TUTORIAL 3

Matrices

I Multiplication

II Rank

III Laplace Expansion

IV Cramer's Rule

V A^{-1} (formal method)

VI A^{-1} (elementary row operations)

VII Eigenvalues and Eigenvectors

EXERCISES

SECTION I (MATRIX MULTIPLICATION)

If you need some practice at matrix multiplication then try as many of the examples below as you need:

1. Find AB and AC if $A=\begin{bmatrix}4&2\\2&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\2&1\end{bmatrix}$, $C=\begin{bmatrix}2&2\\0&-1\end{bmatrix}$

2. $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $B=\begin{bmatrix}5&6\\7&8\end{bmatrix}$, $C=\begin{bmatrix}-1&0&1\\3&-2&1\end{bmatrix}$,

   $D=\begin{bmatrix}1&1\\-1&2\\2&-2\end{bmatrix}$, $E=\begin{bmatrix}-2&2&1\\0&-2&-1\\1&0&1\end{bmatrix}$, $F=\begin{bmatrix}0&1&2\\-1&-1&0\\1&2&3\end{bmatrix}$,

   $X=\begin{bmatrix}1&-2\end{bmatrix}$, $Y=\begin{bmatrix}1&2&1\end{bmatrix}$

   Find AB, BA, AC, BC,
   CB, XA, XB, XC,
   AX, CD, DC, YD, YC,
   DX, XD, EF, FE, YF

3. Find AB where $A=\begin{bmatrix}2&6\\3&9\end{bmatrix}$, $B=\begin{bmatrix}3&-6\\-1&2\end{bmatrix}$

SECTION II (RANK)

4. Determine the rank of the following matrices (by calculation of determinants):

   (a) $\begin{bmatrix}5&3\\2&1\end{bmatrix}$,

   (b) $\begin{bmatrix}4&2\\10&5\end{bmatrix}$,

   (c) $\begin{bmatrix}0&1\\0&0\end{bmatrix}$,

   (d) $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$,

   (e) $\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$.

For the following examples, apply rank tests to determine the nature of the solutions. Do <u>not</u> solve the sets of equations

5.(a) $$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ 1 & 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

5.(b) $$\begin{bmatrix} 2 & -1 & 7 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Now we consider specific solutions ...

6. Consider the system
$$\begin{aligned} 2x - y + 3z &= 0 \\ x + 2y - z &= 0 \\ 3x + 4y + z &= 0 \end{aligned}$$

Show that $x=y=z=0$ is the only solution.

7. Consider
$$\begin{aligned} 3x - 2y &= 0 \\ x + 4y &= 0 \\ 2x - y &= 0. \end{aligned}$$

Show that the matrix of coefficients is of rank 2 and that $x=y=0$ is the only solution.

8. Consider
$$\begin{aligned} x - y + z &= 0 \\ 2x + 3y + z &= 0 \\ 3x + 2y + 2z &= 0. \end{aligned}$$

Show why there are solutions <u>different from</u> $x=y=z=0$.
Let $z=k$ and try to generate values of $x = -\frac{4}{5}k$, $y = \frac{k}{5}$.
Verify by substitution that these satisfy all three equations, whatever the choice of k.

9. Determine the rank of A by casting it into echelon form where
$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix}.$$

10. Do the same for
$$B = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{bmatrix}.$$

SECTION III (LAPLACE EXPANSION)

11. Evaluate using Laplace expansion

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & -1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 0 & -3 \\ 1 & 4 & 2 \\ -1 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 4 & -2 & 1 \\ 5 & 0 & -1 \\ 2 & 3 & -3 \end{vmatrix}$$

12. Given that the determinant of a single element matrix is the value of that element, e.g. $A = [5] \Rightarrow |A| = 5$, use Laplace expansion to show that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

SECTION IV (CRAMER'S RULE)

13. Find x and y of the system
$$2x - 3y = 4$$
$$3x - y = 1$$

14. Find solutions of the following systems ...

(a)
$$5x - 4y = 3$$
$$2x + 3y = 7$$

(b)
$$2x + 3y - 2z = 4$$
$$x + y - z = 2$$
$$3x - 5y + 3z = 0$$

(c)
$$3x - 2y = 7$$
$$3y + 2z = 6$$
$$2x + 3z = 1$$

(d)
$$3x + 2y + 2z = 3$$
$$x - 4y + 2z = 4$$
$$2x + y + z = 2$$

SECTION V (A^{-1} by 'the formal method')

15. Solve the following system for x, y and z using a matrix equation $A\underset{\sim}{x} = \underset{\sim}{b}$ and by finding A^{-1} using $\det(A)$ and the transpose of the matrix of cofactors of A,
$$5x + 8y + z = 2$$
$$2y + z = -1$$
$$4x + 3y - z = 3$$

16. (a) Find the inverse of $A = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{pmatrix}$ and check your answer by direct multiplication.
(Note: use the "formal method" to find A^{-1})

(b) Repeat this exercise for the matrix $A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{pmatrix}$.

17. Use the formal method to show that the inverse of
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
Furthermore, verify that $AA^{-1} = I$.

SECTION VI (A^{-1} by elementary row operations)

18. Invert (i.e. find the inverse of):

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$,

by applying elementary row operations to the identity matrix.

19. Solve the following system of equations by finding the inverse of the coefficient matrix:-

$$2x_1 - x_2 - 3x_3 = 1$$
$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - 2x_2 - 5x_3 = 2 .$$

(note: apply elementary row operations to the identity matrix to find the inverse of the coefficient matrix.)

SECTION VII (Eigenvalues and Eigenvectors)

20. Show that $\underset{\sim}{x} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$

21. Is $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

22. Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

23. Find the eigenvalues of $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

24. Find the eigenvalues of $A = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix}$.

25. Find the eigenvalues of $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

26. Find the eigenvalues <u>and</u> the (linearly independent) eigenvectors of:

(a) $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$,

(b) $A = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix}$,

(c) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$,

(d) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$.

27. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

show that the characteristic equation takes the form:

$$\lambda^2 - tr(A)\,\lambda + det(A) = 0 ,$$

where $tr(A)$, 'the trace of A', is defined to be the sum of the diagonal elements of A.

SOLUTIONS

1. $$AB=\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}1&1\\2&1\end{pmatrix}=\begin{pmatrix}4(1)+2(2)&4(1)+2(1)\\2(1)+1(2)&2(1)+1(1)\end{pmatrix}=\begin{pmatrix}8&6\\4&3\end{pmatrix}$$

$$AC=\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}2&2\\0&-1\end{pmatrix}=\begin{pmatrix}4(2)+2(0)&4(2)+2(-1)\\2(2)+1(0)&2(2)+1(-1)\end{pmatrix}=\begin{pmatrix}8&6\\4&3\end{pmatrix}.$$

2. $$AB=\begin{pmatrix}19&22\\43&50\end{pmatrix},\quad BA=\begin{pmatrix}23&34\\31&46\end{pmatrix}$$ ← Note: $AB \neq BA$ in general.

$$BC=\begin{bmatrix}13&-12&11\\17&-16&15\end{bmatrix}$$

$$AC=\begin{pmatrix}5&-4&3\\9&-8&7\end{pmatrix},$$ CB: not defined

$XA=[-5\ \ -6]$, $XB=[-9\ \ -10]$
$XC=[-7\ \ 4\ \ -1]$, AX: not defined

$$CD=\begin{pmatrix}1&-3\\7&-3\end{pmatrix},\quad DC=\begin{pmatrix}2&-2&2\\7&-4&1\\-8&4&0\end{pmatrix}$$ Note: CD and DC are not even the same order!

$YD=[1\ \ 3]$, YC = not defined

DX = not defined, XD = not defined

$$EF=\begin{pmatrix}-1&-2&-1\\1&0&-3\\1&3&5\end{pmatrix},\quad FE=\begin{pmatrix}2&-2&1\\2&0&0\\1&-2&2\end{pmatrix},\quad YF=[-1\ \ 1\ \ 5].$$

3. $$AB=\begin{pmatrix}2&6\\3&9\end{pmatrix}\begin{pmatrix}3&-6\\-1&2\end{pmatrix}=\begin{bmatrix}6-6&-12+12\\9-9&-18+18\end{bmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

<u>Note</u>: Neither A nor B are filled with zeroes, yet their product is.

4. (a) $\begin{bmatrix}5&3\\2&1\end{bmatrix}$

rank = "size" of largest non-zero determinant (that can be formed from the elements in the order that they appear in the original matrix).

Try the full 2x2 matrix and its determinant:

$$\begin{vmatrix}5&3\\2&1\end{vmatrix}=5.1-3.2=5-6=-1\neq 0 \quad \therefore \underline{\text{rank}=2.}$$

(b) $\begin{bmatrix}4&2\\10&5\end{bmatrix}$: $\begin{vmatrix}4&2\\10&5\end{vmatrix}=4.5-2.10=20-20=0$

$\therefore$ rank < 2

Now, try to find any <u>1x1</u> submatrix with non-zero determinant ([a] has |a| = a, by definition).

Testing <u>any</u> element gives a non-zero determinant for the corresponding <u>1</u>x<u>1</u> submatrix, e.g. $[10]\rightarrow|10|=10$ $\therefore \underline{\text{rank}=1.}$

(c) $\begin{bmatrix}0&1\\0&0\end{bmatrix}$: $\begin{vmatrix}0&1\\0&0\end{vmatrix}=0-0=0$ $\therefore$ rank < 2.

while [1] has determinant = $\underline{1}$. $\therefore \underline{\text{rank}=1.}$

(d) $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$: $\begin{vmatrix}0&0&0\\0&0&0\\0&0&0\end{vmatrix}=0$, $\begin{vmatrix}0&0\\0&0\end{vmatrix}=0$ (<u>any</u> square submatrix of "size" 2)

$|0|=0$ (any 1x1 submatrix determinant). $\therefore \underline{\text{rank}=0.}$

(e) $\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$: $\begin{vmatrix}0&0&1\\0&0&0\\1&0&0\end{vmatrix}=0-0+1.(0-0.1)=0-0+1.(0)=0.$
(along top row)

But $\begin{matrix}0&\cancel{0}&1\\\cancel{0}&\cancel{0}&\cancel{0}\\1&\cancel{0}&0\end{matrix}$ $\begin{bmatrix}0&1\\1&0\end{bmatrix}\equiv$ 2x2 submatrix <u>formed from elements in the order</u> that they appear

$$\begin{vmatrix}0&1\\1&0\end{vmatrix}=0-1=-1\neq 0 \quad \therefore \underline{\text{rank}=2.}$$

5.(a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ 1 & 4 & 3 \end{bmatrix}$, $A_b = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 4 & 2 & -2 \\ 1 & 4 & 3 & 3 \end{bmatrix}$

$(n=3)$

$$|A| = 1.(4.3-2.4) - 2.(3.3-2.1) + (-1).(3.4-4.1)$$
$$= (12-8) - 2(9-2) - (12-4) = 4-14-8 = -18$$

$|A| \neq 0 \Rightarrow A$ is of rank 3, A_b contains $A \Rightarrow A_b$ rank 3

$\therefore$ rank(A) = rank$(A_b) = n \Rightarrow$ unique solution

and system inhomogeneous $\Rightarrow$ this solution is non-trivial

5.(b) $A = \begin{bmatrix} 2 & -1 & 7 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

$A_b = \begin{bmatrix} 2 & -1 & 7 & 2 \\ 4 & 2 & 2 & 5 \\ 3 & 1 & 3 & 1 \end{bmatrix}$

$(n=3)$

$$|A| = 2(2.3-2.1) - (-1).(4.3-2.3) + 7(4.1-2.3)$$
$$= 2(6-2) + (12-6) + 7(4-6)$$
$$= 8+6-14 = 0 \qquad \therefore \text{rank}(A) < 3$$

Try $\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = 4-(-4) = 8 \neq 0 \quad \therefore \underline{\text{rank}(A) = 2}$

For A_b try ...

$$\begin{vmatrix} -1 & 7 & 2 \\ 2 & 2 & 5 \\ 1 & 3 & 1 \end{vmatrix} = -1(2-15) - 7(2-5) + 2(6-2)$$
$$= +13+21+8 \neq 0 \quad \therefore \underline{\text{rank}(A_b) = 3}$$

We thus have

rank(A) < rank(A_b)

$\Rightarrow$ <u>no solution exists</u>.

