TUTORIAL 2

This tutorial firstly deals with the divergence of a vector field $\underset{\sim}{A}$:

$$\operatorname{div}\underset{\sim}{A} = \underset{\sim}{\nabla}\cdot\underset{\sim}{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad , \text{ where } \underset{\sim}{A} = A_x\underset{\sim}{i} + A_y\underset{\sim}{j} + A_z\underset{\sim}{k},$$

and then returns to consider the curl of $\underset{\sim}{A}$:

$$\operatorname{curl}\underset{\sim}{A} = \underset{\sim}{\nabla}\times\underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}.$$

The emphasis here is in the relationship between: the divergence and the flux $\int_S \underset{\sim}{A}\cdot d\underset{\sim}{S}$ through surfaces S, and the curl and its connection with the rotational character of the field (and hence the circulation $\oint_C \underset{\sim}{A}\cdot d\underset{\sim}{r}$).

The two main theorems that demonstrate these physical relations are:

the divergence theorem : $\displaystyle\int_V \underset{\sim}{\nabla}\cdot\underset{\sim}{A}\, dV = \oint_S \underset{\sim}{A}\cdot d\underset{\sim}{S}$ (surface S enclosing volume V)

and Stoke's theorem : $\displaystyle\int_S (\underset{\sim}{\nabla}\times\underset{\sim}{A})\cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{A}\cdot d\underset{\sim}{r}$

(open surface S with bounding curve C)

Exercises

1. Which of the following vector fields have constant divergence?

   (a) $\underset{\wedge}{V} = 3x\,\underset{\wedge}{i} + y\,\underset{\wedge}{j} + 2z\,\underset{\wedge}{k}$

   (b) $\underset{\wedge}{V} = x^2\,\underset{\wedge}{i} + y^2\,\underset{\wedge}{j} + z^2\,\underset{\wedge}{k}$

   (c) $\underset{\wedge}{V} = xy\,\underset{\wedge}{i} + yz\,\underset{\wedge}{j} + zx\,\underset{\wedge}{k}$

2. Determine the constant a for which $\underset{\wedge}{V}$ has zero divergence, where $\underset{\wedge}{V} = (x+3y)\,\underset{\wedge}{i} + (y-2z)\,\underset{\wedge}{j} + (x+az)\,\underset{\wedge}{k}$.

3. Determine the divergence of the following vector fields at the given point:

   (a) $\underset{\wedge}{V} = xy^2\,\underset{\wedge}{i} - 2yz\,\underset{\wedge}{j} + xyz\,\underset{\wedge}{k}$ at point $(1,-1,2)$

   (b) $\underset{\wedge}{A} = x^2y\,\underset{\wedge}{i} + (xy+yz)\,\underset{\wedge}{j} + xz^2\,\underset{\wedge}{k}$ at point $(1,2,1)$

   (c) $\underset{\wedge}{A} = xz\,\underset{\wedge}{i} - y^2\,\underset{\wedge}{j} + 2x^2y\,\underset{\wedge}{k}$ at point $(3,2,1)$

4. Show that the divergence of $\underset{\wedge}{V} = \dfrac{\underset{\wedge}{r}}{r^{3/2}}$ is $\dfrac{3}{2}\dfrac{1}{r^{3/2}}$,

   where $\underset{\wedge}{r}$ is a position vector and $r = |\underset{\wedge}{r}|$.

5. Determine which of the following vector fields are irrotational:

   (a) $\underset{\wedge}{V} = 3x\,\underset{\wedge}{i} + 2y\,\underset{\wedge}{j} - 4z\,\underset{\wedge}{k}$

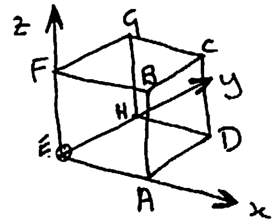
   (b) $\underset{\wedge}{V} = x\,\underset{\wedge}{i} + 5xy\,\underset{\sim}{j}$

   (c) $\underset{\sim}{A} = (4xy-z^3)\,\underset{\wedge}{i} + 2x^2\,\underset{\sim}{j} - 3xz^2\,\underset{\sim}{k}$.

6. (a) Show that curl $(-y\,\underset{\wedge}{i} + x\,\underset{\sim}{j})$ is a constant vector.

   (b) Show that the vector field $yz\,\underset{\wedge}{i} + zx\,\underset{\wedge}{j} + xy\,\underset{\wedge}{k}$ has zero divergence and zero curl. Interpret what these properties tell us about the character of this vector field.

7. The six planes $x=0$, $y=0$, $z=0$, $x=a$, $y=a$, $z=a$ form the faces of a cube of side a



In the above diagram, the corners of this cube are labelled as A, B, C, D, E, F and G.

If $\underset{\wedge}{A} = x\,\underset{\wedge}{i} + y\,\underset{\wedge}{j} + z^2\,\underset{\wedge}{k}$, evaluate the flux of $\underset{\wedge}{A}$, namely $\int \underset{\wedge}{A}\cdot d\underset{\wedge}{S}$ through each of the six faces. Hence, determine the flux of A through the entire surface of the cube.

8. Now, find the divergence of the vector field $\underset{\wedge}{A}$ of the previous question and integrate it through the volume of the cube to verify the divergence theorem, i.e that

$$\int_V \nabla\cdot\underset{\wedge}{A}\, dV = \oint_S \underset{\wedge}{A}\cdot d\underset{\sim}{S}.$$

9. Repeat questions 7. and 8. for the vector field $\underline{A} = x\underline{i} + y\underline{j} + z\underline{k}$ and the cube of side $a = 1$ that is bounded by the six planes $x=0$, $y=0$, $z=0$, $x=1$, $y=1$ and $z=1$.

