DR GRAHAM McDONALD

MATHEMATICAL METHODS
AND APPLICATIONS

(AKA THEORETICAL PHYSICS I)

TUTORIAL 4

This Tutorial includes:

- exam-type questions for o.d.e.'s
- exam-type questions for p.d.e.'s
  (don't miss the important technique of separation of variables covered in the last five exercises)

Exercises

ODE's

1. A circuit consists of a resistance R and an inductance L connected in series to a battery of constant voltage V. By considering the voltage dropped across R and L, one arrives at an ordinary differential equation for the current $I(t)$, where t is time:

$$L\frac{dI}{dt} + RI = V.$$

Show that the general solution of this differential equation is given by

$$I(t) = I(0)e^{-\frac{R}{L}t} + \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

Find also the particular solution when $I(0) = 0$ and identify the long-term steady-state and transient components of this solution, giving a physical interpretation of your results.

2. A circuit consists of a resistance R and a capacitance C connected in series to a battery of constant voltage V. By considering the voltage dropped across R and C, one arrives at an ordinary differential equation for the charge stored $Q(t)$, where t is time:

$$R\frac{dQ}{dt} + \frac{Q}{C} = V.$$

Show that the general solution of this differential equation is given by

$$Q(t) = Q(0)e^{-t/RC} + VC\left(1 - e^{-t/RC}\right).$$

Find also the particular solution when $Q(0) = 0$ and identify the long-term steady-state and transient components of this solution, giving a physical interpretation of your results.

3. Exam 1999/2000

(a) Use the integrating factor method to show that the solution of the differential equation

$$\frac{dx}{dt} + \frac{x}{\tau} = Ae^{i\Omega t}$$

is

$$x(t) = x(0)e^{-t/\tau} + B\left(e^{i\Omega t} - e^{-t/\tau}\right),$$

where $B = A/(i\Omega + 1/\tau)$ and A, τ and Ω are constants.

(15 marks)

(b) Distinguish between the transient and the long-term steady-state components of the solution $x(t)$ in part (a). If $T = 2\pi/\Omega$ is the period of the driving term $Ae^{i\Omega t}$, what is the phase relationship between the steady-state component and the driving term when $T \gg \tau$ and when $T \ll \tau$?

(10 marks)

4. Exam 1998/1999

(a) Radium disintegrates with a comparatively long half-life into radon, which in turn disintegrates to polonium, with a comparatively short half-life. If N_2 is the amount of radon present at any instant, t, N_1 is the amount of radium at any instant t and λ_2, λ_1 are their respective disintegration constants, then the rate at which radon accumulates is

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

Show that, using the integrating factor method,

$$N_2 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

where N_0 is the number of radium atoms at $t = 0$ and $N_2 = 0$ at $t = 0$. (20 marks)

— continued —

4. continued

If the amount of radium is assumed to be constant show that the amount of radon present, after a time t, is:

$$N_2 = \frac{\lambda_1}{\lambda_2} N_1\left(1 - e^{-\lambda_2 t}\right).$$

(5 marks)

PDE's

5. Show that $v = f(x+ct)$, where f is an arbitrary differentiable function and c is a constant, is a solution of the equation

$$\frac{\partial v}{\partial x} - \frac{1}{c}\frac{\partial v}{\partial t} = 0.$$

6. Show that $v = F(y-3x)$, where F is an arbitrary differentiable function, is a solution of the equation

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0$$

7. Show that $y = F(2x+5t) + G(2x-5t)$, where F and G are arbitrary differentiable functions, is a solution of the equation

$$4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2}$$

8. Derive the general solution of the partial differential equation

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y .$$

Find the particular solution for which $z(x, y=0) = x^2$ and $z(x=1, y) = \cos y$.

9. Derive the general solution of $t\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial u}{\partial x} = x^2$.

10. Find the general solution of $2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 2u = 0$.

11 Find the general solution of $4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.

12. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 10e^{2x+y}$ (POISSON'S EQUATION).

SEPARATION OF VARIABLES

13. Solve the boundary-value problem

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} , \quad u(0,y) = 8e^{-3y} + 4e^{-5y} .$$

14. Solve the boundary-value problem

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 , \quad u(x,0) = 4e^{-x} .$$

15. Solve the boundary-value problem

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u , \quad u(x,0) = 3e^{-5x} + 2e^{-3x} .$$

