Tutorial B

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES Intermediate Mathematics

Divergence and Curl

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The aim of this package is to provide a short self assessment programme for students who would like to be able to calculate divergences and curls in vector calculus.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction (Grad)

The vector differential operator ∇ , called "del" or "nabla", is **defined** in three dimensions to be:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\,.$$

Note that these are *partial derivatives*!

If a scalar function, $f(x, y, z)$, is defined and differentiable at all points in some region, then f is a differentiable scalar field. The del vector operator, ∇, may be applied to scalar fields and the result, ∇f, is a vector field. It is called the *gradient* of f (see the package on **Gradients and Directional Derivatives**).

Quiz As a revision exercise, choose the gradient of the scalar field $f(x, y, z) = xy^2 - yz$.

(a) $\boldsymbol{i} + (2x - z)\boldsymbol{j} - y\boldsymbol{k}$,
(b) $2xy\boldsymbol{i} + 2xy\boldsymbol{j} + y\boldsymbol{k}$,
(c) $y^2\boldsymbol{i} - z\boldsymbol{j} - y\boldsymbol{k}$,
(d) $y^2\boldsymbol{i} + (2xy - z)\boldsymbol{j} - y\boldsymbol{k}$.

The vector operator ∇ may also be allowed to act upon vector fields. Two different ways in which it may act, the subject of this package, are extremely important in mathematics, science and engineering. We will first briefly review some useful properties of vectors.

Consider the (three dimensional) vector, $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$. We may also write this as $\boldsymbol{a} = (a_1, a_2, a_3)$. If we multiply it by a constant c, then every component of the vector is multiplied by c:

$$c\boldsymbol{a} = \boldsymbol{a}c = (ca_1, ca_2, ca_3)\,.$$

If we introduce a second vector, $\boldsymbol{b} = (b_1, b_2, b_3)$, then we recall that there are two different ways of multiplying vectors together, the scalar and vector products.

The **scalar product** (also called dot product) is defined by:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3\,.$$

It is a scalar (as the name scalar product implies).

Quiz Select the scalar product of $\boldsymbol{a} = (1, 2, 3)$ and $\boldsymbol{b} = (3, -2, 1)$.

(a) 2, (b) 10, (c) $3x - 4y + 3z$, (d) 4.

The **vector product** (or cross product) is defined by:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} . \end{aligned}$$

It is a vector (as the name vector product implies). Note that the second line is a *useful shorthand* for the first.

Quiz Choose the vector product of $\boldsymbol{a} = (1, 2, 3)$ and $\boldsymbol{b} = (3, -2, 1)$.

(a) $8\boldsymbol{i} - 8\boldsymbol{j} - 8\boldsymbol{k}$, (b) $-4\boldsymbol{i} - 10\boldsymbol{j} + 4\boldsymbol{k}$,

(c) $8\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k}$, (d) $8\boldsymbol{i} - 10\boldsymbol{j} - 8\boldsymbol{k}$.

2. Divergence (Div)

If $\boldsymbol{F}(x, y)$ is a vector field, then its **divergence** is written as $\text{div}\, \boldsymbol{F}(x, y) = \boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{r})$ which in two dimensions is:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{F}(x, y) &= (\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}) \cdot (F_1(x, y)\boldsymbol{i} + F_2(x, y)\boldsymbol{j}), \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} . \end{aligned}$$

It is obtained by taking the *scalar product* of the vector operator $\boldsymbol{\nabla}$ applied to the vector field $\boldsymbol{F}(x, y)$. The *divergence* of a vector field is a *scalar* field.

Example 2 The divergence of $\boldsymbol{F}(x, y) = 3x^2\boldsymbol{i} + 2y\boldsymbol{j}$ is:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{F}(x, y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2y) = 6x + 2 . \end{aligned}$$

Quiz Select the divergence of $\boldsymbol{F}(x, y) = \dfrac{x}{y}\boldsymbol{i} + (2x - 3y)\boldsymbol{j}$.

(a) $\dfrac{1}{y} - 3$, (b) $-\dfrac{x}{y^2} + 2$, (c) $\dfrac{1}{y} - \dfrac{x}{y^2}$, (d) -2.