6. $$A\underset{\sim}{x} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{i.e. } A\underset{\sim}{x} = \underset{\sim}{0}$$

$\therefore$ System is <u>homogeneous</u> $\Rightarrow$

<u>either</u> $\det A = 0$ and infinite no. solutions

<u>or</u> $\det A \neq 0$ and only the trivial solution

$\rightarrow$ Test $\det A$, $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 3 & 4 & 1 \end{vmatrix}$

(use column 1 for a change ... just for a little variety!)

$$= 2\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} - 1.\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$$
$$= 2(2+4) - (-1-12) + 3(1-6)$$
$$= 12+13-15 = 10$$

i.e. $|A| \neq 0 \Rightarrow \underset{\sim}{x} = 0$ is the <u>only</u> solution.

7. $3x-2y = 0$, $x+4y = 0$, $2x-y = 0$ — We could be considered as

$$3x-2y+0.z = 0$$
$$x+4y+0.z = 0$$
$$2x-y+0.z = 0$$

and, since determinants are only defined for <u>square</u> matrices,

- we could test $\begin{vmatrix} 3 & -2 & 0 \\ 1 & 4 & 0 \\ 2 & -1 & 0 \end{vmatrix} = 0\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix}$

$= 0$ (by expanding down the last column)

$\therefore$ system has coefficient matrix of rank less than 3.

7. (continued)

- now test a square submatrix,

$$\begin{vmatrix}3 & -2\\ 1 & 4\end{vmatrix} = 3.4-(-2.1) = 12+2 \neq 0$$

$\therefore$ the coefficient matrix has rank 2

(Since A_b contains A, A_b also has rank 2)

The rank of A tells us that we have 2 linearly independent equations in the two unknowns (x and y). These 2 equations could be found by reducing the system to echelon form to give the two non-zero rows

i.e. $\left(\begin{array}{cc|c}3 & -2 & 0\\ 1 & 4 & 0\\ 2 & -1 & 0\end{array}\right)$. Then, $\begin{array}{l}r_2 \to 3r_2 - r_1\\ r_3 \to \frac{3}{2}r_3 - r_1\end{array}$ gives $\left(\begin{array}{cc|c}3 & -2 & 0\\ 0 & 14 & 0\\ 0 & \frac{1}{2} & 0\end{array}\right)$

and $r_3 \to r_3 - \frac{1}{14}r_2$ yields $\left(\begin{array}{cc|c}3 & -2 & 0\\ 0 & 14 & 0\\ 0 & 0 & 0\end{array}\right)$.

- However, knowing that the rank of A is 2 (= number of unknowns) is sufficient for us to deduce that the homogeneous system only has the trivial solution $x=y=0$.

8. $A\underset{\sim}{x} = \underset{\sim}{0}$ i.e. $\begin{pmatrix}1 & -1 & 1\\ 2 & 3 & 1\\ 3 & 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

But, $\det(A) = \begin{vmatrix}1 & -1 & 1\\ 2 & 3 & 1\\ 3 & 2 & 2\end{vmatrix} = \begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix} + \begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix} + \begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}$ (using the top row for expansion)

$= (6-2) + (4-3) + (4-9) = 0.$

8. (continued)

$\therefore$ det(A) = 0 and so there are non-trivial solutions of this homogeneous system i.e. solutions that have $x \neq 0, y \neq 0, z \neq 0$,

So suppose that $z = k$, first two equations give ...

$x - y + k = 0$ ①

$2x + 3y + k = 0$ ②. Then, ② − 2×① gives

$5y - k = 0 \quad \therefore y = k/5$

Substitute in ① to find that $x = -k + \frac{k}{5} = -\frac{4k}{5}$

(Third equation gives $-3\left(\frac{4}{5}k\right) + \frac{2}{5}k + 2k = 0$, thus these values of x and y are consistent with the third equation.)

9.

$$\begin{pmatrix}1 & 3 & 4\\ 2 & -1 & 1\\ 3 & 2 & 5\\ 5 & 15 & 20\end{pmatrix} \to \begin{pmatrix}1 & 3 & 4\\ 0 & -7 & -7\\ 3 & 2 & 5\\ 5 & 15 & 20\end{pmatrix}$$ by adding 2nd row to (-2) times first row i.e. $r_2 \to r_2 - 2r_1$

$$\to \begin{pmatrix}1 & 3 & 4\\ 0 & -7 & -7\\ 0 & -7 & -7\\ 5 & 15 & 20\end{pmatrix}$$ by adding to 3rd row (-3) times first row i.e. $r_3 \to r_3 - 3r_1$

$$\to \begin{pmatrix}1 & 3 & 4\\ 0 & -7 & -7\\ 0 & -7 & -7\\ 0 & 0 & 0\end{pmatrix}$$ by adding to 4th row (-5) times first row i.e. $r_4 \to r_4 - 5r_1$

$$\to \begin{pmatrix}1 & 3 & 4\\ 0 & -7 & -7\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$ by adding to 3rd row (-1) times second row i.e. $r_3 \to r_3 - r_2$

- This is in echelon form because there are only zeroes below the main diagonal i.e. it is of the form $\begin{bmatrix}* & * & *\\ 0 & * & *\\ 0 & 0 & *\\ 0 & 0 & 0\end{bmatrix}$

9. (continued)

$\therefore$ rank of $A = 2$

<u>Note</u> This follows from the fact that :

number of non-zero rows of A in echelon form

$=$ rank of A

$=$ number of linearly independent row vectors

10.