16. Solve the boundary-value problem

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} , \quad u(0,t) = 0$$

$$u(\pi,t) = 0$$

$$u(x,0) = 2\sin 3x - 4\sin 5x .$$

17. Solve the boundary-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} , \quad \frac{\partial u}{\partial x}(0,t) = 0$$

$$u(2,t) = 0$$

$$u(x,0) = 8\cos\frac{3\pi x}{4} - 6\cos\frac{9\pi x}{4}$$

→ FULL WORKED SOLUTIONS FOLLOW ←

1. $$\frac{dI}{dt} + \left(\frac{R}{L}\right) I = \frac{V}{L}$$

$$IF = e^{\int P\,dt} = e^{\frac{R}{L}\int dt} = e^{\frac{R}{L}t}$$

Multiply equation

$$e^{\frac{R}{L}t}\frac{dI}{dt} + e^{\frac{R}{L}t}\left(\frac{R}{L}\right) I = e^{\frac{R}{L}t}\left(\frac{V}{L}\right)$$

$$\therefore \frac{d}{dt}\left[e^{\frac{R}{L}t} I\right] = \frac{V}{L} e^{\frac{R}{L}t}$$

$$\therefore e^{\frac{R}{L}t} I = \frac{V}{L}\cdot\frac{L}{R} e^{\frac{R}{L}t} + C \quad \text{(integrating wrt } t\text{)}$$

$$\therefore I(t) = \frac{V}{R} + C e^{-\frac{R}{L}t} \quad \left(\text{dividing through by } e^{\frac{R}{L}t}\right)$$

Determine C physically

at $t=0$ $$I(0) = \frac{V}{R} + C$$

$$\therefore C = I(0) - \frac{V}{R}$$

$\therefore$ General solution is $$I(t) = \frac{V}{R} + \left(I(0) - \frac{V}{R}\right) e^{-\frac{R}{L}t}$$

i.e. $$I(t) = \frac{V}{R} + I(0) e^{-R/L\,t} - \frac{V}{R} e^{-R/L\,t}$$

i.e. $$I(t) = I(0) e^{-R/L\,t} + \frac{V}{R}\left[1 - e^{-R/L\,t}\right].$$

1. continued

Particular solution when $I(0)=0$ i.e. switch closed at $t=0$
i.e. $I=0$ at $t=0$

$$I(t) = \frac{V}{R}\left[1 - e^{-R/L\,t}\right] = \underbrace{\frac{V}{R}}_{\text{steady-state}} - \underbrace{\frac{V}{R} e^{-R/L\,t}}_{\text{transient}}$$

i.e. decays to zero as $t \to \infty$

Ultimately, $I \to \frac{V}{R}$ (Ohm's law) and all voltage dropped across R i.e. voltage is only non-zero across L when current is varying : $V_L = L\frac{dI}{dt}$.

2. $$\frac{dQ}{dt} + \left(\frac{1}{RC}\right) Q = \frac{V}{R}$$

$$IF = e^{\frac{1}{RC}\int dt} = e^{\frac{t}{RC}}$$. Now multiply equation.

$$e^{\frac{t}{RC}}\frac{dQ}{dt} + \frac{e^{t/RC}}{RC} Q = e^{t/RC}\frac{V}{R}$$

i.e. $$\frac{d}{dt}\left[e^{t/RC} Q\right] = e^{t/RC}\cdot\frac{V}{R}$$

Integrate $$e^{t/RC} Q = \frac{V}{R}\cdot\frac{1}{\left(\frac{1}{RC}\right)} e^{t/RC} + A$$

2. continued

i.e. $Q(t) = \frac{V}{R}(RC) + Ae^{-t/RC}$ (multiplying through by $e^{-t/RC}$)

i.e. $Q(t) = VC + Ae^{-t/RC}$

Identify physical character of A

at $t=0$, $Q(0) = VC + A$

$\therefore A = Q(0) - VC$

$\therefore$ General solution is $Q(t) = VC + \left[Q(0) - VC\right]e^{-t/RC}$

i.e. $Q(t) = VC + Q(0)e^{-t/RC} - VCe^{-t/RC}$

i.e. $Q(t) = Q(0)e^{-t/RC} + VC(1 - e^{-t/RC})$.

Particular solution — Uncharged at $t=0$ i.e. $Q(0)=0$ since switch closed

$\therefore$ $Q(t) = VC(1 - e^{-t/RC})$

$Q(t) = \underbrace{VC}_{\text{steady-state}} - \underbrace{VCe^{-t/RC}}_{\text{transient}}$ i.e. $\to 0$ as $t \to \infty$

As $t \to \infty$, all voltage across C since voltage across R is IR

i.e. $\frac{dQ}{dt}R$ and requires time-varying charge.

3. $$\boxed{\frac{dx}{dt} + \frac{x}{\tau} = Ae^{i\Omega t}}$$

driving term

DRIVEN / DAMPED DYNAMICS

where τ, A, Ω are constants.

Sol^{n} We have $\frac{dx}{dt} + P(t)x = Q(t)$

where $P(t) = \frac{1}{\tau}$ and $Q(t) = Ae^{i\Omega t}$.

Integrating factor is $e^{\int P(t)\,dt} = e^{\frac{1}{\tau}\int dt} = e^{t/\tau}$.