The definition of the **divergence** may be directly extended to vector fields defined in three dimensions, $\boldsymbol{F}(x, y, z) = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} .$$

EXERCISE 1. Calculate the divergence of the vector fields $\boldsymbol{F}(x, y)$ and $\boldsymbol{G}(x, y, z)$ (click on the green letters for the solutions).

(a) $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$, (b) $\boldsymbol{F} = y^3\boldsymbol{i} + xy\boldsymbol{j}$,

(c) $\boldsymbol{F} = 3x^2\boldsymbol{i} - 6xy\boldsymbol{j}$, (d) $\boldsymbol{G} = x^2\boldsymbol{i} + 2z\boldsymbol{j} - y\boldsymbol{k}$,

(e) $\boldsymbol{G} = \dfrac{4y}{x^2}\boldsymbol{i} + \sin(y)\boldsymbol{j} + 3\boldsymbol{k}$, (f) $\boldsymbol{G} = e^x\boldsymbol{i} + \ln(xy)\boldsymbol{j} + e^{xyz}\boldsymbol{k}$.

3. Curl

The **curl** of a vector field, $\boldsymbol{F}(x, y, z)$, in three dimensions may be written $\text{curl}\, \boldsymbol{F}(x, y, z) = \boldsymbol{\nabla} \times \boldsymbol{F}(x, y, z)$, i.e.:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F}(x, y, z) &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} . \end{aligned}$$

It is obtained by taking the *vector product* of the vector operator $\boldsymbol{\nabla}$ applied to the vector field $\boldsymbol{F}(x, y, z)$. The second line is again a formal shorthand. The *curl* of a vector field is a *vector* field.

N.B. $\boldsymbol{\nabla} \times \boldsymbol{F}$ is sometimes called the *rotation* of $\boldsymbol{F}$ and written $\text{rot}\, \boldsymbol{F}$.

Example 3 The curl of $\boldsymbol{F}(x,y,z) = 3x^2\boldsymbol{i} + 2z\boldsymbol{j} - x\boldsymbol{k}$ is:

$$\begin{aligned} \nabla \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 2z & -x \end{vmatrix} \\ &= (\frac{\partial(-x)}{\partial y} - \frac{\partial(2z)}{\partial z})\boldsymbol{i} - (\frac{\partial(-x)}{\partial x} - \frac{\partial(3x^2)}{\partial z})\boldsymbol{j} \\ &\quad + (\frac{\partial(2z)}{\partial x} - \frac{\partial(3x^2)}{\partial y})\boldsymbol{k} \\ &= (0-2)\boldsymbol{i} - (-1-0)\boldsymbol{j} + (0-0)\boldsymbol{k} \\ &= -2\boldsymbol{i} + \boldsymbol{j}\,. \end{aligned}$$

Quiz Which of the following is the curl of $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$?

(a) $2\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, (b) $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, (c) 0, (d) $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

EXERCISE 2. Calculate the curl of the following vector fields $\boldsymbol{F}(x,y,z)$ (click on the green letters for the solutions).

(a) $\boldsymbol{F} = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$, (b) $\boldsymbol{F} = y^3\boldsymbol{i} + xy\boldsymbol{j} - z\boldsymbol{k}$,

(c) $\boldsymbol{F} = \dfrac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}}$, (d) $\boldsymbol{F} = x^2\boldsymbol{i} + 2z\boldsymbol{j} - y\boldsymbol{k}$.

Here is a review exercise before the final quiz.

EXERCISE 3. Let f be a scalar field and $\boldsymbol{F}(x,y,z)$ and $\boldsymbol{G}(x,y,z)$ be vector fields. What, if anything, is wrong with each of the following expressions (click on the green letters for the solutions)?

(a) $\nabla f = x^3 - 4y$, (b) $\nabla \cdot \boldsymbol{F} = \boldsymbol{i} - x^2y\boldsymbol{j} - z\boldsymbol{k}$,

(c) $\nabla \times \boldsymbol{G} = \nabla \cdot \boldsymbol{F}$.