10. Use the divergence theorem to show that if S is a closed surface enclosing a volume V then

$$\oint_S r^2 \underline{r}.d\underline{S} = 5 \int_V r^2 dV ,$$

where $\underline{r}$ is a position vector and $r = |\underline{r}|$.

Hence, deduce the value of the surface integral $\oint_S \underline{A}.d\underline{S}$ where

$$\underline{A} = (x^2+y^2+z^2)(x\underline{i}+y\underline{j}+z\underline{k})$$

and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

11. Find $\nabla.\underline{A}$ and $\nabla\times\underline{A}$ for the vector field

$$\underline{A} = (3x-2y)\,\underline{i} + x^2 z\,\underline{j} + (1-2z)\,\underline{k}.$$

For this vector field, evaluate

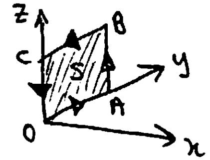
(a) $\int \underline{A}.d\underline{S}$ over the circular region in the $z=0$ plane bounded by $x^2+y^2=a^2$. Count the area element as positive in the $+z$-direction.

11. continued

(b) $\int (\nabla\times\underline{A}).d\underline{S}$ over the same region.

12. Consider the closed circuit OABCO defined with respect to Cartesian axes as:

(0,0,0) to (0,1,0) to (0,1,1) to (0,0,1) to (0,0,0)



If $\underline{A} = xy\,\underline{i} + (2y-xz)\,\underline{j} + xz\,\underline{k}$, find curl $\underline{A}$.

Hence, find $\int_S (\nabla\times\underline{A}).d\underline{S}$ over the plane bounded by the above circuit. Consider $d\underline{S}$ as positive in the $+x$-direction.

Now find the loop integral $\oint \underline{A}.d\underline{r}$ round the circuit and check that your answers are consistent with Stokes theorem.

[A further similar example appears in Stroud's 'Further Engineering Mathematics', p726, Ex 2]

13. {Extract from Jan 2002 exam: "bookwork"} 5b

Describe the property of a vector field $\mathbf{A}$ that $div\,\mathbf{A} = \nabla.\mathbf{A}$ expresses (make reference to the divergence theorem and give one, or more, physical examples in your answer).

Describe the property of a vector field $\mathbf{A}$ that $curl\,\mathbf{A} = \nabla \times \mathbf{A}$ expresses (make reference to Stokes' theorem and give one, or more, physical examples in your answer).

Discuss the concept of *vector area* (making reference to the magnitude and to the direction of the cross product of two vectors in your answer.

(16 marks)

FULL WORKED SOLUTIONS

1. (a) $\underset{\sim}{V} = 3x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + 2z\,\underset{\sim}{k} \equiv V_x\,\underset{\sim}{i} + V_y\,\underset{\sim}{j} + V_z\,\underset{\sim}{k}$

$$\text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}.\underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

{Recall that this is scalar – as a dot product should be}

$$= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(2z)$$

$$= 3 + 1 + 2 = 6$$

(i.e. independent of x, y, z and constant everywhere).

(b) $\underset{\sim}{V} = x^2\,\underset{\sim}{i} + y^2\,\underset{\sim}{j} + z^2\,\underset{\sim}{k} \equiv V_x\,\underset{\sim}{i} + V_y\,\underset{\sim}{j} + V_z\,\underset{\sim}{k}$

$$\text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}.\underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z = 2(x+y+z)$$

[i.e. given by a (scalar) function of x, y, z and not constant everywhere].

(c) $\underset{\sim}{V} = xy\,\underset{\sim}{i} + yz\,\underset{\sim}{j} + zx\,\underset{\sim}{k} \equiv V_x\,\underset{\sim}{i} + V_y\,\underset{\sim}{j} + V_z\,\underset{\sim}{k}$

$$\text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}.\underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$= y + z + x$$

(not constant everywhere in x, y, z).

2. $\underset{\sim}{V} = (x+3y)\underset{\sim}{i} + (y-2z)\underset{\sim}{j} + (x+az)\underset{\sim}{k}$, where a is constant

i.e. $\underset{\sim}{V} \equiv V_x \underset{\sim}{i} + V_y \underset{\sim}{j} + V_z \underset{\sim}{k}$

$$\text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az)$$

$$= 1 + 1 + a$$

$$= 2 + a.$$

For zero divergence everywhere, we need $\underset{\sim}{\nabla}\cdot\underset{\sim}{V} = 0$

i.e. $2 + a = 0$

i.e. $a = -2.$

3. (a) $\underset{\sim}{V} = xy^2\underset{\sim}{i} - 2yz\underset{\sim}{j} + xyz\underset{\sim}{k} \equiv V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}$

$$\text{div}\,\underset{\sim}{V} = \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-2yz) + \frac{\partial}{\partial z}(xyz)$$

$$= y^2 - 2z + xy$$

At point $(1,-1,2)$, $x=1, y=-1, z=2$,

$$\therefore \underset{\sim}{\nabla}\cdot\underset{\sim}{V} = (-1)^2 - 2(2) + (1)(-1)$$

$$= 1 - 4 - 1 = -4.$$

(b) $\underset{\sim}{A} = (x^2y)\underset{\sim}{i} + (xy+yz)\underset{\sim}{j} + (xz^2)\underset{\sim}{k}$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{A} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy+yz) + \frac{\partial}{\partial z}(xz^2)$$