Multiply to get $e^{t/\tau}\frac{dx}{dt} + e^{t/\tau}\frac{x}{\tau} = e^{t/\tau}Ae^{i\Omega t}$

i.e. $\frac{d}{dt}\left[e^{t/\tau}.x\right] = e^{t/\tau}Ae^{i\Omega t} = Ae^{(i\Omega + \frac{1}{\tau})t}$

Integrate: $\int_0^t \frac{d}{dt}\left[e^{t/\tau}.x\right]dt = \int_0^t Ae^{(i\Omega + \frac{1}{\tau})t}\,dt$

i.e. $e^{t/\tau}.x(t) - x(0) = A\left[\frac{1}{(i\Omega + \frac{1}{\tau})}e^{(i\Omega + \frac{1}{\tau})t}\right]_0^t$

$= \frac{A}{(i\Omega + \frac{1}{\tau})}\left[e^{(i\Omega + \frac{1}{\tau})t} - 1\right]$

Divide by integrating factor: $\boxed{x(t) = x(0)e^{-t/\tau} + B\left(e^{i\Omega t} - e^{-t/\tau}\right)}$, $B = \frac{A}{i\Omega + \frac{1}{\tau}}$

3. continued

Look at the solution in more detail ...

$$\boxed{x(t) = x(0)e^{-t/\tau} + B\left(e^{i\Omega t} - e^{-t/\tau}\right)}$$

where $B = \dfrac{A}{i\Omega + \frac{1}{\tau}}$.

Terms involving $e^{-t/\tau}$ decay to zero when $t \gg \tau$ and these terms are therefore transient.

This leaves the long-term steady-state component of $x(t)$ as simply $x(t) \to Be^{i\Omega t} = \dfrac{A}{i\Omega+\frac{1}{\tau}}e^{i\Omega t}$ (as $t\to\infty$).

If $T = \frac{2\pi}{\Omega}$ = period of driving term

then $\boxed{\text{driving term} = Ae^{i\frac{2\pi}{T}t}}$

$$\boxed{\text{steady-state component} = \frac{Ae^{i\frac{2\pi}{T}t}}{i\frac{2\pi}{T} + \frac{1}{\tau}}}$$

CONSIDER PHASE RELATIONSHIP BETWEEN STEADY-STATE COMPONENT AND DRIVING TERM.

- $T \gg \tau$, steady-state component $\to \tau A e^{i\frac{2\pi}{T}t}$
  i.e. IN PHASE with driving term
- $T \ll \tau$, steady-state component $\to \dfrac{TA}{i2\pi}e^{i\frac{2\pi}{T}t}$
  i.e. $-i\left(\frac{T}{2\pi}\right)Ae^{i\frac{2\pi}{T}t} = e^{-i\frac{\pi}{2}}\left(\frac{T}{2\pi}\right)Ae^{i\frac{2\pi}{T}t}$
  i.e. 90° $\left(\frac{\pi}{2}\right)$ OUT OF PHASE with driving term

4. N_0 = no. Radium atoms at $t=0$

N_1 = no. Radium atoms at time t

N_2 = no. Radon atoms at time t

λ_1, λ_2 = decay constants for Ra and Rn

For radium $\dfrac{dN_1}{dt} = -\lambda_1 N_1$ i.e. $\displaystyle\int \frac{dN_1}{N_1} = -\lambda_1 \int dt$

i.e. $\ln N_1 = -\lambda_1 t + C$

physical character of C? at $t=0$, $\ln N_1 = C$

$\Rightarrow C = \ln N_0$

$\therefore \ln N_1 = -\lambda_1 t + \ln N_0$

i.e. $\ln N_1 - \ln N_0 = -\lambda_1 t$

i.e. $\ln\left(\dfrac{N_1}{N_0}\right) = -\lambda_1 t$ i.e. $\dfrac{N_1}{N_0} = e^{-\lambda_1 t}$

i.e. $\boxed{N_1 = N_0 e^{-\lambda_1 t}}$

We are given that $\dfrac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$

i.e. $\dfrac{dN_2}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2$

4. continued Standard form

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}$$

i.e. $$\frac{dN_2}{dt} + P(t) N_2 = Q(t)$$

$$IF = e^{\int P dt} = e^{\lambda_2 \int dt} = e^{\lambda_2 t}.$$

Multiply equation

$$e^{\lambda_2 t} \frac{dN_2}{dt} + \lambda_2 e^{\lambda_2 t} N_2 = \lambda_1 N_0 e^{-\lambda_1 t} \cdot e^{\lambda_2 t}$$

i.e. $$\frac{d}{dt}\left[e^{\lambda_2 t} N_2\right] = \lambda_1 N_0 e^{(\lambda_2-\lambda_1)t}$$

Integrate

$$e^{\lambda_2 t} N_2 = \frac{\lambda_1 N_0}{(\lambda_2-\lambda_1)} e^{(\lambda_2-\lambda_1)t} + C$$

general solution

$$N_2(t) = \frac{\lambda_1 N_0}{(\lambda_2-\lambda_1)} e^{(\lambda_2-\lambda_1)t} e^{-\lambda_2 t} + C e^{-\lambda_2 t}$$