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. Select the divergence of $\boldsymbol{G}(x,y,z) = 2x^3\boldsymbol{i} - 3xy\boldsymbol{j} + 3x^2z\boldsymbol{k}$?
   (a) $9x^2 - 3x$, (b) $6x^2 + 3x$, (c) 0, (d) $3x^2 - 3x$,
2. Select the divergence of $\boldsymbol{r}/r^3$, where $r = |\boldsymbol{r}|$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
   (a) $\dfrac{-1}{r^3}$, (b) 0, (c) $\dfrac{-2}{r^3}$, (d) $\dfrac{3}{r^3}$.
3. Choose the curl of $\boldsymbol{F}(x,y,z) = x^2\boldsymbol{i} + xyz\boldsymbol{j} - z\boldsymbol{k}$ at the point $(2,1,-2)$.
   (a) $2\boldsymbol{i} + 2\boldsymbol{k}$, (b) $-2\boldsymbol{i} - 2\boldsymbol{j}$, (c) $4\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$, (d) $-2\boldsymbol{i} - 2\boldsymbol{k}$.
4. Select the irrotational vector field (i.e., whose curl is zero)
   (a) $yz\boldsymbol{i} - 2xz\boldsymbol{j} + xyz\boldsymbol{k}$, (b) $yz\boldsymbol{i} + xz\boldsymbol{j} + xz\boldsymbol{k}$,
   (c) $z\boldsymbol{i} - z^2\boldsymbol{j} + yz\boldsymbol{k}$, (d) $y\boldsymbol{i} + (x-z)\boldsymbol{j} - y\boldsymbol{k}$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) The vector field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$ has components

$$F_1 = x\,, \qquad F_2 = y\,,$$

and its divergence is

$$\begin{aligned} \nabla \cdot \boldsymbol{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y = 1 + 1 = 2\,. \end{aligned}$$

Click on the green square to return

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Exercise 1(b) If the vector field is $\boldsymbol{F} = y^3\boldsymbol{i} + xy\boldsymbol{j}$, its components are

$$F_1 = y^3, \qquad F_2 = xy,$$

and its divergence is

$$\begin{aligned}\nabla \cdot \boldsymbol{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}y^3 + \frac{\partial}{\partial y}xy = 0 + x = x.\end{aligned}$$

Click on the green square to return

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Exercise 1(c) If the vector field is $\boldsymbol{F} = 3x^2\boldsymbol{i} - 6xy\boldsymbol{j}$, its components are

$$F_1 = 3x^2, \qquad F_2 = -6xy,$$

and its divergence is

$$\begin{aligned}\nabla \cdot \boldsymbol{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}3x^2 + \frac{\partial}{\partial y}(-6xy) = 6x - 6x = 0.\end{aligned}$$

N.B. A vector field with vanishing divergence is called a solenoidal vector field.

Click on the green square to return

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Exercise 1(d)

The vector field $\boldsymbol{G} = x^2\boldsymbol{i} + 2z\boldsymbol{j} - y\boldsymbol{k}$ has components

$$G_1 = x^2, \qquad G_2 = 2z, \qquad G_3 = -y$$

and its divergence is

$$\begin{aligned}\nabla \cdot \boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(-y) = 2x + 0 + 0 = 2x.\end{aligned}$$

Click on the green square to return

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Exercise 1(e)

Consider the vector field $\boldsymbol{G} = \frac{4y}{x^2}\boldsymbol{i} + \sin(y)\boldsymbol{j} + 3\boldsymbol{k}$. Its components are

$$G_1 = \frac{4y}{x^2}, \qquad G_2 = \sin(y), \qquad G_3 = 3$$

and its divergence is

$$\begin{aligned}\nabla \cdot \boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} \\ &= \frac{\partial}{\partial x}\left(\frac{4y}{x^2}\right) + \frac{\partial}{\partial y}\sin(y) + \frac{\partial}{\partial z}3 \\ &= 4y \times \frac{\partial}{\partial x}x^{-2} + \cos(y) = 4y \times (-2)x^{-2-1} + \cos(y) \\ &= -8yx^{-3} + \cos(y) = -\frac{8y}{x^3} + \cos(y).\end{aligned}$$