$$= 2xy + x + z + 2xz$$

$$= x(2y + 1 + 2z) + z$$

At point $(1,2,1)$, $x=1, y=2, z=1$,

$$\therefore \underset{\sim}{\nabla}\cdot\underset{\sim}{A} = 1(2.2 + 1 + 2.1) + 1$$

$$= 4 + 1 + 2 + 1 = 8.$$

(c) $\underset{\sim}{A} = xz\underset{\sim}{i} - y^2\underset{\sim}{j} + 2x^2y\underset{\sim}{k}$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{A} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y)$$

$$= z - 2y$$

At point $(3,2,1)$, $x=3, y=2, z=1$ but $\underset{\sim}{\nabla}\cdot\underset{\sim}{A}$ does not depend on x, it only depends on y and z, and ...

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{A} = 1 - 2(2) = 1 - 4 = -3.$$

4. $\underset{\sim}{V} = \dfrac{\underset{\sim}{r}}{r^{3/2}}$, where $\underset{\sim}{r}$ is a position vector and $|\underset{\sim}{r}| = r$.

If $\underset{\sim}{r}$ is a position vector, then $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

and $r = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$\therefore \quad \underset{\sim}{V} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{r^{3/2}} \qquad \left\{ r^{3/2} = \left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^{3/2} = (x^2+y^2+z^2)^{3/4} \right\}$$

$$= \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{(x^2+y^2+z^2)^{3/4}}$$

$$\text{i.e.} \quad \underset{\sim}{V} = \frac{x}{(x^2+y^2+z^2)^{3/4}}\underset{\sim}{i} + \frac{y}{(x^2+y^2+z^2)^{3/4}}\underset{\sim}{j} + \frac{z}{(x^2+y^2+z^2)^{3/4}}\underset{\sim}{k}$$

$$\equiv V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}$$

$$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-3/4}\right] = \underbrace{(x^2+y^2+z^2)^{-3/4} + x \cdot \underbrace{\left(-\tfrac{3}{4}\right)(x^2+y^2+z^2)^{-\frac{7}{4}} \cdot 2x}_{\text{chain rule}}}_{\text{product rule}}$$

$$= \frac{(x^2+y^2+z^2)^{4/4}}{(x^2+y^2+z^2)^{7/4}} - \frac{6x^2}{4} \cdot \frac{1}{(x^2+y^2+z^2)^{7/4}}$$

$$= \frac{x^2+y^2+z^2 - \frac{3}{2}x^2}{(x^2+y^2+z^2)^{7/4}}$$

$$= \frac{-\frac{1}{2}x^2+y^2+z^2}{(x^2+y^2+z^2)^{7/4}}$$

4. continued   The symmetry of x, y, z then implies that

$$\frac{\partial V_y}{\partial y} = \frac{x^2 - \frac{1}{2}y^2 + z^2}{(x^2+y^2+z^2)^{7/4}}, \qquad \frac{\partial V_z}{\partial z} = \frac{x^2+y^2-\frac{1}{2}z^2}{(x^2+y^2+z^2)^{7/4}}$$

$$\therefore \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\left(-\frac{1}{2}x^2+y^2+z^2\right) + \left(x^2-\frac{1}{2}y^2+z^2\right) + \left(x^2+y^2-\frac{1}{2}z^2\right)}{(x^2+y^2+z^2)^{7/4}}$$

$$= \frac{3}{2}\,\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/4}}$$

$$= \frac{3}{2}\,\frac{1}{(x^2+y^2+z^2)^{3/4}} = \frac{3}{2}\,\frac{1}{\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^{3/2}}$$

$$\therefore \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \frac{3}{2}\,\frac{1}{r^{3/2}}, \quad \text{since } r = (x^2+y^2+z^2)^{\frac{1}{2}}.$$

5. * If a vector field is IRROTATIONAL then the curl of this field is zero everywhere in this field * *

(a) $\underset{\sim}{V} = 3x\underset{\sim}{i} + 2y\underset{\sim}{j} - 4z\underset{\sim}{k}$

$$\equiv V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}$$

5.(a) continued

$$\text{curl } \underset{\sim}{V} = \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$\therefore \underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x & 2y & -4z \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2y & -4z \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 3x & -4z \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ 3x & 2y \end{vmatrix}$$

$$= \underset{\sim}{i} \left[\frac{\partial}{\partial y}(-4z) - \frac{\partial}{\partial z}(2y) \right] - \underset{\sim}{j} \left[\frac{\partial}{\partial x}(-4z) - \frac{\partial}{\partial z}(3x) \right] + \underset{\sim}{k} \left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(3x) \right]$$

$$= \underset{\sim}{i} \left[0 - 0 \right] - \underset{\sim}{j} \left[0 - 0 \right] + \underset{\sim}{k} \left[0 - 0 \right] = \underset{\sim}{0}.$$

Since $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$ this vector field is irrotational.

(b) $\underset{\sim}{V} = x\underset{\sim}{i} + 5xy\underset{\sim}{j} \equiv V_x \underset{\sim}{i} + V_y \underset{\sim}{j} + V_z \underset{\sim}{k}$,

where $V_z = 0$ everywhere (i.e. for all x, y, z)

5.(b) continued

$$\underset{\sim}{\nabla} \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & 5xy & 0 \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 5xy & 0 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ x & 0 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ x & 5xy \end{vmatrix}$$

$$= \underset{\sim}{i} \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(5xy) \right] - \underset{\sim}{j} \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x) \right] + \underset{\sim}{k} \left[\frac{\partial}{\partial x}(5xy) - \frac{\partial}{\partial y}(x) \right]$$

$$= \underset{\sim}{i} \left[0 - 0 \right] - \underset{\sim}{j} \left[0 - 0 \right] + \underset{\sim}{k} \left[5y - 0 \right] = 5y \underset{\sim}{k}.$$

$\therefore \underset{\sim}{\nabla} \times \underset{\sim}{V}$ requires $y = 0$ to be $\underset{\sim}{0}$ and the vector field is not irrotational.