(dividing by $e^{\lambda_2 t}$)

i.e. $$N_2(t) = \frac{\lambda_1 N_0}{(\lambda_2-\lambda_1)} e^{-\lambda_1 t} + C e^{-\lambda_2 t}$$

4. continued

physical character of c?

at $t=0$. $$N_2(0) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1} + C$$

$$\therefore C = N_2(0) - \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}$$

But $N_2 = 0$ at $t=0$, so $N_2(0) = 0$

$$\therefore C = -\frac{\lambda_1 N_0}{\lambda_2-\lambda_1}$$

$$\therefore N_2(t) = \left(\frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\right) e^{-\lambda_1 t} + \left(-\frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\right) e^{-\lambda_2 t}$$

i.e. $$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

If amount of radium assumed constant i.e. $N_1(t) \to N_1 = \text{constant}$

then $\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$ i.e. $\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1$

$IF = e^{\lambda_2 \int dt} = e^{\lambda_2 t}$ gives $\frac{d}{dt}\left[e^{\lambda_2 t} N_2\right] = e^{\lambda_2 t} \lambda_1 N_1$

i.e. $e^{\lambda_2 t} N_2 = \frac{e^{\lambda_2 t} \lambda_1 N_1}{\lambda_2} + C$, i.e. $N_2(t) = \frac{\lambda_1}{\lambda_2} N_1 + C e^{-\lambda_2 t}$

$t=0$ $N_2(0) = \frac{\lambda_1}{\lambda_2} N_1 + C$ i.e. $C = N_2(0) - \frac{\lambda_1}{\lambda_2} N_1 = -\frac{\lambda_1}{\lambda_2} N_1$ $\therefore N_2 = \frac{\lambda_1}{\lambda_2} N_1 (1 - e^{-\lambda_2 t})$.

5. Let $u = x+ct$ i.e. $v = f(u)$. Use the chain rule.

$$\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \quad ; \quad \frac{\partial v}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = c\frac{\partial f}{\partial u}$$

$$\therefore \frac{\partial v}{\partial x} - \frac{1}{c}\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} - \frac{1}{c}\left(c\frac{\partial f}{\partial u}\right) = 0.$$

6. $\boxed{v = F(y-3x) \text{ and } \frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0}$

Chain rule (function of a function, u)

Set $u = y-3x$.

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} = F'(u).(-3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = F'(u)(1)$$

$$\therefore \frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = F'(u).(-3) + 3F'(u).(1)$$

$$= -3F'(u) + 3F'(u) = 0.$$

General solution: solves the equation and has the required arbitrary function.

7. $\boxed{y = F(2x+5t) + G(2x-5t) \text{ and } 4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2}}$

Chain rule set $u = 2x+5t$, $v = 2x-5t$

7. continued

$$\frac{\partial y}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial t}$$

$$= F'(u)5 + G'(v)(-5)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(F'(u)5 + G'(v)(-5)\right)$$

$$= 5\frac{\partial}{\partial t}F'(u) - 5\frac{\partial}{\partial t}G'(v)$$

$$= 5\frac{\partial F'(u)}{\partial u}\frac{\partial u}{\partial t} - 5\frac{\partial G'(v)}{\partial v}\frac{\partial v}{\partial t}$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = 5F''(u)(5) - 5G''(v)(-5)$$

$$= 25\left(F''(u) + G''(v)\right).$$

$$\frac{\partial y}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = F'(u)(2) + G'(v)(2)$$

$$= 2\left(F'(u) + G'(v)\right)$$

$$\frac{\partial^2 y}{\partial x^2} = 2\frac{\partial}{\partial x}\left(F'(u) + G'(v)\right)$$

$$= 2\left\{\frac{\partial F'(u)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G'(v)}{\partial v}\frac{\partial v}{\partial x}\right\}$$

$$= 2\left\{F''(u)(2) + G''(v)(2)\right\}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = 4\left\{F''(u) + G''(v)\right\}.$$

7. continued

18

$$\Rightarrow 4\frac{\partial^2 y}{\partial t^2} - 25\frac{\partial^2 y}{\partial x^2} = 4\left\{25\left[F''(u)+G''(v)\right]\right\} - 25\left\{4\left[F''(u)+G''(v)\right]\right\}$$

$$= (100-100)\left(F''(u)+G''(v)\right)$$

$$= 0$$

$\therefore$ General solution, i.e. solves the equation and has the required $\underline{2}$ arbitrary functions (equation is second order).

8. Write as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$.

Integrate with respect to (w.r.t.) x, $\frac{\partial z}{\partial y} = \frac{1}{3}x^3 y + F(y)$,

where $F(y)$ is an arbitrary function of y.