Click on the green square to return

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Exercise 1(f) Consider the vector field $\boldsymbol{G} = e^x\boldsymbol{i} + \ln(xy)\boldsymbol{j} + e^{xyz}\boldsymbol{k}$.
Its components are

$$G_1 = e^x\,, \qquad G_2 = \ln(xy)\,, \qquad G_3 = e^{xyz}$$

and its divergence is

$$\begin{aligned}\nabla\cdot\boldsymbol{G} &= \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\\ &= \frac{\partial}{\partial x}e^x + \frac{\partial}{\partial y}\ln(xy) + \frac{\partial}{\partial z}e^{xyz}\\ &= e^x + \frac{\partial}{\partial y}(\ln(x)+\ln(y)) + e^{xyz}\times\frac{\partial}{\partial z}(xyz)\\ &= e^x + \frac{1}{y} + xye^{xyz}\,.\end{aligned}$$

Click on the green square to return

□

Exercise 2(a)
The components of the vector field $\boldsymbol{F} = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$ are

$$F_1 = x\,, \qquad F_2 = -y\,, \qquad F_3 = z$$

and its curl is:

$$\begin{aligned}\nabla\times\boldsymbol{F} &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k}\\ &= (\frac{\partial(z)}{\partial y} - \frac{\partial(-y)}{\partial z})\boldsymbol{i} - (\frac{\partial(z)}{\partial x} - \frac{\partial(x)}{\partial z})\boldsymbol{j} + (\frac{\partial(-y)}{\partial x} - \frac{\partial(x)}{\partial y})\boldsymbol{k}\\ &= 0\boldsymbol{i} - 0\boldsymbol{j} + 0\boldsymbol{k} = 0\,.\end{aligned}$$

Therefore the vector field $\boldsymbol{F} = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$ is an irrotational vector field.

Click on the green square to return

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Exercise 2(b)
The components of the vector field $\boldsymbol{F} = y^3\boldsymbol{i} + xy\boldsymbol{j} - z\boldsymbol{k}$ are

$$F_1 = y^3\,, \qquad F_2 = xy\,, \qquad F_3 = -z$$

and its curl is:

$$\begin{aligned}\nabla\times\boldsymbol{F} &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k}\\ &= (\frac{\partial(-z)}{\partial y} - \frac{\partial(xy)}{\partial z})\boldsymbol{i} - (\frac{\partial(-z)}{\partial x} - \frac{\partial(y^3)}{\partial z})\boldsymbol{j}\\ &\quad + (\frac{\partial(xy)}{\partial x} - \frac{\partial(y^3)}{\partial y})\boldsymbol{k}\\ &= 0\boldsymbol{i} - 0\boldsymbol{j} + (y - 3y^2)\boldsymbol{k} = (y - 3y^2)\boldsymbol{k}\,,\end{aligned}$$

i.e., the curl vector is in the $\boldsymbol{k}$ direction.

Click on the green square to return

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Exercise 2(c) The components of the vector field $\boldsymbol{F} = \dfrac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}}$ are $F_1 = \dfrac{x}{r}$, $F_2 = \dfrac{y}{r}$, $F_3 = \dfrac{z}{r}$, where $r = \sqrt{x^2+y^2+z^2}$.
The $\boldsymbol{i}$ component of $\nabla\times\boldsymbol{F}$, is:

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}(\frac{z}{r}) - \frac{\partial}{\partial z}(\frac{y}{r}) = z\frac{\partial}{\partial y}(\frac{1}{r}) - y\frac{\partial}{\partial z}(\frac{1}{r})$$

The derivative of $\dfrac{1}{r}$ with respect to y is

$$\frac{\partial}{\partial y}(\frac{1}{r}) = \frac{\partial}{\partial y}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} = (-\frac{1}{2})\times\frac{2y}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\frac{y}{r^3}\,.$$

and similarly $\dfrac{\partial}{\partial z}(\dfrac{1}{r}) = -\dfrac{z}{r^3}$. Thus the $\boldsymbol{i}$ component of the curl is $(-\dfrac{zy}{r^3}) - (-\dfrac{yz}{r^3}) = 0$. It may be checked that the $\boldsymbol{j}$ and $\boldsymbol{k}$ components of the curl also vanish.
Click on the green square to return