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & -3 & -12 \\ 3 & -2 & -4 & -2 \end{pmatrix}$$ adding to 2nd row (-2) times first row

i.e. $r_2 \rightarrow r_2 - 2r_1$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & -3 & -12 \\ 0 & -8 & -7 & -11 \end{pmatrix}$$ adding to 3rd row (-3) times first row

i.e. $r_3 \rightarrow r_3 - 3r_1$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & -8 & -7 & -11 \end{pmatrix}$$ by multiplying 2nd row by (-1)

i.e. $r_2 \rightarrow -r_2$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & 0 & 17 & 85 \end{pmatrix}$$ by adding to 3rd row (8) times 2nd row

i.e. $r_3 \rightarrow r_3 + 8r_2$

This is in echelon form i.e. $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$, with 3 non-zero rows.

$\therefore$ Rank $= 3$

11.

- $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$

$= 1\{-2+4\} - 2\{4+3\} + 3\{8+3\}$

$= 2 - 14 + 33 = 35 - 14 = 21$

- $\begin{vmatrix} 2 & 0 & -3 \\ 1 & 4 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & -3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix}$

(Using 1st row is faster)

$= 2\{8-2\} - \{3\} - \{12\} = 12 - 3 - 12 = -3$

- $\begin{vmatrix} 4 & -2 & 1 \\ 5 & 0 & -1 \\ 2 & 3 & -3 \end{vmatrix} = 4\begin{vmatrix} 0 & -1 \\ 3 & -3 \end{vmatrix} - 5\begin{vmatrix} -2 & 1 \\ 3 & -3 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix}$

(OR why not use middle row or column to exploit the zero?)

$= 4\{3\} - 5\{6-3\} + 2\{2\} = 12 - 15 + 4 = 1$

12. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = \sum_{\substack{\text{row} \\ \text{or column}}} a_{ij}A_{ij}$ where A_{ij} = cofactor = signed minor

Sign table is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$. Use row 1 for example.

$A_{11} = +d$ i.e. $\begin{matrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{matrix}$, minor is $|d| = d$

$A_{12} = -c$ i.e. $\begin{matrix} \cancel{a} & \cancel{b} \\ c & \cancel{d} \end{matrix}$, minor is $|c| = c$

$\therefore |A| = a_{11}A_{11} + a_{12}A_{12}$

$= ad + b.(-c)$

$= ad - bc$

You can check that this works for any row or column of A if you like.

13.

$$D=\begin{vmatrix}2&-3\\3&-1\end{vmatrix}=-2+9=7 \quad \leftarrow \quad \begin{bmatrix}2&-3\\3&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\1\end{bmatrix}$$

$$x=\frac{\begin{vmatrix}4&-3\\1&-1\end{vmatrix}}{7}=-\frac{1}{7},\quad y=\frac{\begin{vmatrix}2&4\\3&1\end{vmatrix}}{7}=-\frac{10}{7}$$

(Cramer's rule)

14. (a)

$$\begin{pmatrix}5&-4\\2&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix}$$

$$\therefore x=\frac{\begin{vmatrix}3&-4\\7&3\end{vmatrix}}{\begin{vmatrix}5&-4\\2&3\end{vmatrix}},\quad y=\frac{\begin{vmatrix}5&3\\2&7\end{vmatrix}}{\begin{vmatrix}5&-4\\2&3\end{vmatrix}}$$

$$x=\frac{9+28}{15+8}=\frac{37}{23},\ y=\frac{35-6}{23}=29/23$$

14. (b)

$$\begin{pmatrix}2&3&-2\\1&1&-1\\3&-5&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$$

$$\therefore x=\frac{\begin{vmatrix}4&3&-2\\2&1&-1\\0&-5&3\end{vmatrix}}{\begin{vmatrix}2&3&-2\\1&1&-1\\3&-5&3\end{vmatrix}};\quad D=\begin{vmatrix}2&3&-2\\1&1&-1\\3&-5&3\end{vmatrix}=2\begin{vmatrix}1&-1\\-5&3\end{vmatrix}-\begin{vmatrix}3&-2\\-5&3\end{vmatrix}+\begin{vmatrix}3&-2\\1&-1\end{vmatrix}3$$

$$\therefore\ D=2\{3-5\}-\{9-10\}+3\{-3+2\}=-4+1-3=-6$$

14. (b) [continued]

- $$x=\frac{\begin{vmatrix}4&3&-2\\2&1&-1\\0&-5&3\end{vmatrix}}{D}=\left\{4\begin{vmatrix}1&-1\\-5&3\end{vmatrix}-2\begin{vmatrix}3&-2\\-5&3\end{vmatrix}\right\}\div D$$
$$=\frac{4\{3-5\}-2\{9-10\}}{D}=\frac{-8+2}{D}=\frac{-6}{D}=1$$

- $$y=\frac{\begin{vmatrix}2&4&-2\\1&2&-1\\3&0&3\end{vmatrix}}{D}=\frac{-4\{3+3\}+2\{6+6\}}{D}$$
$$=-\frac{24+24}{D}=0 \quad \therefore y=0$$

- $$z=\frac{\begin{vmatrix}2&3&4\\1&1&2\\3&-5&0\end{vmatrix}}{D}=\frac{4\begin{vmatrix}1&1\\3&-5\end{vmatrix}-2\begin{vmatrix}2&3\\3&-5\end{vmatrix}}{D}$$
$$=-\frac{4\{5-3\}-2\{-10-9\}}{D}$$
$$=\frac{4(-18)-2(-19)}{D}=\frac{-72+78}{D}=\frac{6}{(-6)}$$
$$\therefore z=-1$$

14. (c)

$$\begin{pmatrix}3&-2&0\\0&3&2\\2&0&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}7\\6\\1\end{pmatrix}$$

$$D=3\begin{vmatrix}3&2\\0&3\end{vmatrix}+2\begin{vmatrix}0&2\\2&3\end{vmatrix}=3\{9\}+2\{-4\}=19$$

- $$x=\frac{\begin{vmatrix}7&-2&0\\6&3&2\\1&0&3\end{vmatrix}}{D}=\frac{+2\begin{vmatrix}6&2\\1&3\end{vmatrix}+3\begin{vmatrix}7&0\\1&3\end{vmatrix}}{D}=\frac{2\{18-2\}+3(21)}{35}$$
$$=\frac{32+63}{19}=\frac{95}{19}=5$$