Integrate w.r.t. y, $z(x,y) = \frac{1}{6}x^3y^2 + \int F(y)\,dy + G(x)$,

where $G(x)$ is an arbitrary function of x.

This can be written as $\boxed{z(x,y) = \frac{1}{6}x^3y^2 + H(y) + G(x)}$,

with the required two arbitrary functions.

Requiring that $z(x, y=0) = x^2$ and $z(x=1, y) = \cos y$:

Firstly, $z = x^2 = 0 + H(0) + G(x)$ for $y = 0$

i.e. $G(x) = x^2 - H(0)$

and $\boxed{z = \frac{1}{6}x^3y^2 + H(y) + x^2 - H(0)}$ for any y.

8. continued

19

Secondly, $z = \cos y = \frac{1}{6}y^2 + H(y) + 1 - H(0)$ for $x = 1$

i.e. $H(y) - H(0) = \cos y - \frac{1}{6}y^2 - 1$

Use this to eliminate the arbitrary function $H(y)$ from z

i.e. $\underline{z(x,y) = \frac{1}{6}x^3y^2 + x^2 + \cos y - \frac{1}{6}y^2 - 1}$

This is the required particular solution.

9. Write as $\frac{\partial}{\partial x}\left[t\frac{\partial u}{\partial t} + 2u\right] = x^2$.

Integrate w.r.t. x, $t\frac{\partial u}{\partial t} + 2u = \frac{1}{3}x^3 + F(t)$

i.e. $\frac{\partial u}{\partial t} + \frac{2}{t}u = \frac{1}{3}\frac{x^3}{t} + \frac{F(t)}{t}$.

Integrating factor: $e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$

i.e. $\frac{\partial}{\partial t}(t^2 u) = \frac{1}{3}tx^3 + tF(t)$

Integrate w.r.t. t $t^2 u = \frac{1}{6}t^2x^3 + \int tF(t)\,dt + H(x)$

i.e. $\underline{t^2 u = \frac{1}{6}t^2x^3 + G(t) + H(x)}$

10. $2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 2u$

Set $u = e^{ax+by}$ i.e. $2a + 3b = 2$ i.e. $a = \frac{2-3b}{2}$

and $u = e^{\frac{2-3b}{2}x + by} = e^{x}\underbrace{e^{\frac{b}{2}(-3x+2y)}}_{\text{identify this part as associated with arbitrary } b}$, for any b

General solution is $u = e^{x}F(2y-3x)$.

11. $4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

Set $u = e^{ax+by}$ i.e. $4a^2 - 4ab + b^2 = 0$ i.e. $(2a-b)(2a-b) = 0$

and $b = 2a, 2a$. These repeated roots would give the same arbitrary function contribution to the general solution but the equation is second order and we need two arbitrary functions in the general solution.

As with ordinary differential equations and repeated roots, one finds that either $x\,G(x+2y)$ or $y\,G(x+2y)$ is another solution i.e. multiply the function by the lowest (positive integer) power of an independent variable.

The general solution is $u = F(x+2y) + x\,G(x+2y)$

OR $u = F(x+2y) + y\,G(x+2y)$.

12. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 10e^{2x+y}$

Examine the homogeneous problem first

i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation).

Set $u = e^{ax+by}$ i.e. $a^2 + b^2 = 0$ i.e. $b = \pm ia$

and $e^{ax+by} = e^{ax \pm iay} = e^{a(x \pm iy)}$, for any a.

General solution is thus $u = F(x+iy) + G(x-iy)$.

As with ordinary differential equations, the next step is to use a trial solution that has the same functional form as the right-hand side but with an undetermined coefficients.

Try $u_p = \alpha e^{2x+y}$: $2^2\alpha e^{2x+y} + \alpha e^{2x+y} = 10e^{2x+y}$

i.e. $4\alpha + \alpha = 10$ i.e. $\alpha = 2$

and particular solution is $u_p = 2e^{2x+y}$.

General solution is sum of the complementary solution (with the required two arbitrary functions) and the particular solution

i.e. $u = F(x+iy) + G(x-iy) + 2e^{2x+y}$.

13. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, $u(0,y) = 8e^{-3y} + 4e^{-5y}$.

Separation of variables : $u = X(x)Y(y)$

gives $YX_x = 4XY_y$ (subscript denoting partial derivative)

i.e. $\frac{X_x}{4X} = \frac{Y_y}{Y} = c$ (separation constant)

Each equation can now be treated as an o.d.e. i.e. $\frac{dX}{dx} = 4cX$

and $\frac{dY}{dy} = cY$

Solutions are $X = Ae^{4cx}$, $Y = Be^{cy}$.

A solution is thus $\boxed{u = XY = Ke^{c(4x+y)}}$, $K = AB$.