(c) $\underset{\sim}{A} = (4xy - z^3)\underset{\sim}{i} + 2x^2 \underset{\sim}{j} - 3xz^2 \underset{\sim}{k}$

$$\underset{\sim}{\nabla} \times \underset{\sim}{A} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy - z^3 & 2x^2 & -3xz^2 \end{vmatrix} = \underset{\sim}{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2x^2 & -3xz^2 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 4xy - z^3 & -3xz^2 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ 4xy - z^3 & 2x^2 \end{vmatrix}$$

5. (c) continued

$$\therefore \nabla \times \underset{\sim}{A} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial z}(4xy - z^3)\right]$$

$$+\underset{\sim}{k}\left[\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right]$$

$$= \underset{\sim}{i}[0-0] - \underset{\sim}{j}\left[(-3z^2) - (-3z^2)\right] + \underset{\sim}{k}\left[(4x) - (4x)\right]$$

$$= \underset{\sim}{i}.0 - \underset{\sim}{j}.0 + \underset{\sim}{k}.0 = \underset{\sim}{0}$$

$\therefore$ This vector field $\underset{\sim}{A}$ is irrotational.

6. (a) To show that curl $(-y\underset{\sim}{i} + x\underset{\sim}{j})$ is a constant vector
(b) To show that vector field $yz\underset{\sim}{i} + zx\underset{\sim}{j} + xy\underset{\sim}{k}$ has zero divergence and zero curl. Interpret what these properties tell us about the character of this latter vector field.

Ans (a) Let $\underset{\sim}{V} = -y\underset{\sim}{i} + x\underset{\sim}{j}$

$= V_x\underset{\sim}{i} + V_y\underset{\sim}{j} + V_z\underset{\sim}{k}$, where $V_x = -y$, $V_y = x$, $V_z = 0$

$$\nabla \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]$$

6. (a) continued $\nabla \times \underset{\sim}{V} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + [1-(-1)]\underset{\sim}{k} = 2\underset{\sim}{k}$

i.e. a constant vector $2\underset{\sim}{k}$

i.e. no dependence on x, y, z

i.e. constant vector throughout x, y, z-space.

(b) $\underset{\sim}{V} = yz\underset{\sim}{i} + zx\underset{\sim}{j} + xy\underset{\sim}{k}$

$$\nabla \cdot \underset{\sim}{V} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy)$$

$$= 0 + 0 + 0 = 0$$

This implies no net 'outflow' of flux per unit volume anywhere in x, y, z-space i.e there are no sources or sinks of flux anywhere.

$$\nabla \times \underset{\sim}{V} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right]$$

$$- \underset{\sim}{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right]$$

$$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

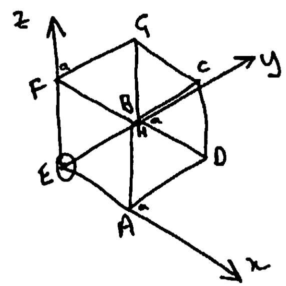
6. (b) continued

i.e. $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{i}\left[x - x\right] - \underset{\sim}{j}\left[y - y\right] + \underset{\sim}{k}\left[z - z\right]$

$= 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k} = \underset{\sim}{0}$

This implies that ~~the vector~~ field is irrotational i.e. there are no vortices in the field. We can also say that ~~the~~ field is conservative.

7\.



$\underset{\sim}{A} = x\underset{\sim}{i} + y\underset{\sim}{j} + z^2\underset{\sim}{k}$

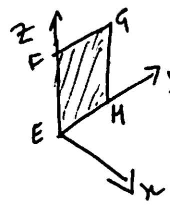
We wish to find $\int \underset{\sim}{A}.d\underset{\sim}{S}$ for each of the six sides.

This is a closed surface so the $d\underset{\sim}{S}$ vector for each surface will point outwards

Take each side in turn

- For this side, what is $\underset{\sim}{A}$?
- work out the direction of the unit vector $\underset{\sim}{\hat{n}}$ such that $d\underset{\sim}{S} = \underset{\sim}{\hat{n}}\, dS$
- work out $\underset{\sim}{A}.d\underset{\sim}{S}$
- integrate over the side if $\underset{\sim}{A}.d\underset{\sim}{S}$ is non-zero

7\. continued



On EFGH, $x=0$ so $\underset{\sim}{A} = y\underset{\sim}{j} + z^2\underset{\sim}{k}$

$\underset{\sim}{\hat{n}}$ points in the -ve x-direction (outwards and normal)

i.e. $\underset{\sim}{\hat{n}}$ is $-\underset{\sim}{i}$

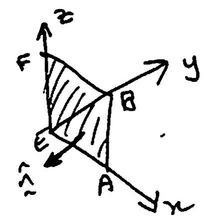
$\therefore \underset{\sim}{A}.d\underset{\sim}{S} = \underset{\sim}{A}.\underset{\sim}{\hat{n}}\, dS$