14. (c) [continued]

- $$y=\frac{\begin{vmatrix}3&7&0\\0&6&2\\2&1&3\end{vmatrix}}{D}=\frac{3\begin{vmatrix}6&2\\1&3\end{vmatrix}+2\begin{vmatrix}7&0\\6&2\end{vmatrix}}{19}$$
$$=\frac{3\{16\}+28}{19}=\frac{48+28}{19}=\frac{76}{19}=4$$

- $$z=\frac{\begin{vmatrix}3&-2&7\\0&3&6\\2&0&1\end{vmatrix}}{19}=\frac{3\begin{vmatrix}3&6\\0&1\end{vmatrix}+2\begin{vmatrix}-2&7\\3&6\end{vmatrix}}{D}=\frac{3(3)+2\{-12-21\}}{19}=\frac{9-66}{19}=\frac{-57}{19}=-3$$

14. (d)

$$\begin{pmatrix}3&2&2\\1&-4&2\\2&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\4\\2\end{pmatrix}$$

$$D=3\begin{vmatrix}-4&2\\1&1\end{vmatrix}-\begin{vmatrix}2&2\\1&1\end{vmatrix}+2\begin{vmatrix}2&2\\-4&2\end{vmatrix}$$
$$=3\{-4-2\}-\{2-2\}+2\{4+8\}$$
$$=-18+24=6$$

- $$x=\frac{\begin{vmatrix}3&2&2\\4&-4&2\\2&1&1\end{vmatrix}}{6}=\frac{3\begin{vmatrix}-4&2\\1&1\end{vmatrix}-4\begin{vmatrix}2&2\\1&1\end{vmatrix}+2\begin{vmatrix}2&2\\-4&2\end{vmatrix}}{6}$$

$$\therefore\ x=\frac{3\{-4-2\}-4\{2-2\}+2\{4+8\}}{6}$$

$$\therefore\ x=\frac{-18+24}{6}=1$$

14. (d) [continued]

- $$y=\frac{\begin{vmatrix}3&3&2\\1&4&2\\2&2&1\end{vmatrix}}{D}=\frac{1}{6}\left\{3\begin{vmatrix}4&2\\2&1\end{vmatrix}-\begin{vmatrix}3&2\\2&1\end{vmatrix}+2\begin{vmatrix}3&2\\4&2\end{vmatrix}\right\}=\frac{1}{6}\{0+1-4\}=-\frac{1}{2}$$

- $$z=\frac{\begin{vmatrix}3&2&3\\1&-4&4\\2&1&2\end{vmatrix}}{6}=\frac{1}{6}\left[3\begin{vmatrix}-4&4\\1&2\end{vmatrix}-\begin{vmatrix}2&3\\1&2\end{vmatrix}+2\begin{vmatrix}2&3\\-4&4\end{vmatrix}\right]$$

$$=\frac{1}{6}\{3(-8-4)-(4-3)+2(8+12)\}$$
$$=\frac{1}{6}\{-36-1+40\}=\frac{3}{6}=\frac{1}{2}.$$

15. $A\underset{\sim}{x}=\underset{\sim}{b}$ where $A=\begin{bmatrix}5&8&1\\0&2&1\\4&3&-1\end{bmatrix}$, $\underset{\sim}{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$

$|A|=5A_{11}+4A_{31}=5\begin{vmatrix}2&1\\3&-1\end{vmatrix}+4\begin{vmatrix}8&1\\2&1\end{vmatrix}=5.(-5)+4.(6)=\underline{-1}$
(down column 1)

cofactor A_{ij} is the signed minor
eg. $\underset{\text{row 1, column 2}}{A_{12}}=-\begin{vmatrix}0&1\\4&-1\end{vmatrix}=+4.$ $\begin{bmatrix}+&-&+\\-&+&-\\+&-&+\end{bmatrix}$ signs

The rest are $A_{21}=-\begin{vmatrix}8&1\\3&-1\end{vmatrix}=11$, $A_{13}=+\begin{vmatrix}0&2\\4&3\end{vmatrix}=-8$

$A_{22}=+\begin{vmatrix}5&1\\4&-1\end{vmatrix}=-9$, $A_{23}=-\begin{vmatrix}5&8\\4&3\end{vmatrix}=17$

$A_{32}=-\begin{vmatrix}5&1\\0&1\end{vmatrix}=-5$, $A_{33}=+\begin{vmatrix}5&8\\0&2\end{vmatrix}=10$

$\Rightarrow$ $C=$ matrix of cofactors $=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}=\begin{bmatrix}-5&4&-8\\11&-9&17\\6&-5&10\end{bmatrix}$

$\Rightarrow$

$$A^{-1}=\frac{1}{|A|}C^T=-1.\begin{bmatrix}-5&11&6\\4&-9&-5\\-8&17&10\end{bmatrix}=\begin{bmatrix}5&-11&-6\\-4&9&5\\8&-17&10\end{bmatrix}.$$

where $C^T=$ the transpose of C (i.e. interchanged rows and columns).