Boundary condition requires $u(0,y) = 8e^{-3y} + 4e^{-5y}$, so take a superposition of solutions i.e. $u = K_1e^{c_1(4x+y)} + K_2e^{c_2(4x+y)}$

and $u(0,y) = K_1e^{c_1y} + K_2e^{c_2y}$

$\therefore K_1 = 8,\ c_1 = -3,\ K_2 = 4,\ c_2 = -5$

and required solution is

$$u(x,y) = 8e^{-3(4x+y)} + 4e^{-5(4x+y)}.$$

---

14. $\boxed{3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \quad ; \quad u(x,0) = 4e^{-x}}$

Set $u = X(x)Y(y)$ and substitute

$3Y\frac{\partial X}{\partial x} + 2X\frac{\partial Y}{\partial y} = 0$. Now, rearrange in terms of functions of only x only.

i.e. $\frac{3}{X}\frac{\partial X}{\partial x} + \frac{2}{Y}\frac{\partial Y}{\partial y} = 0$ (dividing by XY)

i.e. $\frac{3}{X}\frac{\partial X}{\partial x} = -\frac{2}{Y}\frac{\partial Y}{\partial y}$ Here, LHS = function of only x

RHS = function of only y

$\Rightarrow$ each must equal a constant

i.e. $\frac{3}{X}\frac{\partial X}{\partial x} = c = -\frac{2}{Y}\frac{\partial Y}{\partial y}$, c = 'separation constant'

This gives two <u>o.d.e's</u> (individually each has only one indept. variable)

i.e. $\frac{dX}{dx} = \frac{c}{3}X$ and $\frac{dY}{dy} = -\frac{c}{2}Y$

Solve the ode's

$\int\frac{dX}{X} = \frac{c}{3}\int dx$ $\qquad$ $\int\frac{dY}{Y} = -\frac{c}{2}\int dy$

i.e. $\ln X = \frac{c}{3}x + A$ $\qquad$ i.e. $\ln Y = -\frac{c}{2}y + B$

i.e. $X = e^{\frac{c}{3}x + A}$ $\qquad$ i.e. $Y = e^{-\frac{c}{2}y + B}$

14. continued

$\therefore$ A solution is $u = XY$

i.e. $u = e^{\frac{c}{3}x+A} \cdot e^{-\frac{c}{2}y+B}$

i.e. $u = e^{c\left(\frac{x}{3}-\frac{y}{2}\right)} e^{A+B}$

i.e. $u = e^{c'(2x-3y)} e^{A+B}$, where constant redefined as $c' = c/6$

i.e. $\boxed{u = k e^{c'(2x-3y)}}$, $k = e^{A+B}$.

Apply boundary condition at $y=0$

i.e. $u(x,0) = 4e^{-x}$

$y=0$ give $u = ke^{c'2x} = 4e^{-x}$

$\therefore$ $k=4$ and $2c' = -1$ i.e. $c' = -\frac{1}{2}$

and required solution is

$$u = ke^{c'(2x-3y)} \quad \text{with } k=4,\ c' = -\tfrac{1}{2}$$

i.e. $u = 4e^{-\frac{1}{2}(2x-3y)}$

i.e. $\boxed{u = 4e^{(3y-2x)/2}}$.

15. $\boxed{\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u \quad ; \quad u(x,0) = 3e^{-5x} + 2e^{-3x}}$

Set $u = X(x)\,Y(y)$ and substitute

$Y\dfrac{\partial X}{\partial x} = 2X\dfrac{\partial Y}{\partial y} + XY$ i.e. $\dfrac{1}{X}\dfrac{\partial X}{\partial x} = \dfrac{2}{Y}\dfrac{\partial Y}{\partial y} + 1$

(after dividing by XY).

i.e. $\dfrac{1}{X}\dfrac{\partial X}{\partial x} = c = \dfrac{2}{Y}\dfrac{\partial Y}{\partial y} + 1$, where c = separation constant

i.e. $\dfrac{1}{X}\dfrac{dX}{dx} = c$ and $\dfrac{2}{Y}\dfrac{dY}{dy} + 1 = c$ (ode's)

$\int \dfrac{dX}{X} = c\int dx$

i.e. $\ln X = cx + A$

i.e. $X = e^{cx+A}$

$2\int \dfrac{dY}{Y} = (c-1)\int dy$

i.e. $2\ln Y = (c-1)y + B$

i.e. $\ln Y = \dfrac{(c-1)}{2}y + B'$, ($B' = B/2$)

i.e. $Y = e^{\frac{(c-1)}{2}y + B'}$

$\therefore$ A solution is $u = XY = e^{cx+A} e^{\frac{(c-1)}{2}y+B'}$

i.e. $u = e^{cx+\left(\frac{c-1}{2}\right)y} e^{A+B'}$

15. continued (26)

i.e. $\boxed{u = k e^{c(x+\frac{y}{2})-\frac{y}{2}}}$, where $k = e^{A+B}$ (another constant)

Apply boundary condition at $y=0$

i.e. $u(x,y=0) = 3e^{-5x} + 2e^{-3x}$

$y=0$ gives $u = ke^{cx}$ as a suitable solution.