$= (y\underset{\sim}{j} + z^2\underset{\sim}{k}).(-\underset{\sim}{i})\, dS$

$= 0$

$\therefore \int_{EFGH} \underset{\sim}{A}.d\underset{\sim}{S} = 0$

On FBAE



Here, $y=0$ so $\underset{\sim}{A} = x\underset{\sim}{i} + y\underset{\sim}{j} + z^2\underset{\sim}{k}$

becomes $\underset{\sim}{A} = x\underset{\sim}{i} + z^2\underset{\sim}{k}$

$\underset{\sim}{\hat{n}}$ points in -ve y-direction

i.e. $\underset{\sim}{\hat{n}}$ is $-\underset{\sim}{j}$

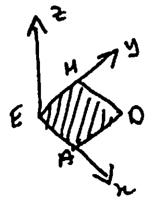
$\therefore \underset{\sim}{A}.d\underset{\sim}{S} = \underset{\sim}{A}.\underset{\sim}{\hat{n}}\, dS$

$= (x\underset{\sim}{i} + z^2\underset{\sim}{k}).(-\underset{\sim}{j})\, dS$

$= 0$

$\therefore \int_{FBAE} \underset{\sim}{A}.d\underset{\sim}{S} = 0$, too.

7. continued **On ADHE**



Here, $z=0$, $\mathbf{A} = x\mathbf{i}+y\mathbf{j}$

$\hat{\mathbf{n}}$ is $-\mathbf{k}$

So, $\mathbf{A}.d\mathbf{S} = \mathbf{A}.\hat{\mathbf{n}}\, dS$

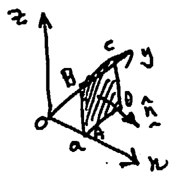
$$= (x\mathbf{i}+y\mathbf{j}).(-\mathbf{k})\, dS$$

$$= 0$$

and $\int_{ADHE} \mathbf{A}.d\mathbf{S} = 0.$

On ABCD

Here, $x=a$, $\mathbf{A} = a\mathbf{i}+y\mathbf{j}+z^2\mathbf{k}$, $\hat{\mathbf{n}}$ is $\mathbf{i}$



So, $\mathbf{A}.d\mathbf{S} = (a\mathbf{i}+y\mathbf{j}+z^2\mathbf{k}).(\mathbf{i})\, dS$

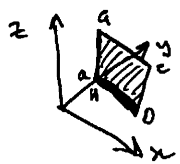
$$= (a\,\mathbf{i}.\mathbf{i} + y\,\mathbf{j}.\mathbf{i} + z^2\,\mathbf{k}.\mathbf{i})\, dS$$

$$= (a+0+0)\, dS = a\, dS$$

$\therefore \int_{ABCD} \mathbf{A}.d\mathbf{S} = \int_{ABCD} a\, dS = a\int_{ABCD} dS = a \cdot a^2$, since each side is a square of area a^2

$\therefore \int_{ABCD} \mathbf{A}.d\mathbf{S} = a^3.$

On GCDH



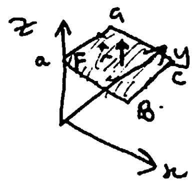
Here $y=a$, $\mathbf{A} = x\mathbf{i}+a\mathbf{j}+z^2\mathbf{k}$, $\hat{\mathbf{n}}$ is $\mathbf{j}$

$\mathbf{A}.d\mathbf{S} = \mathbf{A}.\hat{\mathbf{n}}\, dS = (x\mathbf{i}+a\mathbf{j}+z^2\mathbf{k}).\mathbf{j}\, dS$

$= a\, dS$, $\int_{GCDH} \mathbf{A}.d\mathbf{S} = a\int_{GCDH} dS = a^3.$

(as previously)

7. continued **On GCBF**



Here, $z=a$, $\mathbf{A} = x\mathbf{i}+y\mathbf{j}+a^2\mathbf{k}$, $\hat{\mathbf{n}}$ is $\mathbf{k}$

$\therefore \mathbf{A}.d\mathbf{S} = \mathbf{A}.\hat{\mathbf{n}}\, dS$

$$= (x\mathbf{i}+y\mathbf{j}+a^2\mathbf{k}).\mathbf{k}\, dS = a^2 dS$$

$$\therefore \int_{GCBF} \mathbf{A}.d\mathbf{S} = \int_{GCBF} a^2 dS = a^2\int_{GCBF} dS = a^2.a^2 = a^4.$$

Total flux across whole cube is sum of the fluxes through the sides

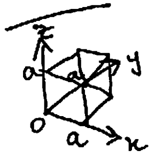
i.e. $\oint_{cube} \mathbf{A}.d\mathbf{S} = \int_{EFGH} + \cdots + \int_{GCBF}$

$$= 0+0+0+a^3+a^3+a^4 = 2a^3+a^4.$$

8. $\mathbf{A} = x\mathbf{i}+y\mathbf{j}+z^2\mathbf{k}$.   Need to show: $\int_V \nabla.\mathbf{A}\, dV = \int_S \mathbf{A}.d\mathbf{S}$

$\therefore \operatorname{div}\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)$

$$= 1+1+2z = 2+2z.$$



$$\int_V \nabla.\mathbf{A}\, dV = \int_V 2+2z\, dV = \int_0^a\int_0^a\int_0^a (2+2z)\,dx\,dy\,dz$$