□

Exercise 2(d)
The components of the vector field $\boldsymbol{F} = x^2\boldsymbol{i} + 2z\boldsymbol{j} - y\boldsymbol{k}$ are

$$F_1 = x^2\,, \qquad F_2 = 2z\,, \qquad F_3 = -y$$

and its curl is:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F} &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k} \\
&= (\frac{\partial(-y)}{\partial y} - \frac{\partial(2z)}{\partial z})\boldsymbol{i} - (\frac{\partial(-y)}{\partial x} - \frac{\partial(x^2)}{\partial z})\boldsymbol{j} \\
&\quad + (\frac{\partial(2z)}{\partial x} - \frac{\partial(x^2)}{\partial y})\boldsymbol{k} \\
&= (-1-2)\boldsymbol{i} - (0-0)\boldsymbol{j} + (0-0)\boldsymbol{k} = -3\boldsymbol{i}\,.
\end{aligned}$$

Click on the green square to return

□

Exercise 3(a)
The formula

$$\boldsymbol{\nabla} f = x^3 - 4y$$

must be incorrect because the gradient of a scalar function is a vector field by definition, while the expression on the right hand side of this equation is a scalar.

Click on the green square to return

□

Exercise 3(b)
The equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{i} - x^2y\boldsymbol{j} - z\boldsymbol{k}$$

must be incorrect, because the divergence of a vector field must be a scalar by definition but the right hand side of the equation is a vector.

Click on the green square to return

□

Exercise 3(c)
The equation

$$\boldsymbol{\nabla} \times \boldsymbol{G} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$$

must be incorrect because its left hand side is a vector field, a curl, while its right hand side is a scalar function, a divergence.

Click on the green square to return

□

Solutions to Quizzes

Solution to Quiz:
If the scalar field is $f(x,y,z) = xy^2 - yz$, its gradient is

$$\begin{aligned}\nabla f &= \frac{\partial}{\partial x}(xy^2 - yz)\boldsymbol{i} + \frac{\partial}{\partial y}(xy^2 - yz)\boldsymbol{j} \\ &\quad + \frac{\partial}{\partial z}(xy^2 - yz)\boldsymbol{k} \\ &= y^2 \times \frac{\partial}{\partial x}(x)\boldsymbol{i} + \left[x \times \frac{\partial}{\partial y}(y^2) - z \times \frac{\partial}{\partial y}(y)\right]\boldsymbol{j} \\ &\quad + (-y) \times \frac{\partial}{\partial z}(z)\boldsymbol{k} \\ &= y^2\boldsymbol{i} + (2xy - z)\boldsymbol{j} - y\boldsymbol{k}\,.\end{aligned}$$

End Quiz

Solution to Quiz:
The scalar product of the two vectors

$$\boldsymbol{a} = (1,2,3) \quad \text{and} \quad \boldsymbol{b} = (3,-2,1)$$

is

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= 1\times 3 + 2\times(-2) + 3\times 1 \\ &= 3 - 4 + 3 \\ &= 2\,.\end{aligned}$$

End Quiz

Solution to Quiz:
The vector product of two vectors

$$\boldsymbol{a} = (1,2,3) \quad \text{and} \quad \boldsymbol{b} = (3,-2,1)$$

is

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} \\ &= (2\times 1 - 3\times(-2))\,\boldsymbol{i} - (1\times 1 - 3\times 3)\,\boldsymbol{j} \\ &\quad + (1\times(-2) - 2\times 3)\boldsymbol{k} \\ &= (2+6)\,\boldsymbol{i} - (1-9)\,\boldsymbol{j} + (-2-6)\boldsymbol{k} \\ &= 8\boldsymbol{i} - (-8)\boldsymbol{j} - 8\boldsymbol{k} = 8\boldsymbol{i} + 8\boldsymbol{j} - 8\boldsymbol{k}\,.\end{aligned}$$

End Quiz

Solution to Quiz:
The vector field

$$\boldsymbol{F}(x,y) = \frac{x}{y}\boldsymbol{i} + (2x - 3y)\boldsymbol{j}$$

has components $F_1(x,y) = \dfrac{x}{y}$ and $F_2 = 2x - 3y$, so its divergence is

$$\begin{aligned}\nabla\cdot\boldsymbol{F}(x,y) &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial}{\partial y}(2x - 3y) \\ &= \frac{1}{y} - 3\,.\end{aligned}$$

N.B. The divergence of a vector is a scalar.