Superposition principle Since equation is linear, the sum of solutions is also a solution.

Boundary condition suggests taking two terms of the form ke^{cx}

i.e. try $u = k_1 e^{c_1 x} + k_2 e^{c_2 x}$ at $y=0$

whereby $k_1 = 3$, $k_2 = 2$ and $c_1 = -5$, $c_2 = -3$

Then, required solution ($y=0$ and $y \neq 0$) is

$$u = k_1 e^{c_1(x+\frac{y}{2})-\frac{y}{2}} + k_2 e^{c_2(x+\frac{y}{2})-\frac{y}{2}}$$

$$= 3e^{-5(x+\frac{y}{2})-\frac{y}{2}} + 2e^{-3(x+\frac{y}{2})-\frac{y}{2}}$$

$$\therefore \quad u = 3e^{-5x-3y} + 2e^{-3x-2y}.$$

(27)

16. $\boxed{\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2} \; ; \; u(0,t)=0, \; u(\pi,t)=0, \; u(x,0) = 2\sin 3x - 4\sin 5x}$

Set $u = X(x)T(t)$ and substitute

i.e. $X\dfrac{\partial T}{\partial t} = 4T\dfrac{\partial^2 X}{\partial x^2}$ i.e. $\dfrac{1}{T}\dfrac{\partial T}{\partial t} = \dfrac{4}{X}\dfrac{\partial^2 X}{\partial x^2}$

(after dividing by XT).

i.e. $\dfrac{1}{T}\dfrac{\partial T}{\partial t} = c$ i.e. $\dfrac{1}{T}\dfrac{dT}{dt} = c$ and $\dfrac{4}{X}\dfrac{d^2X}{dx^2} = c$

(c = separation constant)

$$\int \frac{dT}{T} = c\int dt$$

i.e. $\ln T = ct + A$

i.e. $T = e^{ct+A}$: suggests that we look for a physical solution that doesn't tend to infinity as $t \to \infty$. Set $c = -\lambda^2$ i.e. choose a negative separation constant.

i.e. $T = e^{-\lambda^2 t + A}$.

While, $\dfrac{4}{X}\dfrac{d^2X}{dx^2} = -\lambda^2$ and $\dfrac{d^2X}{dx^2} = -\dfrac{\lambda^2}{4}X$

i.e. $\dfrac{d^2X}{dx^2} + \dfrac{\lambda^2}{4}X = 0$ Homogeneous 2nd order linear, constant coefficients Set $X = e^{mx}$

$\rightarrow$ Aux. Equ^{n} $\quad m^2 + \frac{\lambda^2}{4} = 0 \quad$ i.e. $m^2 = -\frac{\lambda^2}{4}$

i.e. $m = \pm\sqrt{-\frac{\lambda^2}{4}}$ $\quad$ i.e. $m = \pm\sqrt{-1}\sqrt{\frac{\lambda^2}{4}}$

i.e. $m = \pm i\frac{\lambda}{2}$

$\rightarrow$ complex roots $p \pm iq$ with $p=0$ and $q = \frac{\lambda}{2}$

$\rightarrow$ general sol^{n} of A.E. $\quad X = e^{px}(B_1\cos qx + B_2 \sin qx)$

(have chosen to label constants B_1, B_2 to avoid confusion with the "A" and "C" already defined here)

i.e. $X = e^{0x}\left(B_1\cos\frac{\lambda x}{2} + B_2\sin\frac{\lambda x}{2}\right)$

i.e. $X = B_1\cos\frac{\lambda x}{2} + B_2\sin\frac{\lambda x}{2}$.

$\therefore$ A solution is $u = XT = e^{-\lambda^2 t + A}\left(B_1\cos\frac{\lambda x}{2} + B_2\sin\frac{\lambda x}{2}\right)$

Apply boundary conditions

$u(0,t) = 0$ $\quad$ i.e. $u=0$ when $x=0$

i.e. $0 = e^{-\lambda^2 t + A}(B_1\cos 0 + B_2\sin 0)$

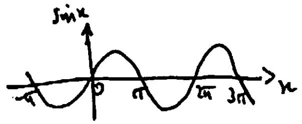
i.e. $0 = B_1\cos 0 + B_2 \sin 0$

i.e. $0 = B_1$

We then have $\quad u = e^{-\lambda^2 t + A} B_2 \sin\frac{\lambda x}{2}$

$u(\pi, t) = 0$. $\quad$ i.e. $u = 0$ when $x = \pi$

i.e. $0 = e^{-\lambda^2 t + A} B_2 \sin\left(\frac{\lambda\pi}{2}\right)$



$\sin x = 0$ when $x = n\pi$ $\quad (n = 0, \pm1, \pm2, \ldots)$

$\therefore \frac{\lambda\pi}{2} = n\pi$ $\quad$ i.e. $\lambda = 2n$

We then have $\quad u = e^{-4n^2 t + A}.B_2 \sin nx$

i.e. $u = e^{A}.B_2 e^{-4n^2 t}\sin nx$

i.e. $u = De^{-4n^2 t}\sin nx$, where D = constant.