From $A\underset{\sim}{x}=\underset{\sim}{b}$, $A^{-1}A\underset{\sim}{x}=A^{-1}\underset{\sim}{b}\Rightarrow\underset{\sim}{x}=A^{-1}\underset{\sim}{b}=\begin{pmatrix}5&-11&-6\\-4&9&5\\8&-17&-10\end{pmatrix}\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}3\\-2\\3\end{pmatrix}\equiv\begin{pmatrix}x\\y\\z\end{pmatrix}$

i.e. $x=3$, $y=-2$, $z=3$

16.(a) $A = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{pmatrix}$

Cofactors

$A_{11} = +\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7$, $A_{12} = -\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} = -14$, $A_{13} = +\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = -7$

$A_{21} = -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} = 6$, $A_{22} = +\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = -2$, $A_{23} = -\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} = -11$

$A_{31} = +\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 2$, $A_{32} = -\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = 11$, $A_{33} = +\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 8$

$\therefore C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{pmatrix}$, $|A| = 3A_{11} - 2A_{12} + 2A_{13}$

$= 3.7 + 2.14 - 2.7 = 21 + 28 - 14$.

i.e. $\underline{|A| = 35}$

$A^{-1} = \frac{1}{|A|} C^T = \underline{\frac{1}{35}\begin{pmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{pmatrix}}$; $AA^{-1} = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{pmatrix}\frac{1}{35} = \begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix}\frac{1}{35}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

16.(b) $A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{pmatrix}$

Cofactors $A_{11} = +\begin{vmatrix} 2 & 1 \\ -2 & -5 \end{vmatrix} = -8$, $A_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} = 7$, $A_{13} = +\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -6$,

$A_{21} = -\begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1$, $A_{22} = +\begin{vmatrix} 2 & -3 \\ 2 & -5 \end{vmatrix} = -4$, $A_{23} = -\begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$,

$A_{31} = +\begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} = 5$, $A_{32} = -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$, $A_{33} = +\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$.

$\therefore$ Matrix of cofactors, $C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} -8 & 7 & -6 \\ 1 & -4 & 2 \\ 5 & -5 & 5 \end{pmatrix}$

$|A| = 2A_{11} - 1.A_{12} - 3.A_{13} = -16 - 7 + 18 = -5$.

$A^{-1} = \frac{1}{|A|} C^T = -\frac{1}{5}\begin{pmatrix} -8 & 1 & 5 \\ 7 & -4 & -5 \\ -6 & 2 & 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 8 & -1 & -5 \\ -7 & 4 & 5 \\ 6 & -2 & -5 \end{pmatrix}$

Check: $AA^{-1} = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 8 & -1 & -5 \\ -7 & 4 & 5 \\ 6 & -2 & -5 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} 16+7-18 & -2-4+6 & -10-5+15 \\ 8-14+6 & -1+8-2 & -5+10-5 \\ 16+14-30 & -2-8+10 & -10-10+25 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{I}$.

17. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad - bc$.

Matrix of cofactors, $C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$: sign table is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$A_{11} = +d$, $A_{12} = -c$, $A_{21} = -b$, $A_{22} = +a$

i.e. $C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$. $A^{-1} = \frac{1}{|A|} C^T = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Verify that $AA^{-1} = I$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{(ad-bc)}$

$$= \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+ad \end{bmatrix} \frac{1}{(ad-bc)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

18. Note: you merely place **A** and **I** side by side – not multiply them – hence the notation ...

$$\mathbf{A} \,|\, \mathbf{I}$$

left-hand side   right-hand side

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, Now work on column 1 of A first.

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$ by adding to 2nd row (-3) times the first row "$r_3 \to r_3 - 3r_1$"

$\to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$ by multiplying 2nd row by $\left(-\frac{1}{2}\right)$, "$r_2 \to -\frac{1}{2} r_2$"

$\to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$ by adding to 1st row (-2) times 2nd row (i.e. $r_1 \to r_1 - 2r_2$) i.e. $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$. Consider $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$ and work on column 1 first.

$r_2 \to r_2 + 2r_1$, $r_3 \to r_3 - r_1$ gives $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \end{array}\right]$. Now do column 2, $r_3 \to r_3 + r_2$ gives $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$.

Finally, do column 3 : $r_1 \to r_1 + r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ and $r_2 \to r_2 + 2r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ i.e. $B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

19. Combined coefficient matrix is

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{array}{l} r_1 \leftrightarrow r_2 \\ r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - 2r_2 \end{array} \longrightarrow \left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -5 & -5 & 1 & -2 & 0 \\ 0 & -6 & -7 & 0 & -2 & 1 \end{array}\right]$$

$$\to \quad r_2 \to r_2 - r_3 \to \left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & -6 & -7 & 0 & -2 & 1 \end{array}\right] \to r_3 \to r_3 + 6r_2 \left[\begin{array}{cccccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 5 & 6 & -2 & -5 \end{array}\right]$$

$$\to \begin{array}{l} r_1 \to r_1 - 2r_3 \\ r_3 \to r_3/5 \end{array} \to \left[\begin{array}{cccccc} 1 & 0 & -3 & -2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & \underline{1} & \frac{6}{5} & -\frac{2}{5} & -1 \end{array}\right] \to \begin{array}{l} r_1 \to r_1 + 3r_3 \\ r_2 \to r_2 - 2r_3 \end{array} \to \left[\begin{array}{cccccc} 1 & 0 & 0 & 8/5 & -1/5 & -1 \\ 0 & 1 & 0 & -7/5 & 4/5 & 1 \\ 0 & 0 & 1 & 6/5 & -2/5 & -1 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{A^{-1}}$$

$\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$ i.e. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 & -1 & -5 \\ -7 & 4 & 5 \\ 6 & -2 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ i.e. $x_1 = -1$, $x_2 = 3$, $x_3 = -2$ //

20.

$$\mathbf{Ax} = \begin{pmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

Thus $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ and $\lambda = 0$ is an eigenvalue.

21.

$$\mathbf{Ax} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

for $\mathbf{x}$ to be an eigenvector of $\mathbf{A}$ there must exist a scalar λ such that $\mathbf{Ax} = \lambda\mathbf{x}$ i.e.

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$$

- it is easily verified that no such λ exists
- hence $\mathbf{x}$ is not an eigenvalue of $\mathbf{A}$.

(i.e. we would need <u>both</u> $3 = \lambda$ <u>and</u> $7 = \lambda$)

22.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$\therefore$ det $(\mathbf{A} - \lambda\mathbf{I}) = 0$ gives

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)(4-\lambda) - 6 = 0$

$\therefore 4 - \lambda - 4\lambda + \lambda^2 - 6 = 0$

$\therefore \lambda^2 - 5\lambda - 2 = 0$

$$\lambda_{1,2} = \frac{5}{2} \pm \frac{1}{2}\sqrt{25+8} = \frac{5}{2} \pm \frac{1}{2}\sqrt{33}$$

23.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}$$

det $(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(1-\lambda) + 2 = \lambda^2 - 2\lambda + 3$

23. (continued)

Hence,

$$\lambda^2 - 2\lambda + 3 = 0$$

$\therefore \quad \lambda_{1,2} = 1 \pm \frac{1}{2}\sqrt{4-12}$

$\therefore \quad \lambda_1 = 1 + \frac{i}{2}\sqrt{8} = 1 + i\sqrt{2}$

$\lambda_2 = 1 - i\sqrt{2}$

- these are the eigenvalues of **A**.