8. continued

i.e. $$\int_V \nabla\cdot\mathbf{A}\,dV = \int_0^a\int_0^a \left[(2+2z)x\right]_0^a dy\,dz$$

$$= \int_0^a\int_0^a \left\{(2+2z)a - 0\right\} dy\,dz$$

$$= a\int_0^a\int_0^a 2+2z \; dy\,dz$$

$$= a\int_0^a \left[(2+2z)y\right]_0^a dz$$

$$= a\int_0^a (2+2z)a \; dz$$

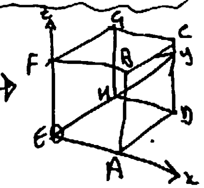
$$= a^2 \int_0^a (2+2z)\,dz = a^2\left[2z + \frac{2z^2}{2}\right]_0^a$$

$$= a^2\left\{2a + a^2 - 0\right\} = 2a^3 + a^4.$$

Since question 7. gave $\int_S \mathbf{A}.d\mathbf{S} = 2a^3 + a^4$, we have verified the divergence theorem for this example

i.e. that $\int_V \nabla.\mathbf{A}\,dV = \int_S \mathbf{A}.d\mathbf{S}$.

9. As 7. with $a=1$ $\quad \mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



1) EFGH $\quad x=0$, $\mathbf{A} = y\mathbf{j} + z\mathbf{k}$, $\hat{\mathbf{n}} = -\mathbf{i}$

$\mathbf{A}.d\mathbf{S} = \mathbf{A}.\hat{\mathbf{n}}\,dS = 0 \quad \therefore \int_{EFGH} \mathbf{A}.d\mathbf{S} = 0.$

2) FBAE $\quad y=0$, $\mathbf{A} = x\mathbf{i} + z\mathbf{k}$, $\hat{\mathbf{n}} = -\mathbf{j}$, $\mathbf{A}.d\mathbf{S} = \mathbf{A}.\hat{\mathbf{n}}\,dS = (x\mathbf{i} + z\mathbf{k})(-\mathbf{j})\,dS = 0$

$\therefore \int_{FBAE} \mathbf{A}.d\mathbf{S} = 0.$

3) ADHE $\quad z=0$, $\mathbf{A} = x\mathbf{i} + y\mathbf{j}$, $\hat{\mathbf{n}} = -\mathbf{k}$, $\mathbf{A}.\hat{\mathbf{n}} = (x\mathbf{i} + y\mathbf{j}).(-\mathbf{k}) = 0 \quad \therefore \int_{ADHE} \mathbf{A}.d\mathbf{S} = 0.$

4) ABCD $\quad x=1$, $\mathbf{A} = \mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\hat{\mathbf{n}} = \mathbf{i}$, $\mathbf{A}.\hat{\mathbf{n}} = (\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(\mathbf{i}) = 1$

$\therefore \int_{ABCD} \mathbf{A}.d\mathbf{S} = \int_{ABCD} \mathbf{A}.\hat{\mathbf{n}}\,dS = \int_{ABCD} (1)\,dS = \int_{ABCD} dS = 1$ (Area of each side is $1\times1=1$)

5) GCDH $\quad y=1$, $\mathbf{A} = x\mathbf{i} + \mathbf{j} + z\mathbf{k}$, $\hat{\mathbf{n}} = \mathbf{j}$, $\mathbf{A}.\hat{\mathbf{n}} = (x\mathbf{i} + \mathbf{j} + z\mathbf{k}).\mathbf{j} = 1$

$\therefore \int_{GCDH} \mathbf{A}.d\mathbf{S} = \int_{GCDH} \mathbf{A}.\hat{\mathbf{n}}\,dS = \int_{GCDH} dS = 1$ (area of the side is 1).

6) GCBF $\quad z=1$, $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$, $\hat{\mathbf{n}} = \mathbf{k}$, $\mathbf{A}.\hat{\mathbf{n}} = (x\mathbf{i} + y\mathbf{j} + \mathbf{k}).(\mathbf{k}) = 1$

$\therefore \int_{GCBF} \mathbf{A}.d\mathbf{S} = \int_{GCBF} dS = 1.$

Total flux, $\oint_S \mathbf{A}.d\mathbf{S} = 0+0+0+1+1+1 = 3.$

$\int_V \mathrm{div}\,\mathbf{A}\,dV$? $\quad \mathrm{div}\,\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3.$

$$\int_0^1\int_0^1\int_0^1 \mathrm{div}\,\mathbf{A}\,dV = \int_0^1\int_0^1\int_0^1 3\,dx\,dy\,dz = \int_0^1\int_0^1 [3x]_0^1\,dy\,dz = \int_0^1\int_0^1 3\,dy\,dz$$

$$= \int_0^1 [3y]_0^1\,dz = \int_0^1 3\,dz = [3z]_0^1 = 3.$$

$\therefore \int_V \mathrm{div}\,\mathbf{A}\,dV = \int_S \mathbf{A}.d\mathbf{S}$ and div. theorem verified.

10. We wish to use $\int_S \underset{\sim}{A} \cdot d\underset{\sim}{S} = \int_V \operatorname{div} \underset{\sim}{A} \, dV$

to show that $\int_S r^2 \underset{\sim}{r} \cdot d\underset{\sim}{S} = 5 \int_V r^2 \, dV$

Comparing the left-hand-sides, try setting $\underset{\sim}{A} = r^2 \underset{\sim}{r}$.

Then, by the divergence theorem $\operatorname{div} \underset{\sim}{A} = 5r^2$

If this is true, i.e. $\operatorname{div} \underset{\sim}{A} = 5r^2$, then we have used the divergence theorem to prove that $\int_S r^2 \underset{\sim}{r} \cdot d\underset{\sim}{S} = 5 \int_V r^2 dV$.

