0}.$$

11. (b) $\underset{\sim}{F} = (y^2\cos x + z^3)\,\underset{\sim}{i} + (2y\sin x + 4)\,\underset{\sim}{j} + (3xz^2+2)\,\underset{\sim}{k}$

$\underset{\sim}{F}$ conservative $\Longleftrightarrow$ $\underset{\sim}{\nabla}\times\underset{\sim}{F} = \underset{\sim}{0}$

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \underset{\sim}{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \underset{\sim}{i}\left[0-0\right] - \underset{\sim}{j}\left[3z^2 - 3z^2\right] + \underset{\sim}{k}\left[2y\cos x - 2y\cos x\right]$$

$$= \underset{\sim}{0}$$

12. (a) $\underset{\sim}{F} = (2xy+z)\,\underset{\sim}{i} + (x^2+2yz)\,\underset{\sim}{j} + (x+y^2)\,\underset{\sim}{k}$

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \underset{\sim}{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \underset{\sim}{i}\left[2y - 2y\right] - \underset{\sim}{j}\left[1-1\right] + \underset{\sim}{k}\left[2x - 2x\right] = \underset{\sim}{0}.$$

$\therefore$ $\underset{\sim}{F}$ conservative

(b) $\underset{\sim}{F} = (yz+2y)\,\underset{\sim}{i} + (xz+2x)\,\underset{\sim}{j} + (xy+3)\,\underset{\sim}{k}$

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \underset{\sim}{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \underset{\sim}{i}\left[x - x\right] - \underset{\sim}{j}\left[y-y\right] + \underset{\sim}{k}\left[(z+2) - (z+2)\right] = \underset{\sim}{0}$$

$\therefore$ $\underset{\sim}{F}$ is conservative.

(c) $\underset{\sim}{F} = (yz^2+3)\,\underset{\sim}{i} + (xz^2+2)\,\underset{\sim}{j} + (2xyz+4)\,\underset{\sim}{k}$

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \underset{\sim}{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \underset{\sim}{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \underset{\sim}{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \underset{\sim}{i}\left[2xz - 2xz\right] - \underset{\sim}{j}\left[2yz - 2yz\right] + \underset{\sim}{k}\left[z^2 - z^2\right]$$

$$= \underset{\sim}{0}$$

$\therefore$ $\underset{\sim}{F}$ is a conservative vector field.