End Quiz

Solution to Quiz:
The components of the vector field $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ are

$$F_1 = x\,, \qquad F_2 = y\,, \qquad F_3 = z$$

and its curl is:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F} &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\boldsymbol{i} - (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z})\boldsymbol{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\boldsymbol{k} \\ &= (\frac{\partial (z)}{\partial y} - \frac{\partial (y)}{\partial z})\boldsymbol{i} - (\frac{\partial (z)}{\partial x} - \frac{\partial (x)}{\partial z})\boldsymbol{j} + (\frac{\partial (y)}{\partial x} - \frac{\partial (x)}{\partial y})\boldsymbol{k} \\ &= 0\boldsymbol{i} - 0\boldsymbol{j} + 0\boldsymbol{k} = \boldsymbol{0}\,. \end{aligned}$$

N.B. A vector field with vanishing curl is called an irrotational vector field.

End Quiz

Divergence & Curl (mini-tutorial)

Theoretical Physics I
GS McDonald

FINAL QUIZ SOLUTIONS

1. $\underset{\sim}{g} = 2x^3\underset{\sim}{i} - 3xy\underset{\sim}{j} + 3x^2z\underset{\sim}{k}$. $\underset{\sim}{\nabla}\cdot\underset{\sim}{g} = \frac{\partial}{\partial x}(2x^3) + \frac{\partial}{\partial y}(-3xy) + \frac{\partial}{\partial z}(3x^2z) = 6x^2 - 3x + 3x^2$
$= 9x^2 - 3x.$

2. $\underset{\sim}{\nabla}\cdot\left(\frac{\underset{\sim}{r}}{r^3}\right) = \underset{\sim}{\nabla}\cdot\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\cdot(x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k})\right]$,

where $\frac{\partial}{\partial x}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\cdot x\right] = \left(-\frac{3}{2}\cdot\frac{1}{(x^2+y^2+z^2)^{5/2}}\cdot 2x\right)\cdot x + \left(\frac{1}{(x^2+y^2+z^2)^{3/2}}\right)\cdot 1$ [product rule]

$= -\frac{3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}$ [to take common denominator]

similarly, $\frac{\partial}{\partial y}[\ldots y] = \frac{-3y^2 + (\quad)}{(\quad)^{5/2}}$ and $\frac{\partial}{\partial z}[\ldots z] = \frac{-3z^2 + (x^2+y^2+z^2)}{(\quad)^{5/2}}$.

Add three together: $\underset{\sim}{\nabla}\cdot\left(\frac{\underset{\sim}{r}}{r^3}\right) = \frac{-3x^2 - 3y^2 - 3z^2 + 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0.$

3. $\underset{\sim}{F} = x^2\underset{\sim}{i} + xyz\underset{\sim}{j} + (-z)\underset{\sim}{k}$.

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xyz & -z \end{vmatrix} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(-z) - \frac{\partial}{\partial z}(xyz)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial z}(x^2)\right] + \underset{\sim}{k}\left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(x^2)\right]$$

$= \underset{\sim}{i}(0 - xy) - \underset{\sim}{j}(0-0) + \underset{\sim}{k}(yz - 0) = -xy\underset{\sim}{i} + yz\underset{\sim}{k}.$

Then, $x=2, y=1, z=-2$ gives $\underset{\sim}{\nabla}\times\underset{\sim}{F} = -2.1.\underset{\sim}{i} + 1.(-2).\underset{\sim}{k} = -2\underset{\sim}{i} - 2\underset{\sim}{k}$.

4. For (d) we have

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-z & -y \end{vmatrix} = \underset{\sim}{i}\left[\frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial z}(x-z)\right] - \underset{\sim}{j}\left[\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial z}(y)\right] + \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x-z) - \frac{\partial}{\partial y}(y)\right]$$

$= \underset{\sim}{i}\left[-1-(-1)\right] - \underset{\sim}{j}\left[0-0\right] + \underset{\sim}{k}\left[1-1\right] = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}.$