24. $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} t-\lambda & 2t \\ 2t & -t-\lambda \end{pmatrix}$

$\det(\mathbf{A} - \lambda\mathbf{I}) = (t-\lambda)(-t-\lambda) - 4t^2 = \lambda^2 - 5t^2$

The characteristic equation is therefore

$$\lambda^2 - 5t^2 = 0$$

The eigenvalues are $\lambda_1 = \sqrt{5}t,\ \lambda_2 = -\sqrt{5}t$.

Note: if **A** depends upon a parameter (in this case the parameter is t) then the eigenvalues may also depend on the parameter.

25. $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2-\lambda & -1 & 1 \\ 3 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix}$ Then, Laplace expansion along row 3 gives $\det(A - \lambda I) = 0 - 0 + (1-\lambda)\left[(2-\lambda)(-2-\lambda) + 3\right]$

i.e. $\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)[(2-\lambda)(-2-\lambda)+3] = (1-\lambda)(\lambda^2-1)$

The characteristic equation is

$$(1-\lambda)(\lambda^2-1) = 0$$

The eigenvalues are $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = -1$.

26. (a) $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$: Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2 = 0$$

i.e. $6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

i.e. $\lambda^2 - 5\lambda + 4 = 0$

By inspection, $\left.\begin{matrix} \lambda_1 + \lambda_2 = +5 \\ \lambda_1\lambda_2 = 4 \end{matrix}\right\} \rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 4. \end{matrix}$

Eigenvectors Since $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, these satisfy $A\underset{\sim}{x} - \lambda I \underset{\sim}{x} = \underset{\sim}{0}$

i.e. $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

i.e. $\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Denote $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ as the eigenvector associated with λ_1.

Then, $\begin{pmatrix} 3-\lambda_1 & 2 \\ 1 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix} (3-\lambda_1)x_1 + 2y_1 = 0 \\ x_1 + (2-\lambda_1)y_1 = 0 \end{matrix}$

i.e. $\begin{matrix} (3-1)x_1 + 2y_1 = 0 \\ x_1 + (2-1)y_1 = 0 \end{matrix}$ (since $\lambda_1 = 1$)

i.e. $\left.\begin{matrix} 2x_1 + 2y_1 = 0 \\ x_1 + y_1 = 0 \end{matrix}\right\} \Rightarrow x_1 = -y_1$

Eigenvectors are only defined in terms of the ratio of the components (to within an undetermined scalar)

$\therefore \quad \underset{\sim}{x}_1 = \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, α = undetermined scalar.

26. (a) continued...

$$\begin{pmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Denote $\underset{\sim}{x_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ as the eigenvector associated with $\lambda_2 = 4$.

Then,

$$\begin{pmatrix} 3-\lambda_2 & 2 \\ 1 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} (3-\lambda_2)x_2 + 2y_2 &= 0 \\ x_2 + (2-\lambda_2)y_2 &= 0 \end{aligned}$$

$\therefore$ i.e. $$\begin{aligned} (3-4)x_2 + 2y_2 &= 0 \\ x_2 + (2-4)y_2 &= 0 \quad (\text{since } \lambda_2 = 4) \end{aligned}$$

i.e. $$\begin{aligned} -x_2 + 2y_2 &= 0 \\ x_2 - 2y_2 &= 0 \end{aligned}$$

Both equations imply $x_2 = 2y_2$.

$\therefore \quad \underset{\sim}{x_2} = \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where β is an undetermined scalar.

26. (b) $A = \begin{bmatrix} 8 & -2 \\ 4 & 2 \end{bmatrix}$. Eigenvalues are given by the characteristic equation: $\det|A - \lambda I| = 0$

Here, $\begin{vmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{vmatrix} = 0$

i.e. $(8-\lambda)(2-\lambda) + 8 = 0$

i.e. $16 - 2\lambda - 8\lambda + \lambda^2 + 8 = 0$

i.e. $\lambda^2 - 10\lambda + 24 = 0$

i.e. $$\left.\begin{aligned} \lambda_1 + \lambda_2 &= +10 \\ \lambda_1\lambda_2 &= 24, \text{ by inspection.} \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= 4 \\ \lambda_2 &= 6. \end{aligned}$$

26. (b) continued... Eigenvectors $\underset{\sim}{x_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{x_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Satisfy $$\begin{pmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = 4$

$$\begin{pmatrix} 8-\lambda_1 & -2 \\ 4 & 2-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} (8-\lambda_1)x_1 - 2y_1 &= 0 \\ 4x_1 + (2-\lambda_1)y_1 &= 0 \end{aligned}$$

i.e. $$\begin{aligned} 4x_1 - 2y_1 &= 0 \\ 4x_1 - 2y_1 &= 0 \quad (\text{since } \lambda_1 = 4) \end{aligned}$$

i.e. $2x_1 = y_1$.

$\therefore \quad \underset{\sim}{x_1} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, α undetermined scalar.

$\lambda_2 = 6$

$$\begin{pmatrix} 8-\lambda_2 & -2 \\ 4 & 2-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} (8-\lambda_2)x_2 - 2y_2 &= 0 \\ 4x_2 + (2-\lambda_2)y_2 &= 0 \end{aligned}$$

i.e. $$\begin{aligned} 2x_2 - 2y_2 &= 0 \\ 4x_2 - 4y_2 &= 0 \quad (\text{since } \lambda_2 = 6) \end{aligned}$$

i.e. $x_2 = y_2$.

$\therefore \quad \underset{\sim}{x_2} = \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

where β is an undetermined scalar.