Since $\underset{\sim}{r}$ is a position vector, $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

and $r^2 = x^2 + y^2 + z^2$.

$$\therefore \underset{\sim}{A} = r^2 \underset{\sim}{r} = (x^2+y^2+z^2)(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})$$

$$= (x^2+y^2+z^2)\,x\underset{\sim}{i} + (x^2+y^2+z^2)\,y\underset{\sim}{j} + (x^2+y^2+z^2)\,z\underset{\sim}{k}$$

i.e. $\underset{\sim}{A} \equiv A_x \underset{\sim}{i} + A_y \underset{\sim}{j} + A_z \underset{\sim}{k}$

$$\operatorname{div} \underset{\sim}{A} = \underset{\sim}{\nabla} \cdot \underset{\sim}{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z, \text{ where } \begin{cases} A_x = x^3 + xy^2 + xz^2 \\ A_y = x^2 y + y^3 + yz^2 \\ A_z = x^2 z + y^2 z + z^3 \end{cases}$$

$$\left.\begin{aligned} \frac{\partial A_x}{\partial x} &= 3x^2 + y^2 + z^2 \\ \frac{\partial A_y}{\partial y} &= x^2 + 3y^2 + z^2 \\ \frac{\partial A_z}{\partial z} &= x^2 + y^2 + 3z^2 \end{aligned}\right\} \Rightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{A} = 5x^2 + 5y^2 + 5z^2 = 5(x^2+y^2+z^2)$$

$$= 5r^2, \text{ as required.}$$

10. continued. So we have $\int_S \underset{\sim}{A} \cdot d\underset{\sim}{S} = 5 \int_V r^2 dV$,

where $\underset{\sim}{A} = r^2 \underset{\sim}{r} = (x^2+y^2+z^2)(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})$

You should realise now that the surface integral (flux) calculation can be much more work than the volume integral (see previous examples).

To work out $\int_S \underset{\sim}{A} \cdot d\underset{\sim}{S}$, where S is the surface of the cube with sides $x=0, x=1$, $y=0, y=1$, $z=0, z=1$,

Work out instead $5 \int_0^1 \int_0^1 \int_0^1 r^2 dV$, where $r^2 = x^2 + y^2 + z^2$

$$\text{i.e. } 5 \int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\, dx\,dy\,dz = 5 \int_0^1 \int_0^1 \left[\frac{x^3}{3} + y^2 x + z^2 x \right]_0^1 dy\,dz$$

$$= 5 \int_0^1 \int_0^1 \frac{1}{3} + y^2 + z^2 \, dy\,dz = 5 \int_0^1 \left[\frac{y}{3} + \frac{y^3}{3} + yz^2 \right]_0^1 dz$$

$$= 5 \int_0^1 \frac{1}{3} + \frac{1}{3} + z^2 \, dz = 5 \left[\frac{z}{3} + \frac{z}{3} + \frac{z^3}{3} \right]_0^1$$

$$= 5 \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right) = 5 \times 1 = 5.$$

11. $A = (3x-2y)\,i + x^2 z\,j + (1-2z)\,k \equiv A_x\,i + A_y\,j + A_z\,k$

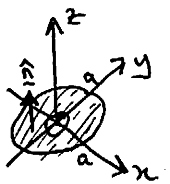
$$\nabla \cdot A = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z = 3 + 0 - 2 = 1$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x-2y & x^2 z & 1-2z \end{vmatrix} = i\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ x^2 z & 1-2z \end{vmatrix} - j\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 3x-2y & 1-2z \end{vmatrix} + k\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ 3x-2y & x^2 z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(1-2z) - \frac{\partial}{\partial z}(x^2 z)\right] - j\left[\frac{\partial}{\partial x}(1-2z) - \frac{\partial}{\partial z}(3x-2y)\right] + k\left[\frac{\partial}{\partial x}(x^2 z) - \frac{\partial}{\partial y}(3x-2y)\right]$$

$$= i\left[0 - x^2\right] - j\left[0-0\right] + k\left[2xz - (-2)\right]$$

$$\therefore \nabla \times A = -x^2\,i + 2(xz+1)\,k.$$



For this circular region, $z=0$.

So, $A = (3x-2y)\,i + 0\,j + (1-0)\,k$

i.e. $A = (3x-2y)\,i + k$

Question states $\hat{n}$ in positive z-direction, i.e. unit vector, $\hat{n} = k$

$\therefore dS = \hat{n}\,dS = k\,dS$.

i.e. $A \cdot dS = \left((3x-2y)\,i + k\right) \cdot k\,dS$

$= (3x-2y)\,i \cdot k\,dS + k \cdot k\,dS = 0 + dS$

$\therefore \int_{\text{region}} A \cdot dS = \int_{\text{region}} dS = \pi a^2$ (the area of the circular disk).

11. continued (b)

$$\nabla \times A = -x^2\,i + 2(xz+1)\,k$$

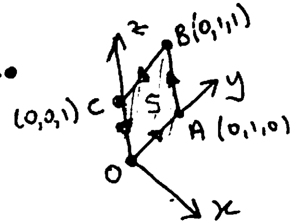
In same region, $z=0$. So, $\nabla \times A = -x^2\,i + 2k$

We still have $dS = k\,dS$.

$$\therefore \int_{\text{region}} (\nabla \times A) \cdot dS = \int_{\text{region}} (-x^2\,i + 2k) \cdot k\,dS = \int_{\text{region}} 2\,dS$$

$$= 2\int_{\text{region}} dS = 2 \times \pi a^2 = 2\pi a^2.$$

12.