Final boundary condition: $u(x, t=0) = 2\sin 3x - 4\sin 5x$

Need superposition principle: $\quad u = D_1 e^{-4n_1^2 t}\sin n_1 x + D_2 e^{-4n_2^2 t}\sin n_2 x$

$u(x, t=0) = D_1 \sin n_1 x + D_2 \sin n_2 x$

$= 2\sin 3x - 4\sin 5x$

$\therefore D_1 = 2,\ D_2 = -4,\ n_1 = 3,\ n_2 = 5$ giving ...

$u(x,t) = 2e^{-4\times 9t}\sin 3x - 4e^{-4\times 25 t}\sin 5x$

i.e. $u(x,t) = 2e^{-36t}\sin 3x - 4e^{-100t}\sin 5x$.

17. $$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \;;\; u_x(0,t)=0,\; u(2,t)=0,\; u(x,0) = 8\cos\frac{3\pi x}{4} - 6\cos\frac{9\pi x}{4}}$$

$u = X(x)\,T(t)$ gives $X\frac{\partial T}{\partial t} = T\frac{\partial^2 X}{\partial x^2}$ i.e. $\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2}$

i.e. $\frac{1}{T}\frac{dT}{dt} = -\lambda^2$ and $\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$ (as in previous example)

$\downarrow$

$T = e^{-\lambda^2 t + A}$

$\downarrow$

$\frac{d^2X}{dx^2} + \lambda^2 X = 0$, A.E. $m^2 + \lambda^2 = 0$

i.e. $m^2 = -\lambda^2$

i.e. $m = \pm i\lambda$

i.e. $X = e^{0x}(B_1\cos\lambda x + B_2\sin\lambda x)$

i.e. $X = B_1\cos\lambda x + B_2\sin\lambda x$.

A solution is $u = XT = e^{-\lambda^2 t + A}(B_1\cos\lambda x + B_2\sin\lambda x)$

Apply boundary conditions

$u(2,t)=0$ i.e. $x=2$ gives $0 = e^{-\lambda^2 t + A}(B_1\cos\lambda 2 + B_2\sin\lambda 2)$

i.e. $B_1\cos 2\lambda + B_2\sin 2\lambda = 0$.

$u_x(0,t)=0$ $\quad \frac{\partial u}{\partial x} = e^{-\lambda^2 t + A}(-B_1\lambda\sin\lambda x + B_2\lambda\cos\lambda x)$

17. continued

Then, $u_x(0,t)=0$, i.e. $x=0$, gives

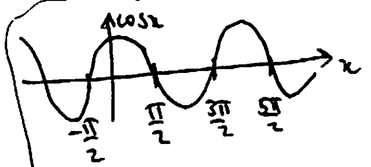
$$e^{-\lambda t^2 + A}(-B_1\lambda\sin 0 + B_2\lambda\cos 0) = 0$$

i.e. $-B\lambda\sin 0 + B_2\lambda\cos 0 = 0$ i.e. $B_2\lambda = 0$

i.e. $B_2 = 0$.

Return to $B_1\cos 2\lambda + B_2\sin 2\lambda = 0$ condition

This gives $B_1\cos 2\lambda = 0$ i.e. $2\lambda = (2n+1)\frac{\pi}{2}$



$\cos x = 0$ when $x = \frac{(2n+1)}{2}\pi$, $n = 0, \pm1, \pm2, \ldots$

i.e. $\lambda = (2n+1)\frac{\pi}{4}$

We then have $u = e^{-(2n+1)^2\frac{\pi^2}{16}t + A}\, B_1\cos(2n+1)\frac{\pi}{4}\cdot\frac{x}{2}$

i.e. $u = e^A B_1 e^{-(2n+1)^2\frac{\pi^2}{16}t}\cos\left(\frac{2n+1}{8}\right)\pi x$

$u = D e^{-(2n+1)^2\frac{\pi^2}{16}t}\cos\left(\frac{2n+1}{8}\right)\pi x$.

Final boundary condition and superposition principle

$u(x,t=0) = D_1\cos\left(\frac{2n_1+1}{8}\right)\pi x + D_2\cos\left(\frac{2n_2+1}{8}\right)\pi x$

$= 8\cos\left(\frac{3\pi}{4}x\right) - 6\cos\left(\frac{9\pi x}{4}\right)$: $D_1 = 8, D_2 = -6$; $2n_1+1 = 6$; $2n_2+1 = 18$

$\therefore u(x,t) = 8e^{-\frac{36\pi^2 t}{16}}\cos\left(\frac{3\pi x}{4}\right) - 6e^{-\frac{324\pi^2 t}{16}}\cos\left(\frac{9\pi}{4}x\right)$.