26. (c) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$: Eigenvalues are given by $\det(A-\lambda I)=0$

where $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)(3-\lambda)-8$$
$$= 3-3\lambda-\lambda+\lambda^2-8$$
$$= \lambda^2-4\lambda-5$$

$|A-\lambda I|=0$ thus gives $\lambda^2-4\lambda-5=0$

Solution by inspection: $\left.\begin{matrix}\lambda_1+\lambda_2=4 \\ \lambda_1\lambda_2=-5\end{matrix}\right\} \Rightarrow \begin{matrix}\lambda_1=-1 \\ \lambda_2=5\end{matrix}$.

Let eigenvectors be $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

These satisfy $\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda_1=-1$ $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix}2x_1+2y_1=0 \\ 4x_1+4y_1=0\end{matrix}$ i.e. $x_1=-y_1$

$\therefore \underset{\sim}{x}_1 = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, α undetermined scalar.

$\lambda_2=5$ $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $\begin{matrix}-4x_2+2y_2=0 \\ 4x_2-2y_2=0\end{matrix}$ i.e. $2x_2=y_2$

$\therefore \underset{\sim}{x}_2 = \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, β an undetermined scalar.

26. (d) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$. Eigenvalues from characteristic equation: $\det(A-\lambda I)=0$

where $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 5 \\ 0 & -1 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ -1 & -2-\lambda \end{vmatrix} + 0 + 0$

(expanding along row 1)

$\therefore (2-\lambda)\left[(2-\lambda)(-2-\lambda)+5\right]=0$

i.e. $(2-\lambda)\left[-4+2\lambda-2\lambda+\lambda^2+5\right]=0$ i.e. $(2-\lambda)\left[\lambda^2+1\right]=0$

i.e. $\lambda_1=2$ and $\lambda_{2,3}^2=-1$ i.e. $\lambda_1=2$, $\lambda_2=i$, $\lambda_3=-i$.

$\{$recall that $(-i)(-i)=(-1)^2i^2=i^2=-1\}$

Denote eigenvectors $\underset{\sim}{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $\underset{\sim}{x}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, $\underset{\sim}{x}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$

In each case $\begin{pmatrix} \underset{\sim}{x}\to\underset{\sim}{x}_1, & \lambda\to\lambda_1 \\ \underset{\sim}{x}\to\underset{\sim}{x}_2, & \lambda\to\lambda_2 \\ \underset{\sim}{x}\to\underset{\sim}{x}_3, & \lambda\to\lambda_3 \end{pmatrix}$, we have $\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 5 \\ 0 & -1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e. $\begin{matrix}(2-\lambda)x+0+0=0 \\ 0+(2-\lambda)y+5z=0 \\ 0-y-(2+\lambda)z=0\end{matrix}$ i.e. $\begin{matrix}(2-\lambda)x=0 \\ (2-\lambda)y+5z=0 \\ y+(2+\lambda)z=0\end{matrix}$ $(*)$

$\lambda_1=2$ $\begin{matrix}(2-2)x_1=0 \\ (2-2)y_1+5z_1=0 \\ y_1+(2+2)z_1=0\end{matrix}$ i.e. $\begin{matrix}0=0 \\ 5z_1=0 \\ y_1+4z_1=0\end{matrix}$ i.e. $\begin{matrix}x_1 \text{ undetermined} \\ z_1=0 \\ y_1=0\end{matrix}$

(using $z_1=0$)

26.(d) continued ...

$\lambda_1 = 2$ (continued) $\quad \therefore \quad \underset{\sim}{x}_1 = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, α undetermined scalar.

$\lambda_2 = i$ $\quad$ (*) gives

$$(2-i)x_2 = 0 \qquad x_2 = 0$$
$$(2-i)y_2 + 5z_2 = 0 \qquad \text{i.e.} \quad y_2 + \left(\frac{5}{2-i}\right)z_2 = 0$$
$$y_2 + (2+i)z_2 = 0 \qquad y_2 + (2+i)z_2 = 0$$

{Note: 2^{nd} equation gives $y_2 + \frac{5(2+i)}{(2-i)(2+i)}z_2 = y_2 + \frac{5(2+i)}{4+1}z_2 = 0$, i.e. 3^{rd} equation.}

i.e. $\quad x_2 = 0$
$\quad y_2 = -(2+i)z_2 \qquad \therefore \quad \underset{\sim}{x}_2 = \beta \begin{pmatrix} 0 \\ -(2+i) \\ 1 \end{pmatrix}$, β a scalar.

$\lambda_3 = -i$ $\quad$ (*) gives

$$(2+i)x_3 = 0 \qquad x_3 = 0$$
$$(2+i)y_3 + 5z_3 = 0 \qquad \text{i.e.} \quad y_3 + \frac{5(2-i)}{(2+i)(2-i)}z_3 = 0$$
$$y_3 + (2-i)z_3 = 0 \qquad y_3 + (2-i)z_3 = 0$$

{Again, the 2^{nd} equation reduces to the 3^{rd} equation}

i.e. $\quad x_3 = 0$
$\quad y_3 = -(2-i)z_3$
$\quad\quad = (i-2)z_3. \qquad \therefore \quad \underset{\sim}{x}_3 = \gamma \begin{pmatrix} 0 \\ i-2 \\ 1 \end{pmatrix}$,

where γ is an undetermined scalar.

27. $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The characteristic equation is $\quad \det(A - \lambda I) = 0$

$$\text{i.e.} \quad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\text{i.e.} \quad (a-\lambda)(d-\lambda) - bc = 0$$
$$\text{i.e.} \quad ad - \lambda d - \lambda a + \lambda^2 - bc = 0$$
$$\text{i.e.} \quad \lambda^2 - \lambda a - \lambda d + ad - bc = 0$$
$$\text{i.e.} \quad \lambda^2 - (a+d)\lambda + ad - bc = 0$$
$$\text{i.e.} \quad \lambda^2 - tr(A)\lambda + \det(A) = 0,$$

where $tr(A) = a+d$ (sum of the diagonal elements of A)

$\det(A) = ad - bc$.