$A = xy\,i + (2y - xz)\,j + xz\,k$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2y-xz & xz \end{vmatrix}$$

i.e. $\nabla \times A = i\left[0 - (-x)\right] - j\left[z - 0\right] + k\left[-z - x\right]$

i.e. $\nabla \times A = +x\,i - z\,j - (x+z)\,k$.

For $\int_S (\nabla \times A) \cdot dS$, we are told to take $\hat{n}$ in +ve x-direction i.e. $\hat{n} = i$

$\therefore dS = \hat{n}\,dS = i\,dS$.

12. continued Don't forget to simplify $\underset{\sim}{\nabla} \times \underset{\sim}{A}$!

On this surface, we have $x=0$.

So, $\underset{\sim}{\nabla} \times \underset{\sim}{A} = 0\hat{i} - z\hat{j} - (0+z)\hat{k} = -z\hat{j} - z\hat{k}$

$\therefore \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{A}) \cdot d\underset{\sim}{S} = \int_S (-z\hat{j} - z\hat{k}) \cdot \hat{i}\, dS = 0$, since $\hat{j} \cdot \hat{i} = \hat{k} \cdot \hat{i} = 0$

The loop integral

$$\boxed{\oint_{OABCO} \underset{\sim}{A} \cdot d\underset{\sim}{r} = \oint_{OABCO} A_x dx + A_y dy + A_z dz}$$

(i.e. $(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$)

where, in general, $A_x = xy$, $A_y = 2y - xz$, $A_z = xz$

$$\oint_{OABCO} \rightarrow \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Consider each part of the path separately

Along OA We have both $x=0$ and $z=0$

$\therefore A_x = 0$, $A_y = 2y$, $A_z = 0$

We also only move along (i.e. change) y, so $dx = dz = 0$

$$\therefore \int_{OA} A_x dx + A_y dy + A_z dz = \int_{OA} A_y dy = \int_0^1 2y\, dy = 2\left[\frac{y^2}{2}\right]_0^1 = 1.$$

12. continued

Along AB We have both $y=1$ and $x=0$

$\therefore A_x = 0$, $A_y = 2$, $A_z = 0$

We also have that only dz is non-zero.

$$\text{So, } \int_{AB} A_x dx + A_y dy + A_z dz = \int_{AB} A_z dz = \int_{AB} 0\, dz = 0.$$

Along BC We have both $z=1$ and $x=0$.

$\therefore A_x = 0$, $A_y = 2y$, $A_z = 0$

We also have that only dy is non-zero.

$$\text{So, } \int_{BC} A_x dx + A_y dy + A_z dz = \int_{BC} A_y dy = \int_1^0 2y\, dy = -\int_0^1 2y\, dy$$

$$= -2\left[\frac{y^2}{2}\right]_0^1 = -1.$$

Along CO We have both $y=0$ and $x=0$

$\therefore A_x = 0$, $A_y = 0$, $A_z = 0$

$$\text{So, } \int_{CO} A_x dx + A_y dy + A_z dz = 0.$$

$$\therefore \oint_{OABCO} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} = 1 + 0 - 1 + 0 = 0 \equiv \int_S (\underset{\sim}{\nabla} \times \underset{\sim}{A}) \cdot d\underset{\sim}{S}$$

consistent with Stokes' Theorem ✓

13. div $\underset{\sim}{A}$ $\equiv$ net volume density of sources and sinks of flux ("net outflow")

$$\int_V \underbrace{\mathrm{div}\,\underset{\sim}{A}}_{\text{volume density}}\, dV = \underbrace{\oint_S \underset{\sim}{A}\cdot d\underset{\sim}{S}}$$

net flux through closed surface S

e.g. $\pm$ point charges $\rightarrow$ $\underset{\sim}{E}$-flux,
lack of magnetic monopoles $\rightarrow$ div $\underset{\sim}{B} = 0$,
incompressible fluid with no sources|sinks $\rightarrow$ div $\underset{\sim}{v} = 0$

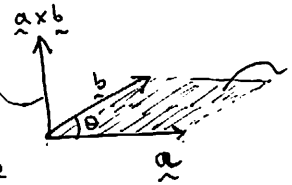
curl $\underset{\sim}{A}$ $\equiv$ twist|swirl|rotation|vorticity|circulation about a point

$$\int_S \mathrm{curl}\,\underset{\sim}{A}\cdot d\underset{\sim}{S} = \underbrace{\oint_C \underset{\sim}{A}\cdot d\underset{\sim}{r}}$$

circulation around C

e.g. electrostatics : $\underset{\sim}{\nabla}\times\underset{\sim}{E} = \underset{\sim}{0}$ (no vortices, conservative),
solenoid, current-carrying wire : loops/vortices in $\underset{\sim}{B}$-field,
non-uniform fluid flow ("paddle wheel"), $\underset{\sim}{\nabla}\times\underset{\sim}{B} \neq \underset{\sim}{0}$,
tornadoes, hurricanes, plug-holes, fluid vortices, ..

vector area



direction of $\underset{\sim}{a}\times\underset{\sim}{b}$ gives direction to the normal to this surface i.e. direction of $d\underset{\sim}{S}$

area $= |\underset{\sim}{a}\times\underset{\sim}{b}|$
$= ab\sin\theta$
$\equiv |d\underset{\sim}{S}|$

– could also mention conventions of :

- outward normals for a closed surface
- clockwise sense of bounding curve C for element normals on open surface.

* [A SUFFICIENT SELECTION OF ABOVE, UNDER EACH HEADING, FOR FULL MARKS] *